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Research Article

Complexity of the paired domination subdivision problem

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Abstract: The paired domination subdivision number of a graph G is the minimum number of edges that must be subdivided (where each edge in G can be subdivided at most once) in order to increase the paired domination number of G. In this note, we show that the problem of computing the paired domination subdivision number is NP-hard for bipartite graphs.

Keywords: paired domination, paired domination subdivision number, complexity

AMS Subject classification: 05C69

1. Terminology and introduction

Paired domination in graphs was introduced by Haynes and Slater [5], and is now well studied. For more details on paired domination, we refer the reader to the recent book chapter [1]. Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. If v is a vertex in V, then *open neighborhood* of v is $N(v) = \{u \in V(G) | uv \in E(G)\}$. A *paired dominating set,* abbreviated PD-set, of a graph G is a set S of vertices such that every vertex is adjacent to some vertex in S and the subgraph $G[S]$ induced by S contains a perfect matching (not necessarily induced). If S is a PD-set of G with a perfect matching M, then two vertices $u, v \in S$ are said to be *partners* or *paired* in S if the edge $uv \in M$. Since the end vertices of any maximal matching in G form a PD-set, every graph G without isolated vertices has a PD-set.

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In [3], Favaron et al. introduced the *paired domination subdivision number* $\mathrm{sd}_{\gamma_{pr}}(G)$ of a graph G defined as the minimum number of edges that must be subdivided (where each edge in G can be subdivided at most once) in order to increase the paired domination number of G. Many results have been established on this parameter (see [2–4, 7–9]). But the question of the complexity of the paired domination subdivision problem has not been addressed and therefore remained open.

The paired domination subdivision problem, to which we shall refer as PD-Subdivision problem, is the following:

PD-Subdivision

Instance: *A nonempty graph G and a positive integer k.*

Question: *Is* $\mathrm{sd}_{\gamma_{pr}}(G) \leq k$?

Our aim in this paper is to show the NP-hardness of the PD-Subdivision problem even for bipartite graphs.

2. NP-hardness result

Following Garey and Johnson's techniques for proving NP-completeness given in [6], we prove our result by providing a polynomial transformation from the well-known NP-complete 3-satisfiability problem, 3-SAT, (see Theorem 3.1 in [6]). Before stating the 3-SAT problem, we recall some terms.

Let U be a set of Boolean variables. A truth assignment for U is a mapping $t : U \longrightarrow \{T, F\}$. If $t(u) = T$, then u is said to be "true" under t; if $t(u) = F$, then u is said to be "false" under t. If u is a variable in U, then u and $\bar{u}$ are *literals* over U. The literal u is true under t if and only if, the variable $\bar{u}$ is false under t; the literal $\bar{u}$ is true if and only if, the variable u is false.

A *clause* over U is a set of literals over U. It represents the disjunction of these literals and is satisfied by a truth assignment if and only if, at least one of its members is true under that assignment. A collection $\mathscr{C}$ of clauses over U is satisfiable if and only if, there exists some truth assignment for U that simultaneously satisfies all the clauses in $\mathscr{C}$. Such a truth assignment is called a *satisfying truth assignment* for $\mathscr{C}$. The 3-SAT problem is specified as follows.

3-SAT

Instance: A collection $\mathscr{C} = \{C_1, C_2, \ldots, C_m\}$ of clauses over a finite set U of variables such that $|C_j| = 3$ for $j = 1, 2, \ldots, m$.

Question: *Is there a truth assignment for U that satisfies all the clauses in $\mathscr{C}$?*

Now we are ready to state our main result.

Theorem 1. *PD-Subdivision problem is NP-hard for bipartite graphs.*

Proof. The transformation is from 3-SAT to PD-Subdivision. Let $U = \{u_1, u_2, \ldots, u_n\}$ and $\mathscr{C} = \{C_1, C_2, \ldots, C_m\}$ be an arbitrary instance of 3SAT. We will construct a bipartite graph G and choose an integer k such that $\mathscr{C}$ is satisfiable if and only if $\mathrm{sd}_{\gamma_{pr}}(G) \leq k$. The graph G is constructed as follows.

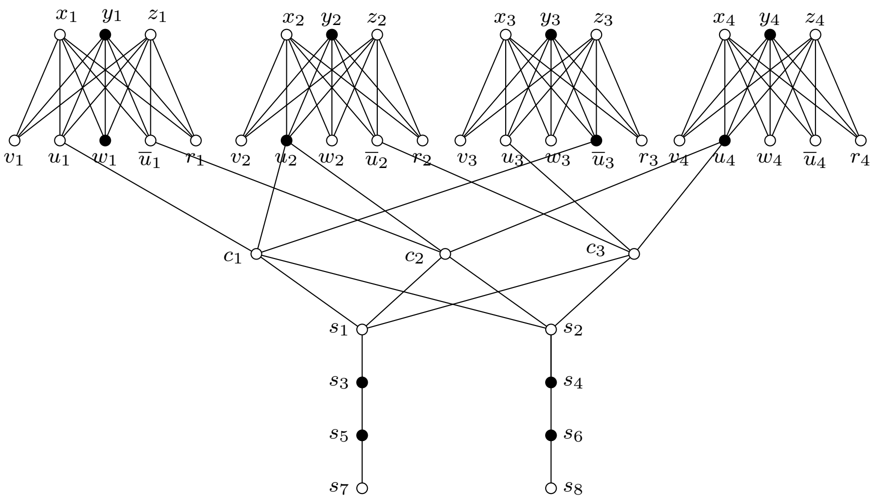

Figure 1. An instance of the paired domination subdivision problem resulting from an instance of 3SAT. Here $\gamma_{pr}(G) = 12$, where $S = \{y_1, y_2, y_3, y_4, w_1, u_2, \overline{u}_3, w_4, s_3, s_4, s_5, s_6\}$ is a paired dominating set of G and $|S| = \gamma_{pr}(G)$.

For each $i = 1, 2, \ldots, n$, corresponding to the variable $u_i \in U$, we associate a complete bipartite graph $H_i = K_{3,5}$ with bipartite sets $X = \{x_i, y_i, z_i\}$ and $Y = \{v_i, u_i, w_i, \overline{u}_i, r_i\}$. For each $j = 1, 2, \ldots, m$, corresponding to the clause $C_j = \{p_j, q_j, r_j\} \in \mathscr{C}$, we associate a single vertex c_j by adding the edge $c_j u_i$ if $u_i \in C_j$ and the edge $c_j \bar{u}_i$ if $\bar{u}_i \in C_j$. Finally, add a graph H with vertex set $V(H) = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8\}$ and edge set $E(H) = \{s_1s_3, s_3s_5, s_5s_7, s_2s_4, s_4s_6, s_6s_8\}$, by joining s_1 and s_2 to each vertex c_j with $1 \leq j \leq m$. Set $k = 1$. Figure 1 shows an example of the graph obtained from the instance $U = \{u_1, u_2, u_3, u_4\}$ and $\mathscr{C} = \{C_1, C_2, C_3\}$, where $C_1 = \{u_1, u_2, \bar{u}_3\}, C_2 = \{\bar{u}_1, u_2, u_4\}, C_3 = \{\bar{u}_2, u_3, u_4\}$.
To prove that this is indeed a transformation, we only need to show that $\mathrm{sd}_{\gamma_{pr}}(G) = 1$ if and only if, there is a truth assignment for U that satisfies all clauses in $\mathscr{C}$. For this purpose, we need to prove the following four claims.

Claim 1. $\gamma_{pr}(G) \geq 2n + 4$. Moreover, if $\gamma_{pr}(G) = 2n + 4$, then for every γ_{pr}(G)-set S of G, $|H_i \cap S| = 2$ and $\big(V(H) \cup \{c_1, c_2, \ldots, c_m\}\big) \cap S = \{s_3, s_4, s_5, s_6\}$.

Proof of Claim. Let S be a γ_{pr}-set of G. For each $i \in \{1, 2, \ldots, n\}$, it is clear that $|V(H_i) \cap S| \geq 2$ and $|H \cap S| \geq 4$, implying that $\gamma_{pr}(G) \geq 2n + 4$.
Suppose that $\gamma_{pr}(G) = 2n+4$. Since $H_i := K_{m,n}$ with $m, n \geq 3$, $|V(H_i) \cap S| = 2$ and so $|\big(V(H) \cup \{c_1, c_2, \ldots, c_m\}\big) \cap S| = 4$. Now we show that $\big(V(H) \cup \{c_1, c_2, \ldots, c_m\}\big) \cap S = \{s_3, s_4, s_5, s_6\}$. Suppose for the sake of contradiction that $s_3 \notin S$. Then to paired

dominate s_3 we must have either $s_1 \in S$ paired in S with some c_j or $s_5 \in S$ paired in S with s_7. In the former case, we must have in addition in S, s_5 and s_7 as partners. But then $|S \cap \big(V(H) \cup \{c_1, c_2, \ldots, c_m\}\big)| \geq 6$ implying that $|S| \geq 2n+6$, which is a contradiction. In the later case, we need that S contains some vertex c_j to paired dominate s_1. But then $|S \cap \big(V(H) \cup \{c_1, c_2, \ldots, c_m\}\big)| \geq 5$ implying that $|S| \geq 2n+5$, which is a contradiction too. Therefore $s_3 \in S$ and to paired dominate s_7, we must have $s_5 \in S$ as a partner with s_3. Hence $s_1 \notin S$. Likewise, $s_4, s_6 \in S$ and are partners and thus $s_2 \notin S$. From this, we deduce that $S \cap \{c_1, c_2, \ldots, c_m\} = \emptyset$. Therefore $|H_i \cap S| = 2$ and $\big(V(H) \cup \{c_1, c_2, \ldots, c_m\}\big) \cap S = \{s_3, s_4, s_5, s_6\}$ as desired.♦

Claim 2. $\mathscr{C}$ is satisfiable if and only if $\gamma_{pr}(G) = 2n + 4$.

Proof of Claim. Suppose that $\gamma_{pr}(G) = 2n+4$ and let S be a γ_{pr}-set of G. By Claim 1, $\{s_3, s_4, s_5, s_6\} \subset S$ and $|V(H_i) \cap S| = 2$ for each $i \in \{1, 2, \ldots, n\}$. Moreover, since $N(c_j) \cap \{s_3, s_4, s_5, s_6\} = \emptyset$ for every j and H_i is a complete bipartite graph, only one of u_i and $\bar{u}_i$ is in S and is paired with one vertex of $\{x_i, y_i, z_i\}$, for each i. Define a mapping $t : U \longrightarrow \{T, F\}$ by

$$t(u_i) = \begin{cases} T & \text{if } u_i \in S, \\ F & \text{if } \bar{u}_i \in S, \end{cases} \tag{1}$$

for $i \in \{1, \ldots, n\}$. We now show that t is a satisfying truth assignment for $\mathscr{C}$. It is sufficient to show that every clause in $\mathscr{C}$ is satisfied by t. To this end, we arbitrarily choose a clause $C_j \in \mathscr{C}$ for $j \in \{1, \ldots, m\}$. By Claim 1, since $|H_i \cap S| = 2$, and $\big(\{s_1, s_2\} \cup \{c_1, c_2, \ldots, c_m\}\big) \cap S = \emptyset$, there exists some $i \in \{1, \ldots, n\}$ such that c_j is adjacent to u_i or $\bar{u}_i$. Suppose that c_j is adjacent to u_i where $u_i \in S$. Since u_i is adjacent to c_j in G, the literal u_i is in the clause C_j by the construction of G. Since $u_i \in S$, it follows that $t(u_i) = T$ by (1), which implies that the clause C_j is satisfied by t. Suppose that c_j is adjacent to $\bar{u}_i$ where $\bar{u}_i \in S$. Since $\bar{u}_i$ is adjacent to c_j in G, the literal $\bar{u}_i$ is in the clause C_j. Since $\bar{u}_i \in S$, it follows that $t(u_i) = F$ by (1). Thus, t assigns $\bar{u}_i$ the truth value T, that is, t satisfies the clause C_j. By the arbitrariness of j, with $1 \leq j \leq m$, it follows that t satisfies all the clauses in $\mathscr{C}$, so $\mathscr{C}$ is satisfiable. Conversely, suppose that $\mathscr{C}$ is satisfiable, and let $t : U \longrightarrow \{T, F\}$ be a satisfying truth assignment for $\mathscr{C}$. Create a subset S of $V(G)$ as follows: if $t(u_i) = T$, then let $u_i \in S$, and if $t(u_i) = F$, then let $\bar{u}_i \in S$. Let $\{y_1, y_2, \ldots, y_n, s_3, s_4, s_5, s_6\} \subseteq S$ and the intersection of the set of the remaining vertices of G with S is empty. Clearly, $|S| = 2n + 4$. Since t is a satisfying truth assignment for $\mathscr{C}$, for each $j \in \{1, \ldots, m\}$, at least one of literals in C_j is true under the assignment t. It follows that the corresponding vertex c_j in G is adjacent to at least one vertex p of S. Since c_j is adjacent to each literal in C_j by the construction of G, thus S is a paired dominating set of G, and so $\gamma_{pr}(G) \leq |S| = 2n + 4$. By Claim 1, $\gamma_{pr}(G) \geq 2n + 4$, and thus $\gamma_{pr}(G) = 2n + 4$. ♦

Claim 3. Let G' be obtained from G by subdividing any edge e of $E(G)$, then $\gamma_{pr}(G') \leq 2n+6$.

Proof of Claim. Let $e = uv \in E(G)$ and let G' be the graph obtained from G by subdividing the edge e with new vertex w. Now, consider the following cases:

Case 1. If $e = s_1s_3$, then consider the set $S = \{s_1, w, s_5, s_7, s_4, s_6\} \cup \{v_i, x_i \mid 1 \leq i \leq n\}$.

Case 2. If $e = s_3s_5$, then consider the set $S = \{s_1, s_3, s_5, s_7, s_4, s_6\} \cup \{v_i, x_i \mid 1 \leq i \leq n\}$.

Case 3. If $e = s_5s_7$, then consider the set $S = \{s_1, s_3, w, s_7, s_4, s_6\} \cup \{v_i, x_i \mid 1 \leq i \leq n\}$. (The case $e \in \{s_2s_4, s_4s_6, s_6s_8\}$ is similar).

Case 4. If $e = s_1c_j$ for any $j \in \{1, 2, \ldots, m\}$, then consider the set $S = \{s_1, w, s_5, s_7, s_4, s_6\} \cup \{v_i, x_i \mid 1 \leq i \leq n\}$.

Case 5. If $e = c_ju_i$ for some $1 \leq j \leq m$ and $1 \leq i \leq n$, then consider the set $S = \{s_1, c_j, s_5, s_7, s_4, s_6\} \cup \{x_i, u_i \mid 1 \leq i \leq n\}$. (The case $e = c_j\overline{u}_i$ is similar).

Case 6. If $e = uv$ such that $u \in \{x_i, y_i, z_i\}$ and $v \in \{u_i, \overline{u}_i\}$, say, without loss of generality, $u = x_r, v = u_r$ and u_r is adjacent to c_j for some $1 \leq r \leq n$ and $1 \leq j \leq m$, then consider the set $S = \{s_1, c_j, s_5, s_7, s_4, s_6\} \cup \{x_i, v_i \mid 1 \leq i \leq n\}$.

Case 7. If $e = uv$ such that $u \in \{x_i, y_i, z_i\}$ and $v \in \big(\{v_i, w_i, r_i\}\big)$, say, without loss of generality, $u = x_r, v = v_r$ for some $1 \leq r \leq n$, then consider the set $S = \{s_3, s_5, s_4, s_6, w, x_r\} \cup \{y_i \mid 1 \leq i \leq n\}$ and add, as a partner of y_i in S, the vertex u_i if $u_i \in C_j$ and $\overline{u}_i$ if $\overline{u}_i \in C_j$.
Clearly, in either of the above cases, S is a paired dominating set of G' with $|S| = 2n+6$, and therefore $\gamma_{pr}(G') \leq 2n+6$. ♦

Claim 4. $\gamma_{pr}(G) = 2n+4$ *if and only if,* $\mathrm{sd}_{\gamma_{pr}}(G) = 1$.

Proof of Claim. Assume $\gamma_{pr}(G) = 2n+4$. Let G' be the graph obtained from G by subdividing the edge $e = s_3s_5$ with new vertex w. Suppose, for a contradiction, that $\gamma_{pr}(G) = \gamma_{pr}(G')$, and let S' be a $\gamma_{pr}(G')$-set whose subgraph has a perfect matching M.
First assume that $w \notin S'$. Then to paired dominate s_7, we have $\{s_5, s_7\} \subseteq S'$. Now, if $s_3 \notin S'$, then $s_1 \in S'$ and is paired in S' with some vertex c_j. Hence S' is a $\gamma_{pr}(G)$-set containing c_j, contradicting Claim 1. Assume now that $s_3 \in S'$. Then $s_1 \in S'$ and since $|S' \cap \{s_2, s_4, s_6, s_8\}| \geq 2$ and $|V(H_i) \cap S'| \geq 2$ for each $i \in \{1, ..., n\}$, we deduce that $|S'| \geq 2n+6$, contradicting the fact that $|S'| = 2n+4$.
Secondly, assume that $w \in S'$. Then to paired dominate s_7, we must have $s_5 \in S'$. Now, if $s_1 \notin S'$, then $s_3 \notin S'$ and to paired dominate s_1, we must have $c_j \in S'$ for some $j \in \{1, \ldots, m\}$. In this case, let $S = (S' \setminus \{w\}) \cup \{s_3\}$, where s_3 and s_5 are partners in S. Clearly, S is a paired dominating set of G, and since $|S| = 2n+4$,

we obtain a contradiction with Claim 1. Therefore we assume that $s_1 \in S'$. Then $S = (S' \setminus \{w\}) \cup \{s_7\}$ is a paired dominating set of G yielding a contradiction with Claim 1, as above. Hence, $\gamma_{pr}(G) < \gamma_{pr}(G')$, and therefore $\mathrm{sd}_{\gamma_{PR}}(G) = 1$.
Conversely, assume that $\mathrm{sd}_{\gamma_{PR}}(G) = 1$. By Claim 1, we have $\gamma_{pr}(G) \geq 2n+4$. Let G' be the graph obtained from G by subdividing an edge e such that $\gamma_{pr}(G) < \gamma_{pr}(G')$. By Claim 3, we have $\gamma_{pr}(G') \leq 2n + 6$. Thus, $2n + 4 \leq \gamma_{pr}(G) < \gamma_{pr}(G') \leq 4n + 6$, which yields $\gamma_{pr}(G) = 2n + 4$, as desired.♦

By Claims 2 and 4, we prove that $\mathrm{sd}_{\gamma_{PR}}(G) = 1$ if and only if there is a truth assignment for U that satisfies all clauses in $\mathscr{C}$. Since the construction of the paired subdivision number instance is straightforward from a 3-satisfiability instance, the size of the paired domination subdivision number instance is bounded above by a polynomial function of the size of 3-satisfiability instance. It follows that this is a polynomial reduction and the proof is complete. □

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