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Research Article

Regular graphs with large Italian domatic number

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Abstract: For a graph G, an Italian dominating function is a function $f : V(G) \to \{0, 1, 2\}$ such that for each vertex $v \in V(G)$ either $f(v) \neq 0$, or $\sum_{u \in N(v)} f(u) \geq 2$. If a family $\mathcal{F} = \{f_1, f_2, \ldots, f_t\}$ of distinct Italian dominating functions satisfy $\sum_{i=1}^{t} f_i(v) \leq 2$ for each vertex v, then this is called an Italian dominating family. In [L. Volkmann, The Roman {2}-domatic number of graphs, Discrete Appl. Math. **258** (2019), 235–241], Volkmann defined the *Italian domatic number* of G, $d_I(G)$, as the maximum cardinality of any Italian dominating family.

In this same paper, questions were raised about the Italian domatic number of regular graphs. In this paper, we show that two of the conjectures are false, and examine some exceptions to a Nordhaus-Gaddum type inequality.

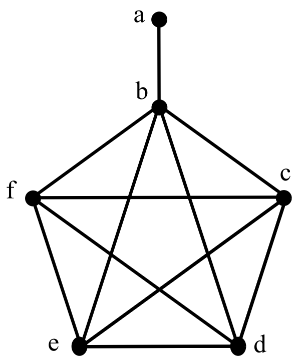
Keywords: Italian domination, Nordhaus-Gaddum, Domatic number

AMS Subject classification: 05C69

1. Introduction

Roman domination first originated with Hedetniemi et al. in a series of papers ([3],[4]), written in response to an article by Stewart [10] regarding a strategy to defend the Roman Empire. Since then, many variations of Roman domination have been considered, including Italian domination. Italian domination first appeared in a paper by Chellali et al. [2], though it was called Roman {2}-domination at the time. In [5], the same concept was termed Italian domination.

An Italian dominating function is a function $f : V(G) \to \{0, 1, 2\}$ (an assignment of weights to vertices), such that for every vertex $v \in V(G)$ such that $f(v) = 0$, then $\sum_{w \in N(v)} f(w) \geq 2$. The weight of an Italian dominating function f is the sum of the values of f on the vertices of G, i.e. $\sum_{v \in V(G)} f(v)$. The Italian domination number of a graph G, denoted $\gamma_I(G)$, is the minimum weight of any Italian dominating function. In [12], Volkmann defined the Italian domatic number for graphs (and also for digraphs in [11]) as follows: Let $\mathcal{F} = \{f_1, f_2, \ldots, f_t\}$ be a set of *distinct* Italian dominating



	a	b	c	d	e	f
f_1	1	0	0	1	0	1
f_2	1	0	1	0	1	0
f_3	0	2	0	1	1	0

Figure 1. The functions f_1, f_2, and f_3 shown in the table form an Italian dominating family.

functions. If $\sum_{i=1}^{t} f_i(v) \leq 2$ for each vertex v, then this is called an Italian dominating family (see Figure 1 for an example). Then the *Italian domatic number* of G, $d_I(G)$, is the maximum cardinality of any such family.
In [12], Volkmann proved the following Nordhaus-Gaddum type inequality regarding the Italian domatic number.

Theorem 1. *If G is a graph of order n, then*

$$d_I(G) + d_I(\overline{G}) \leq n + 2.$$

with the exception of the cases in which G is 4-regular of order 9, 7-regular of order 18, or 16-regular of order 45.

The exceptions to Theorem 1 in [12] are due to limitations on the proof, not necessarily because the result does not hold in these cases. In [12], Volkmann conjectured that these limitations could be removed (i.e., that Theorem 1 holds for all graphs), and specifically conjectured the following two statements, which would imply that result.

Conjecture 1. If G is a regular graph of order n,

$$d_I(G) + d_I(\overline{G}) \leq n + 1.$$

Conjecture 2. If G is a δ-regular graph of order n,

$$d_I(G) \leq \delta + 1.$$

Similar conjectures were reiterated in [11]. In this note, we will show that these conjectures are false, and partially address the three exceptions to Theorem 1. In particular, there is an infinite family of δ-regular graph such that $d_I(G) = \delta + 2$, and this is shown in Theorem 6. In addition, to partially address the exceptions in Theorem 1, we will look at the case of a 4-regular of order 9 and a 7-regular of order 18, and determine the following result.

Theorem 2. *Let G be a δ-regular graph of order n.*

- *There is exactly one graph with $\delta = 4$ and $n = 9$ such that*

$$d_I(G) + d_I(\overline{G}) = n + 3.$$

- *There is no graph with $\delta = 7$ and $n = 18$ such that*

$$d_I(G) + d_I(\overline{G}) = n + 3.$$

2. Italian dominating functions for graphs with large domatic number

To begin, we can consider some of the properties of Italian dominating functions if the domatic number is as large as possible. In [2], Chellali et al. proved a simple bound that gives a lower bound on $\gamma_I(G)$.

Theorem 3. *For a graph G on n vertices with maximum degree $\Delta(G)$,*

$$\gamma_I(G) \geq \left\lceil \frac{2n}{\Delta(G) + 2} \right\rceil.$$

Regarding the domatic number, the following simple bound was shown in [12] (we include a proof here to highlight the conditions for equality).

Theorem 4. *For any graph G with minimum degree $\delta(G)$,*

$$d_I(G) \leq \delta(G) + 2.$$

Furthermore, if $\mathcal{F} = \{f_1, f_2, \ldots, f_{d_I(G)}\}$ is an Italian dominating family, such that $d_I(G) = \delta(G) + 2$, then for any minimum degree vertex v, the following three statements must be true:

1. *There are exactly $\delta(G)$ Italian dominating functions such that $f_i(v) = 0$, and exactly two Italian dominating functions such that $f_i(v) = 1$.*
2. *For every $u \in N(v)$, if $f_i(v) = 1$, then $f_i(u) = 0$.*
3. *For every Italian dominating function f_i in which $f_i(v) = 0$, $\sum_{u \in N(v)} f_i(u) = 2$.*

Proof. Let $d = d_I(G)$ and $\mathcal{F} = \{f_1, f_2, \ldots, f_d\}$ be the corresponding family of Italian dominating functions, and choose v be a vertex of minimum degree. First, we can determine the number of functions f_i such that $f_i(v) = 0$. WLOG assume $f_1, f_2, \ldots, f_{d'}$ are the Italian dominating functions such that $f_i(v) = 0$ (for some d').

For each i such that $f_i(v) = 0$, the sum of the values of f_i on the neighbors of v must be at least 2, i.e. $\sum_{u\in N(v)} f_i(u) \geq 2$. Also, by definition of the Italian domatic number, $\sum_{i=1}^{d'} f_i(u) \leq 2$. Thus,

$$2d' \leq \sum_{i=1}^{d'} \left(\sum_{u\in N(v)} f_i(u) \right) = \sum_{u\in N(v)} \left(\sum_{i=1}^{d'} f_i(u) \right) \leq 2\delta(G). \tag{1}$$

Thus, there are at most $\delta(G)$ Italian dominating functions such that $f_i(v) = 0$. Note that equality is obtained only when $\delta(G) = d'$, and as a result, when $\sum_{i=1}^{d'} f_i(u) = 2$ and $\sum_{u\in N(v)} f_i(u) = 2$ (Statement 3). Next, consider the two cases for functions that are nonzero at v.

1. There is a function f_i such that $f_i(v) = 2$.
   In this case, there is only one such function, since $\sum_{i=1}^{d} f_i(u) \leq 2$, so we have $d_I(G) \leq \delta(G) + 1$.

2. There are two functions $f_{d'+1}, f_{d'+2}$ such that $f_{d'+1}(v) = 1$ and $f_{d'+2}(v) = 1$.
   This can occur for at most two indices, since $\sum_{i=1}^{d} f_i(v) \leq 2$, so we have $d_I(G) \leq \delta(G) + 2$.

Therefore, $d_I(G) \leq \delta(G) + 2$, and equality is obtained only in case 2, when there are exactly $\delta(G)$ Italian dominating functions in $\mathcal{F}$ such that $f_i(v) = 0$, and two Italian dominating functions in $\mathcal{F}$ in which $f_i(v) = 1$ (Statement 1).
Lastly, consider a neighbor u of the vertex v. To achieve equality in Equation 1, $\sum_{i=1}^{d'} f_i(u) = 2$. Since $\sum_{i=1}^{d'+2} f_i(u) \leq 2$, then this must imply that $f_{d'+1}(u) = f_{d'+2}(u) = 0$ (Statement 2); otherwise $\sum_{i=1}^{d'} f_i(u) < 2$. □

In a regular graph, we can apply the statements about vertices of minimum degree at equality to every vertex, so that if $d_I(G) = \delta(G) + 2$, and $\mathcal{F} = \{f_1, f_2, \ldots, f_d\}$ is a corresponding family of Italian dominating functions, then this would imply each Italian dominating function is a function $f_i : V(G) \to \{0, 1\}$. This allows us to view the Italian dominating functions as indicator functions, and in what follows, it will be convenient to restate the property that $d_I(G) = \delta(G) + 2$ for a regular graph G in terms of a family of sets.

Lemma 1. *Let G be a regular graph.*
Then G satisfies $d_I(G) = \delta(G) + 2$ if and only if there is a family of distinct sets $\mathcal{S} = \{S_1, \ldots, S_{\delta(G)+2}\}$, $S_i \subset V(G)$, that satisfy the following four conditions:

1. *Every vertex v appears in exactly two sets S_i.*

2. *The sets S_i are independent.*

3. *For any vertex $v \notin S_i$, $|N(v) \cap S_i| = 2$.*

4. *Each set S_i satisfies $|S_i| = \frac{2n}{\delta(G)+2} = \gamma_I(G)$.*

Proof. First, suppose that G is a regular graph such that $d_I(G) = \delta(G) + 2$, and let $\mathcal{F} = \{f_1, f_2, \dots, f_{\delta(G)+2}\}$ be a corresponding Italian dominating family. For each dominating function f_i, we define the set $S_i = S(f_i) = \{v \in V(G) : f_i(v) \neq 0\}$.
Since G is a regular graph, the statements about minimum degree vertices at equality in Theorem 4 can be applied to every vertex in the graph. As discussed, one consequence of Statement 1 in Theorem 4, applied to all vertices in G, is that $f_i : V(G) \to \{0, 1\}$ for each i. Conditions 1 through 3 above are, roughly, translations of these statements to the sets S_i, as we will show below.

- First, Statement 1 of Theorem 4 indicates that for each vertex v, there are exactly two Italian dominating functions such that $f_i(v) = 1$. Applying the definition of the sets S_i, this implies that each vertex v appears in exactly two sets, which is Condition 1.

- Second, Statement 2 of Theorem 4 indicates that for each vertex v, and any vertex $u \in N(v)$, if $f_i(v) = 1$, then $f_i(u) = 0$. By definition of the sets S_i, this can be rewritten as follows: if $v \in S_i$, then for any $u \in N(v)$, $u \notin S_i$, implying each set S_i is independent, Condition 2.

- Third, Statement 3 of Theorem 4 indicates that for each vertex v, and every dominating function f_i such that $f_i(v) = 0$, it must be that $\sum_{u \in N(v)} f_i(u) = 2$. This implies that there are exactly two neighbors of v, say u and w, such that $f_i(u) = f_i(w) = 1$. Then, by definition of the sets S_i, this implies that for any vertex $v \notin S_i$, $|N(v) \cap S_i| = 2$,

Lastly, since equality in Theorem 4 implied $f_i : V(G) \to \{0, 1\}$ for each i, then $|S_i| \geq \gamma_I(G)$. Applying Theorem 3 (and noting $\Delta(G) = \delta(G)$ in a regular graph), we get

$$\left\lceil \frac{2n}{\delta(G) + 2} \right\rceil (\delta(G) + 2) \leq \sum_{i=1}^{\delta(G)+2} |S_i| \leq 2n \leq \left\lceil \frac{2n}{\delta(G) + 2} \right\rceil (\delta(G) + 2).$$

Equality is only possible if $2n$ is divisible by $\delta(G) + 2$, and for each set S_i, $|S_i| = 2n/(\delta(G) + 2)$.
In the other direction, suppose that there exists sets $S_1, \dots, S_{\delta(G)+2} \subset V(G)$ satisfying Conditions 1 - 4. Let f_i be the indicator function for each set S_i. By Condition 3, such a function is an Italian dominating function, and by Conditions 1 and 4, these functions form an Italian dominating family with $\delta(G) + 2$ Italian dominating functions. Since $d_I(G) \leq \delta(G) + 2$, this implies that $d_I(G) = \delta(G) + 2$.
Lastly, we note that the weight of any function f_i is $|S_i|$, implying $\gamma_I(G) \leq |S_i|$. From Theorem 3, $\gamma_I(G) \geq 2n/(\delta(G) + 2)$; thus $|S_i| = \gamma_I(G)$. □

3. An Intersection Graph

In order to find counterexamples to Conjectures 1 and 2, it is helpful to look at intersection graphs. Suppose that a regular graph G satisfies $d_I(G) = \delta(G) + 2$

and gives rise to sets S_i satisfying the conditions of Lemma 1. We can construct an "intersection" graph from these sets, i.e. let the vertices s_i of a graph H correspond to the sets S_i, so that two vertices in H are adjacent if the corresponding sets intersect. The goal of this section is to investigate the properties of such a graph. The benefit of doing so is that the intersection graphs contain fewer vertices than the original graphs G. This means that searching for potential intersection graphs may be easier than searching for regular graphs G which satisfy $d_I(G) = \delta(G) + 2$ because the pool of potential candidates is much smaller.
More precisely, let G be a regular graph with a family of sets $\mathcal{S}$ satisfying the conditions of Lemma 1. Then we define the multigraph $H = H(G, \mathcal{S})$ as follows:

1. Each vertex $s_i \in V(H(G, \mathcal{S}))$ corresponds to a set S_i.
2. Consider sets S_i and S_j corresponding to $s_i, s_j \in V(H(G, \mathcal{S}))$. If the intersection $S_i \cap S_j$ is non-empty, there is an edge between the vertices s_i and s_j corresponding to each vertex of the intersection (so each edge in $H(G, \mathcal{S})$ corresponds to a vertex in G).

In what follows, we will frequently write $H(G, \mathcal{S})$ as H (it will be implied that it is an intersection graph), and refer to the graph G as the "host" graph. At this point, we can already note a few properties of the graph $H(G, \mathcal{S})$, implied by Lemma 1.

Observation 1. The graph $H = H(G, \mathcal{S})$ is regular with degree $\gamma_I(G)$ and contains $|V(G)|$ edges.

The graph H is regular simply because each set S_i is the same size ($|S_i| = \gamma_I(G)$), and every vertex of G is contained in exactly two different sets (i.e. in one intersection) so it shows up exactly once as an edge in H.
Another important property to note arises from thinking about how to reconstruct the host graph from the intersection graph. The vertices of the host graph correspond to the edges of the intersection graph, but what about the adjacencies of the host graph? Let $v \in V(G)$ be a vertex corresponding to an edge $e = (s_i, s_j) \in E(H)$ (i.e. $v \in S_i, S_j$ in the host graph). In the following list, we give the properties of the neighborhood of v in the host graph G, followed by the corresponding properties in H.

1. *The vertex v has exactly $\delta(G)$ neighbors overall in the host graph.*
   In H, this translates to a collection of $\delta(G)$ edges.
2. *The vertex v is not adjacent to any other vertices in S_i, S_j.*
   The collection of edges from the previous point cannot involve any edges incident with s_i and s_j in H.
3. *The vertex v has exactly two neighbors in each set S_k, (for $k \neq i, j$) in the host graph.*
   In H, this translates to exactly two edges that are incident with a vertex s_k (for all $k \neq i, j$).

Suppose that we remove the vertices s_i and s_j from the graph H. This would leave exactly $\delta(G)$ vertices. Then in H, the neighborhood of the vertex $v \in V(G)$ corresponds to a set of edges in $H - \{s_i, s_j\}$ so that every vertex s_k meets two edges. In other words, a perfect 2-factor [1] consisting of $\delta(G)$ edges in the graph $H - \{s_i, s_j\}$.
Since the host graph G, was undirected, we can strengthen this slightly. If a graph H is an intersection graph, then there must be a perfect 2-factor in every subgraph $H - \{s_i, s_j\}$, where $(s_i, s_j) \in E(H)$, and this family of 2-factors must satisfy what we will call the *symmetric property for edge-removed 2-factors*, or the "SPER-2 property".

Definition 1. We say that a graph H satisfies the "SPER-2 property" if there is a family of two-factors $\mathcal{M} = \{M_e : e \in E(H)\}$ such that for every edge $(s_i, s_j) \in E(H)$, the corresponding $M_{(s_i,s_j)}$ is a perfect 2-factor in the graph $H - \{s_i, s_j\}$, and for all edges $e, e' \in V(H)$, $e' \in M_e$ if and only if $e \in M_{e'}$.

We can now show that if a graph H satisfies the *SPER-2 property*, then it is an intersection graph for some graph G and some family $\mathcal{S}$.

Theorem 5. *Consider a regular graph H with degree k on n vertices. The graph H satisfies the SPER-2 property, if and only if it is an intersection graph $H(G, \mathcal{S})$ for a regular graph G of degree $n-2$ on $kn/2$ vertices such that $d_I(G) = n = \delta(G) + 2$.*

Proof. Let H be a regular graph of degree k with vertices $s_1, s_2, \ldots, s_n$, and let $\mathcal{M} = \{M_e : e \in E(H)\}$ be a family of 2-factors that demonstrates that H satisfies the *SPER-2 property*. Since H is regular with degree k on n vertices, then by the handshaking lemma, there are $k|V(H)|/2$ edges in the graph H.
We can reconstruct the graph G as follows. Let $\{v_{e_1}, v_{e_2}, \ldots, v_{e_{|E(H)|}}\}$ be the vertices of G so that each vertex corresponds to an edge $e_i \in E(H)$ (so there are a total of $kn/2$ vertices in $V(G)$). To define the edges, we say that $(v_{e_i}, v_{e_j}) \in E(G)$ if and only if $v_{e_j} \in M_{e_i}$ (and by the SPER-2 property, $v_{e_i} \in M_{e_j}$). Since each 2-factor M_{e_i} contains $|V(H)| - 2$ edges, each vertex in $V(G)$ has degree $n-2$.
To show that $d_I(G) = \delta(G) + 2$, we can explicitly define a family of sets to satisfy Lemma 1. First, associate a set S with each vertex in H. In particular, for a vertex $s_k \in V(H)$ we define S_k such that

$$S_k = \{v_e \in V(G) : e \text{ is incident with } s_k\}.$$

Then we can show that the four conditions from Lemma 1 are satisfied for this family of sets.

1. Each vertex $v_e \in V(G)$ such that $e = (s_i, s_j)$ is contained in exactly two sets, namely S_i and S_j.

[1] *A perfect 2-factor is a union of vertex-disjoint cycles.*

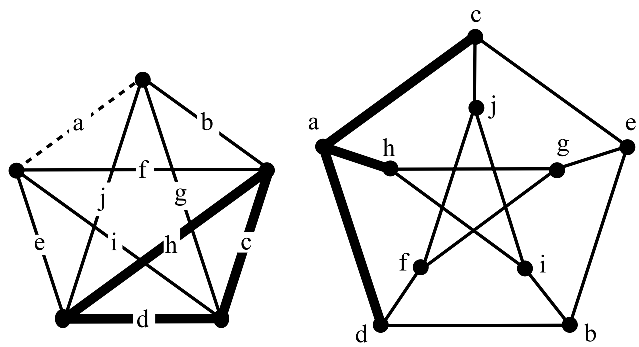


Figure 2. The complete graph K_5 satisfies the conditions of Theorem 5. The 2-factor M_a is shown in bold on K_5; in the host graph G, edges from a to the corresponding vertices are also shown in bold. The 5 Italian dominating functions are the indicator functions for the sets $\{a, j, g, b\}$, $\{b, f, g, e\}$, $\{c, g, i, d\}$, $\{d, h, j, e\}$, $\{e, i, f, a\}$.

2. Each set S_i is independent; If $e = (s_i, s_j)$ so that $v_e \in S_i$, all neighbors of v_e are vertices corresponding to edges from the 2-factor from $H - \{s_i, s_j\}$, so it has no neighbors $v_{e'}$ where e' contains s_i.

3. Since $M_{(s_i,s_j)}$ is a perfect 2-factor of $H - \{s_i, s_j\}$, the vertex $v_{(s_i,s_j)}$ will have two neighbors in each set S_k for $k \neq i, j$.

4. $|S_u| = \delta(H) = k = \dfrac{2\left(\frac{kn}{2}\right)}{n} = \dfrac{2|V(G)|}{\delta(G)+2}$

Therefore, H is an intersection graph for a regular graph G of degree $n-2$ on $kn/2$ vertices with $d_I(G) = \delta(G) + 2$.

In the other direction, suppose that H is an intersection graph for a regular graph G of degree $n-2$ on $kn/2$ vertices such that $d_I(G) = \delta(G)+2$. Let $\mathcal{S} = \{S_1, \ldots, S_{\delta(G)+2}\}$, $S_i \subset V(G)$, be the family of sets guaranteed by Lemma 1. Consider a vertex $v \in V(G)$ such that $v \in S_i, S_j$ (then for any $k \neq i, j$, $v \notin S_k$ by Condition 1). In the graph H, the vertex v corresponds to an edge $e_v = (s_i, s_j) \in E(H)$. Then, for each $v \in V(G)$, define the set $M_{e_v} = \{e_u : u \in N_G(v)\}$. Since v is adjacent to exactly two vertices in every set S_k $(k \neq i, j)$, then there are exactly two edges in M_{e_v} with s_k as an endpoint. Thus, M_{e_v} is a perfect 2-factor of $H - \{s_i, s_j\}$. Furthermore, since G is undirected, $u \in N(v)$ if and only if $v \in N(u)$. This implies that $e_u \in M_{e_v}$ if and only if $e_v \in M_{e_u}$. □

As an example, consider the complete graph $H = K_5$. For each edge (u, v), let $\mathcal{M}_{(u,v)}$ be the triangle on the vertices of $H - \{u, v\}$. It is easily verified that for this family of two factors, $e' \in \mathcal{M}_e$ if and only if $e \in \mathcal{M}_{e'}$. The resulting host graph (the Petersen graph) is shown in Figure 2.

Recall that given any graph G, its square graph G^2 is the graph with the same vertex set of G such that two vertices are adjacent in G^2 whenever they are distance 1 or 2 in G. The graph K_5 can also be thought of as the square of C_5, and we can show that this is one example of a family of graphs that satisfy the conditions of Theorem 5.

Lemma 2. *The square of a cycle C_n, $n \geq 5$, satisfies the SPER-2 property.*

Proof. Let $G = C_n$ with vertices $v_0, \ldots, v_{n-1}$ (To simplify notation and avoid adding "mod n" to the index of each vertex, assume that all arithmetic in the index of a vertex is done modulo n.) The vertices of G^2 are also $v_0, \ldots, v_{n-1}$ and the edges of G^2 can be partitioned into two sets, E_1 and E_2, such that $E_1 = E(G)$ (i.e., edges of the form (v_i, v_{i+1})) and $E_2 = E(G^2) - E(G)$ (i.e. edges of the form (v_i, v_{i+2})).
For each edge e in G^2, we explicitly describe the two-factor $\mathcal{M}_e$.

1. Suppose $e = (v_i, v_{i+1}) \in E_1$.

   In this case, choose $\mathcal{M}_e$ as follows: First, choose edges (v_{i-2}, v_{i-1}) and (v_{i+2}, v_{i+3}) from E_1, and all edges from E_2 except the four that involve v_i and v_{i+1}.

   Then, any vertex outside of v_{i-2}, v_{i-1}, v_{i+2}, and v_{i+3} is contained in two edges from E_2, and each of these four is contained in one edge from E_1 and one from E_2, forming a 2-factor (in fact, the 2-factor these edges form is simply a single cycle involving all vertices except v_i and v_{i+1}).

2. Suppose $e = (v_i, v_{i+2}) \in E_2$.

   In this case, choose $\mathcal{M}_e$ as follows: First, choose edges (v_{i-1}, v_{i+1}) and (v_{i+1}, v_{i+3}) from E_2, and all edges from E_1 except the four that involve v_i and v_{i+2}.

   Then, any vertex outside of v_{i-1}, v_{i+1}, and v_{i+3} is contained in two edges from E_1, the vertex v_{i+1} is in both edges from E_2, and the remaining two are contained in one edge from E_1 and one from E_2, forming a 2-factor (as in the previous case, the 2-factor these edges form is a single cycle involving all vertices except v_i and v_{i+2}).

Now consider the edge $e = (v_j, v_{j+1}) \in E_1$. This edge occurs in the two-factor $\mathcal{M}_{e'}$ for $e' = (v_i, v_{i+1})$ where $i = j + 2$ and also when $i = j - 2$. In addition, e occurs in the two factor $\mathcal{M}_{e'}$ for $e' = (v_i, v_{i+2})$ in any case when $i, i+2 \neq j$ and $i, i+2 \neq j+1$. Thus, $e \in \mathcal{M}_{e'}$ for any edge $e' \in \mathcal{M}_e$.
Next consider the edge $e = (v_j, v_{j+2}) \in E_2$. This edge occurs in the two-factor $\mathcal{M}_{e'}$ for $e' = (v_i, v_{i+2})$ where $i = j + 1$ and also when $i = j - 1$. In addition, e occurs in the two factor $\mathcal{M}_{e'}$ for $e' = (v_i, v_{i+1})$ in any case when $i, i+1 \neq j$ and $i, i+1 \neq j+2$. Thus, $e \in \mathcal{M}_{e'}$ for any edge $e' \in \mathcal{M}_e$. □

Theorem 6. *For any $k \geq 3$, there is a k-regular graph G on $n = 2(k+2)$ vertices such that*

$$d_I(G) = k + 2 = \delta(G) + 2.$$

Proof. For $k \geq 3$, choose H to be the square of the cycle C_r, which is 4-regular on r vertices, where we choose $r = k+2$. By Lemma 2, we know that square of a cycle satisfies the SPER-2 property, so by Theorem 5 there is a regular graph G with degree $(k+2)-2 = k$ on $4(k+2)/2 = 2(k+2)$ vertices, such that $d_I(G) = k+2$. □

4. Cubic Intersection Graphs

In the event that $\gamma_I(G) = 3$, the structure for the intersection graph is particularly nice, and very relevant to discussion of the exceptions to Theorem 1.

Theorem 7. *Let G be a regular graph on n vertices with $d_I(G) = \delta(G)+2$ such that $2n/(\delta(G)+2) = 3$. Then $H = H(G, \mathcal{S})$ satisfies the following properties:*

1. *H is a simple cubic graph*
2. *H is a triangle-free graph.*

Proof. If $2n/(\delta(G)+2) = 3$, then $|S_i| = 3$, and by Observation 1, the graph H is cubic. Now consider two sets S_i and S_j in the host graph, such that $S_i \cap S_j$ is non-empty (i.e. $(s_i, s_j) \in E(H)$). Any vertex $v \in S_j - S_i$ must be adjacent to at least two vertices of S_i, but cannot be adjacent to any vertex in $S_i \cap S_j$. Since $|S_i| = 3$, this implies $|S_i \cap S_j| = 1$, and this would be true about any pair of sets with a non-empty intersection; therefore there is at most one edge between any pair of adjacent vertices s_k and s_ℓ in H.
Next, suppose (for contradiction) that there is a triangle with vertices $\{s_i, s_j, s_k\}$ in H. This implies the pairwise intersections between corresponding sets S_i, S_j, and S_k in the host graph are non-empty. From the previous argument, each pair must intersect in exactly one vertex, and no vertex can intersect all three (by Condition 1); therefore we can write $u = S_i \cap S_j$, $v = S_j \cap S_k$, and $w = S_i \cap S_k$ for some vertices $u \neq v \neq w \in V(G)$. However, the vertex u must be adjacent to two of the three vertices in S_k, but cannot be adjacent to either v or w, a contradiction. □

At this point, we can turn our attention to the exceptions in the statement of Theorem 1. For a regular graph G to satisfy $d_I(G) + d_I(\overline{G}) \geq n+3$, the graph and its complement must satisfy $d_I(G) = \delta(G)+2$ and $d_I(\overline{G}) = \delta(\overline{G})+2$, and consequently $\gamma_I(G) = 2n/(\delta(G)+2)$. For the three exceptions to Theorem 1, we have

1. $n = 9$ and $\delta(G) = 4$
$$\frac{2n}{\delta(G)+2} = \frac{18}{4+2} = 3$$

2. $n = 18$ and $\delta(G) = 7$ (so $\delta(\overline{G}) = 10$)

$$\frac{2n}{\delta(\overline{G})+2} = \frac{36}{10+2} = 3$$

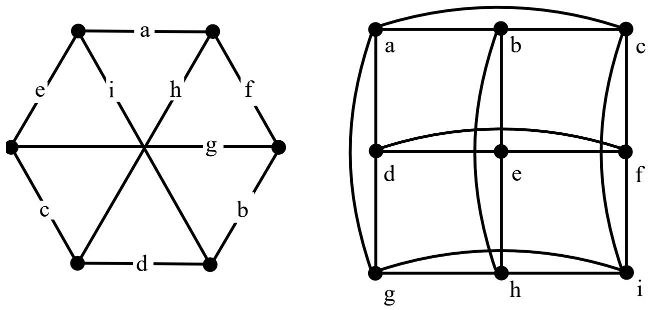


Figure 3. The intersection graph shown on left, and the corresponding host graph G on the right, the 3×3 rooks graph. The 6 Italian dominating functions are the indicator functions for the sets $\{a, e, i\}$, $\{a, f, h\}$, $\{b, f, g\}$, $\{b, d, i\}$, $\{c, d, h\}$, and $\{c, e, g\}$.

3. $n = 45$ and $\delta = 16$ (so $\delta(\overline{G}) = 28$)

$$\frac{2n}{\delta(\overline{G}) + 2} = \frac{90}{28 + 2} = 3$$

We can see that in all three cases, we can narrow the search to a cubic intersection graph.

4.1. A 4-regular graph on 9 vertices such that $d_I(G) + d_I(\overline{G}) = n + 3$.

For the first exception to Theorem 1, we consider 4-regular graphs on 9 vertices. There are 16 such graphs. However, if we consider the intersection graph H, we would be looking for a cubic, triangle-free graph on 6 vertices, and there is only one such graph [1]. This graph satisfies the SPER-2 property; in fact, in this case the 2-factors in $H - \{u, v\}$ were unique for each edge (u, v). The process in the proof of Theorem 5 can be used to reconstruct G and Figure 3 shows both the intersection graph H, and the host graph G.
Furthermore, the 3×3 rook's graph is self-complementary, so that $d_I(G) + d_I(\overline{G}) = 6 + 6 = 9 + 3 = n + 3$. On the basis of this, we can make the following claim.

Claim 1. *The 3×3 rook's graph is the unique 4-regular graph on 9 vertices with $d_I(G) = \delta(G) + 2 = 6$, and for which $d_I(G) + d_I(\overline{G}) = n + 3$.*

4.2. 10-regular graphs on 18 vertices such that $d_I(G) + d_I(\overline{G}) = n + 3$.

For the second exception to Theorem 1, we consider 7-regular graphs on 18 vertices. At this point it would be prohibitive to calculate $d_I(G) + d_I(\overline{G})$ for all such graphs. Instead, we can look at the complement; 10-regular graphs on 18 vertices, and the corresponding intersection graphs. In this case, again $2n/(\delta(G) + 2) = 3$, so we are looking for an intersection graph $H(G, \mathcal{F})$ that is cubic and triangle-free on $\delta(G)+2 = 12$ vertices.

There are 22 connected, cubic, triangle-free graphs on 12 vertices [1], and one disconnected graph (take two disjoint copies of the intersection graph shown in Figure 3). At this point, is is worth noting that 2-factors in cubic graphs are very well-studied, arising from a seminal result of Petersen in 1891 [8], that every 2-connected cubic graph has a perfect matching (the remaining edges forming a perfect 2-factor). However, for most of these graphs, there is some edge (u, v) such that $H - \{u, v\}$ does not contain a 2-factor. The next two lemmas provide simple criteria to exclude graphs from the set of 23 cubic, triangle-free graphs. First, we see that cubic graphs with a particular coloring can be excluded.

Lemma 3. *Suppose that H is a cubic triangle-free graph on n vertices with $\chi(H) = 3$, with a (proper) 3-coloring with color classes of order $(n-2)/2$, $(n-2)/2$, and 2. Then there is an edge $(u, v) \in E(H)$ such that $H - \{u, v\}$ does not contain a perfect 2-factor.*

Proof. Let H be a graph as described above with color classes U_1, U_2, and $\{u, w\}$. If $\chi(H) = 3$, then WLOG we may assume u has neighbors in both U_1 and U_2. Let U_1 be the color class that contains the most neighbors of w. If v is a neighbor of u in U_1, consider the graph $H - \{u, v\}$. This graph contains the independent sets U_2 and $U_1' = U_1 - \{v\}$, where $|U_2| = (n-2)/2$. Thus, all edges in any perfect 2-factor would need to be from vertices in U_2 to vertices in $U_1' + \{w\}$. However, w contains at most one edge into U_2, so no perfect 2-factor is possible. □

Next, we see that cubic graphs containing some forbidden subgraphs can be excluded.

Lemma 4. *Suppose that H is a cubic graph that contains any of the graphs $L_1 - L_6$ shown in Figure 4. Then there is an edge (u, v) such that $H - \{u, v\}$ does not contain a 2-factor.*

Proof. Suppose that H is a cubic graph that contains one of the graphs $L_1 - L_5$, and label vertices of the copy of L_i as in Figure 4. Then consider a perfect 2-factor of the graph $H - \{(u, v)\}$. When removing the vertices u and v from H, an edge was removed from any vertex labeled a, b, c, or d in any of the forbidden subgraphs. As a result, both of the remaining edges of these vertices must be in the 2-factor. In particular, this means that in each of the graph $L_1 - L_5$, both of the edges from the vertex s and the vertex t to the set $\{a, b, c, d\}$ must be part of the 2-factor. However, the vertex labeled w is adjacent to both s and t, and it is not possible for two edges involving w to be in the 2-factor.

Similarly, if H contains a copy of L_6, all edges from $\{a, b, c, d\}$ to s must be present in a perfect 2-factor, but that would imply that s has degree 3 in the 2-factor. □

After applying these two simple lemmas to screen the 23 triangle-free cubic graphs with 12 vertices, there are seven remaining graphs, each of which contain a perfect

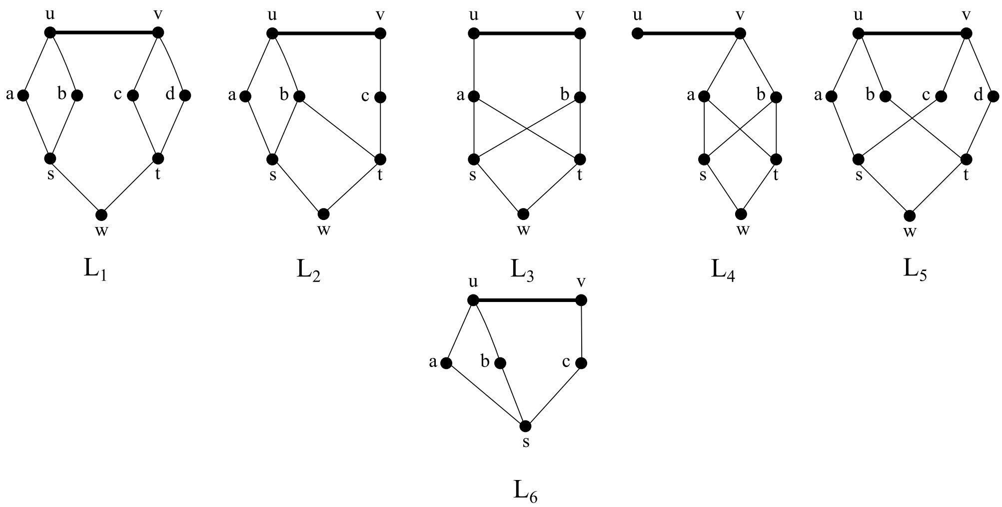


Figure 4. Forbidden subgraphs for any cubic graph G such that $G - e$ always contains a 2-factor.

2-factor in $G - e$ for each edge e (However, this does not imply that the 2-factors satisfy the SPER-2 property). In the table below, the graph6 codes for these seven graphs are given. In addition, the 2-factors associated with each edge were examined to determine when the SPER-2 property was satisfied, and any possible corresponding host graph(s), using the mathematical software system Sage [9] (code for this is available at [6]). The results are provided in Table 1.

Intersection Graph	Corresponding Host Graphs
H_1: KCHY@eAGKOGB	QEYbtZSZrtTlt[tkult]i]ujYlO
H_2: KCHY@eAWCO?F	QEYblZWZrtTlu[tkuxdZm]ujUl_
H_3: KC`Y@aAWHO?X	Does not satisfy SPER-2 property
H_4: KKhY?aAGOE?F	Does not satisfy SPER-2 property
H_5: KhhK?GQ?oa?F	Does not satisfy SPER-2 property
H_6: KMo@_K`@KG@B	QEYdZXtmb^UmucuTuuusuZyjTlO
H_7: K??FFB_F?wB_	QBjB\jWjrtYktZjtNVBtZt\inZ?

Table 1. The triangle-free cubic graphs with 12 vertices which contain a perfect 2-factor in $G - e$ for each edge e (the intersection graphs), and the corresponding host graphs, if they exist. All graphs are displayed in graph6 format [7].

The four host graphs listed in Table 1 are the 10-regular graphs on 18 vertices with $d_I(G) = 12$. However, in each case, $d_I\left(\overline{G}\right) < 9$. On the basis of this, we can make the following claim.

Claim 2. *For any 10-regular graphs on 18 vertices (or 7-regular graph on 18 vertices)* $d_I(G) + d_I\left(\overline{G}\right) \le n + 2$.

Together, Claims 1 and 2 provide the basis for Theorem 2.

5. Open Questions

We conclude by addressing some of the remaining questions. Foremost is the last exception to Theorem 1.

Question . Is there a 28-regular graph G on 45 vertices such that

$$d_I(G) + d_I(\overline{G}) = n + 3?$$

In searching for such graphs, one could look first for the complement: 16-regular graphs on 45 vertices such that $d_I(G) = 18$. This would yield a cubic, triangle-free intersection graph on 30 vertices. However, looking for such graphs that satisfy the SPER-2 property would be daunting. The number of cubic, triangle-free graphs on 30 vertices is 181,492,137,812 [1], which is prohibitively large. Additional progress would be necessary to attack this problem.

In addition to Question 1, there are two other natural questions that arise. In Subsection 4.2, we provided some results to indicate when there is an edge $e = (u, v)$ so that $H - \{u, v\}$ does not have a perfect 2-factor. It would be interesting to find more generally applicable conditions to show that $H - \{u, v\}$ does (or does not) have a perfect 2-factor.

Finally, we focused on intersection graphs which were cubic. A consequence of $|S_i| = 3$ was that the resulting graph was simple and triangle-free. This is not necessarily true for intersection graphs which are not cubic. However, it is likely that there are other structural properties that may be helpful; determining what these structural properties might be is also of interest.

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