Communications in Combinatorics and Optimization
Vol. 7, No. 1 (2022), pp. 81-90
DOI: 10.22049/CCO.2021.26999.1179

CCO
Commun. Comb. Optim.

Research Article

New bounds on the energy of a graph

Hajar Shoshtari[1], Jonnathan Rodríguez[2]*

[1]Department of Mathematics, Esfahan University of Technology, Esfahan, Iran
hajarshooshtary91@gmail.com

[2]Departamento de Matemáticas, Facultad de Ciencias, Básicas, Universidad de Antofagasta, Av. Angamos 601, Antofagasta, Chile
jonnathan.rodriguez@uantof.cl

Received: 14 November 2020; Accepted: 26 April 2021
Published Online: 28 April 2021

Abstract: The energy of a graph G, denoted by $\mathcal{E}(G)$, is defined as the sum of the absolute values of all eigenvalues of G. In this paper, lower and upper bounds for energy in some of the graphs are established, in terms of graph invariants such as the number of vertices, the number of edges, and the number of closed walks.

Keywords: Eigenvalue of graph, Energy, Spectral radius, The number of closed walks

AMS Subject classification: 05C20

1. Introduction

Let $G = (V, E)$ be a simple undirected graph with vertex set $V = V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$, $\mid E(G) \mid= m$. The *order* and *size* of G are $n = |V|$ and $m = |E|$, respectively. A *walk* from a vertex u to a vertex v is a finite alternating sequence $v_0(= u)e_1v_1e_2 \ldots v_{k_1}e_kv_k(= v)$ of vertices and edges such that $e_i = v_{i-1}v_i$ for $i = 1, 2, \ldots, k$. The number k is the *length* of the walk. In particular, if the vertex $v_i, i = 0, 1, \ldots, k$ in the walk are all distinct then the walk is called a *path.* A path of order n is denoted by P_n. A *closed path* or *cycle*, is obtained from a path $v_1, \ldots, v_k$ (where $k \geqslant 3$) by adding the edge v_1v_k. A cycle of order n is denoted by C_n. A graph is *connected* if each pair of vertices in a graph is joined by a walk. A *bipartite* graph is a graph such that its vertex set can be partitioned into two sets X and Y (called the partite sets) such that every edge meet both X and Y. A *complete*

* *Corresponding Author*

bipartite graph is a bipartite graph such that any vertex of a partite set is adjacent to all vertices of the other partite set. A complete bipartite graph with partite set of cardinalities p and q is denoted by $K_{p,q}$. The graph $K_{1,n-1}$ is also called a *star* of order n, denoted by S_n. A simple undirected graph in which every pair of distinct vertices is connected by a unique edge, is the *complete graph* and is denoted by K_n. For other graph theory notation and terminology we refer to [18].
The *adjacency matrix* $A(G)$ of a graph G is defined by its entries as $a_{ij} = 1$ if $v_iv_j \in E(G)$ and 0 otherwise. Let $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{n-1} \geqslant \lambda_n$ denote the *eigenvalues* of $A(G)$. Then λ_1 is called the *spectral radius* of G. When more than one graphs are under consideration, then we write $\lambda_i(G)$ instead of λ_i.
The *energy* of a graph G is defined as

$$\mathcal{E}(G) = \sum_{i=1}^{n} \mid \lambda_i \mid .$$

This concept was introduced by I. Gutman and is intensively studied in chemistry, since it can be used to approximate the total π-*electron* energy of a *molecule* (see, e.g.[7], [10]). In 1971, McClelland [15] discovered the first upper bound for $\mathcal{E}(G)$ as follows:

$$\mathcal{E}(G) \leq \sqrt{2mn}. \tag{1}$$

Since then, numerous other bounds for $\mathcal{E}(G)$ were found (see, e.g. [2], [3], [12] - [14]). Here we just state some upper bounds for $\mathcal{E}(G)$ which were obtained recently. Koolen and Moulton [13] showed that if G is a graph with n vertices, m edges and $2m \geqslant n$ then

$$\mathcal{E}(G) \leq \frac{2m}{n} + \sqrt{(n-1)\Big(2m - (\frac{2m}{n})^2\Big)}, \tag{2}$$

with equality if and only if G is either $\frac{n}{2}K_2$, K_n or a non-complete connected strongly regular graph with two non-trivial eigenvalues both with absolute value $\sqrt{\frac{(2m-(\frac{2m}{n})^2)}{(n-1)}}$. The same authors showed then that if $2m \geqslant n$ and G is a bipartite graph with $n > 2$, then

$$\mathcal{E}(G) \leq 2(\frac{2m}{n}) + \sqrt{(n-1)\Big(2m - 2(\frac{2m}{n})^2\Big)}, \tag{3}$$

with equality if and only if G is either $\frac{n}{2}K_2$, a complete bipartite graph, or the incidence graph of a symmetric 2-(ν, k, λ)-design with $k = \frac{2m}{n}$ and $\lambda = \frac{k(k-1)}{\nu-1}(n = 2\nu)$. In [7, 13], from inequality (2), the authors get the following bound for $\mathcal{E}(G)$:

$$\mathcal{E}(G) \leq \frac{n}{2}(1 + \sqrt{n}). \tag{4}$$

It is not dificult to see that inequality (4) is an improvement of the bound

$$\mathcal{E}(G) \leq n\sqrt{n-1}.$$

The recent mathematical properties of this invariant can be found in e.g. [1, 8, 9, 11, 19].
The paper is organized as follows. In Section 2, we give a list of some previously known results. In Section 3, we present bounds on the energy of graphs.

2. Preliminaries and known results

In this section, we list some known results that will be needed in the next sections. We first state some results on the number of closed walks of a graph.

Proposition 1. *[16] Let G be a graph with n vertices, m edges and $N_G(i)$ be the number of closed walks of length i in G, then*

1. $N_2 = N_2(G) = \sum_{i=1}^{n} \lambda_i^2 = 2m$,
2. $N_3 = N_3(G) = \sum_{i=1}^{n} \lambda_i^3 = 6N_G(K_3)$,
3. $N_4 = N_3(G) = \sum_{i=1}^{n} \lambda_i^4 = 2m + 4N_G(P_3) + 8N_G(C_4)$,
4. $N_5 = N_5(G) = \sum_{i=1}^{n} \lambda_i^5 = 30N_G(K_3) + 10N_G(C_5) + 10N_G(G_1)$,
5. $N_7 = N_7(G) = \sum_{i=1}^{n} \lambda_i^7 = 126N_G(K_3) + 84N_G(G_1) + 14N_G(G_2) + 14N_G(G_3) + 14N_G(G_4) + 28N_G(G_5) + 42N_G(G_6) + 28N_G(G_7) + 112N_G(G_8) + 70N_G(C_5) + 14N_G(C_7)$.

The graphs G_i for $i = 1, 2, \ldots, 8$ are shown in Figure 1.

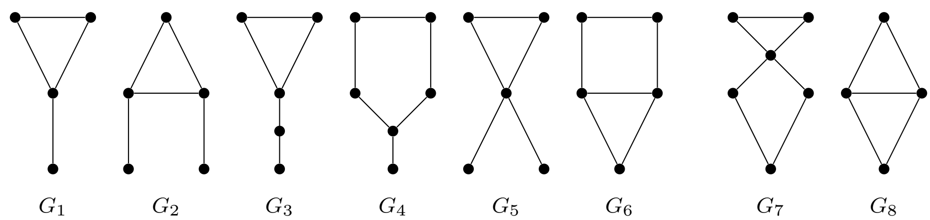

Figure 1. **The graphs G_i for $i = 1, 2, \ldots, 8$ mentioned in Proposition 1**

The next lemma provides a lower bound for the energy of a graph.

Proposition 2. *[4] If G is a graph with m edges, then $\mathcal{E}(G) \geqslant 2\sqrt{m}$, with equality if and only if G is a complete bipartite graph plus arbitrarily many isolated vertices.*

Proposition 3. *[5] A graph G has only one eigenvalue if and only if G is an empty graph. A graph G has two distinct eigenvalues $\mu_1 > \mu_2$ with multiplicities m_1 and m_2 if and only if G is the direct sum of m_1 complete graphs of order $\mu_1 + 1$. In this case, $\mu_2 = -1$ and $m_2 = m_1\mu_1$.*

We now give some known upper bounds for the spectral radius $\lambda_1(G)$.

Proposition 4. *[17] Let G be a connected graph with n vertices and m edges, then*

$$\lambda_1(G) \leqslant n - 1.$$

Proposition 5. *[17] Let G be a tree with n vertices, then*

$$\lambda_1(G) \leqslant \sqrt{n-1}.$$

3. Bounds for the Energy of Graphs

In this section, we obtain some new upper and lower bounds for the energy of graphs. We deal with general graphs, connected graphs, bipartite graphs, connected bipartite graphs, and trees.
We begin with the following upper bound in terms of order and size of a graph.

Theorem 1. *Let G be a non-empty graph with n vertices and m edges. Then*

$$\mathcal{E}(G) \leq \sqrt{2m^2 + \frac{n^2}{2}}, \tag{5}$$

equality holds if and only if $G \cong \frac{n}{2}(K_2), (n = 2m)$.

Proof. Let a_i, b_i, c_i and d_i are sequences of real numbers and p_i, q_i are non-negative for $i = 1, 2, \ldots, n$. Then the following inequality is valid (see [6] p. 7)

$$\sum_{i=1}^{n} p_i a_i^2 \sum_{i=1}^{n} q_i b_i^2 + \sum_{i=1}^{n} p_i c_i^2 \sum_{i=1}^{n} q_i d_i^2 \geqslant 2 \sum_{i=1}^{n} p_i a_i c_i \sum_{i=1}^{n} q_i b_i d_i. \tag{6}$$

For $a_i = b_i = p_i = q_i := 1$ and $c_i = d_i := |\lambda_i|$, $i = 1, 2, \ldots, n$, inequality (6) becomes

$$\sum_{i=1}^{n} 1 \sum_{i=1}^{n} 1 + \sum_{i=1}^{n} |\lambda_i|^2 \sum_{i=1}^{n} |\lambda_i|^2 \geqslant 2 \sum_{i=1}^{n} |\lambda_i| \sum_{i=1}^{n} |\lambda_i|.$$

Notice that

$$\sum_{i=1}^{n} |\lambda_i|^2 = \sum_{i=1}^{n} \lambda_i^2 = 2m,$$

hence

$$n^2 + \Big(\sum_{i=1}^{n} |\lambda_i|^2\Big)^2 \geq 2\Big(\sum_{i=1}^{n} |\lambda_i|\Big)^2.$$

Then, from the above inequality directly follows the assertion of Theorem 1, i.e. the inequality (5). If $G \cong \frac{n}{2}K_2$, then it is easy to check that the equality in (5) holds. Conversely, if the equality in (5) holds, since G is a non-empty graph, by Proposition 3, G has at least two distinct eigenvalues. We consider the following cases.

Case 1. The absolute value of all eigenvalues of G are equal.
Then clearly $\lambda_1 =\mid \lambda_i \mid \quad (2 \leqslant i \leqslant n)$, since G has at least two distinct eigenvalues. By Proposition 3, $\mid \lambda_i \mid= 1 (2 \leqslant i \leqslant n)$. Hence $2m = n$ and also, $\lambda_1 =\mid \lambda_2 \mid= \cdots =\mid \lambda_n \mid= 1$. By applying Proposition 3 again, we obtain that $m_2 = m_1\lambda_1, \lambda_1 = 1$, and therefore $m_1 = m_2$. Then we obtain that $\lambda_1 = 1$ has multiplicity $\frac{n}{2}$, and $\lambda_i = -1 \quad (2 \leqslant i \leqslant n)$ has multiplicity $\frac{n}{2}$. Therefore G is the direct sum of $m_1 = \frac{n}{2}$ complete graphs of order $\lambda_1 + 1 = 2$. Consequently, G is $\frac{n}{2}K_2$.

Case 2. The absolute value of all eigenvalues of G are not equal. Then G has two distinct eigenvalues with different absolute values. By Proposition 3, $\mid \lambda_i \mid= 1 \quad (2 \leqslant i \leqslant n)$. Since, $\sum_{i=1}^{n} \lambda_i = 0$ and $\lambda_2 = \lambda_3 = \cdots = \lambda_n = -1$, we have, $\lambda_1 = n-1$. Hence λ_1 has multiplicity 1 and $\lambda_i = -1$ has multiplicity $n-1$. By Proposition 3, G is the direct sum of a complete graph of order $\lambda_1 + 1 = n$. Consequently, G is K_n. □

Theorem 2. *Let G be a non-empty graph with n vertices and m edges. Then*

$$\mathcal{E}(G) \leq m + \frac{n}{2}, \tag{7}$$

equality holds if and only if $G \cong \frac{n}{2}(K_2), (n = 2m)$.

Proof. Let a_i, b_i, c_i and d_i are sequences of real numbers and p_i, q_i are non-negative for $i = 1, 2, \ldots, n$. Then the following inequality is valid (see [6] p. 7)

$$\sum_{i=1}^{n} p_i a_i^2 \sum_{i=1}^{n} q_i b_i^2 + \sum_{i=1}^{n} p_i c_i^2 \sum_{i=1}^{n} q_i d_i^2 \geqslant 2 \sum_{i=1}^{n} p_i a_i c_i \sum_{i=1}^{n} q_i b_i d_i. \tag{8}$$

For $a_i = b_i = d_i = p_i = q_i := 1$ and $c_i := |\lambda_i|$, $i = 1, 2, \ldots, n$, inequality (8) becomes

$$\sum_{i=1}^{n} 1 \sum_{i=1}^{n} 1 + \sum_{i=1}^{n} |\lambda_i|^2 \sum_{i=1}^{n} 1 \geqslant 2 \sum_{i=1}^{n} |\lambda_i| \sum_{i=1}^{n} 1.$$

So

$$n^2 + \Big(\sum_{i=1}^{n} |\lambda_i|^2\Big) n \geq 2\Big(\sum_{i=1}^{n} |\lambda_i|\Big) n.$$

Then, from the above inequality directly follows the assertion of Theorem 2, i.e. the inequality (7).
If $G \cong \frac{n}{2}K_2$, then it is easy to check that the equality in (7) holds. Conversely, if the equality in (7) holds, since G is a non-empty graph, by Proposition 3, G has at least two distinct eigenvalues. We consider the following cases.

Case 1. The absolute value of all eigenvalues of G are equal.
Then clearly $\lambda_1 =\mid \lambda_i \mid \quad (2 \leqslant i \leqslant n)$, since G has at least two distinct eigenvalues. By Proposition 3, $\mid \lambda_i \mid= 1(2 \leqslant i \leqslant n)$. Hence $2m = n$ and also, $\lambda_1 =\mid \lambda_2 \mid= \cdots =\mid \lambda_n \mid= 1$. By applying Proposition 3 again, we obtain that $m_2 = m_1\lambda_1, \lambda_1 = 1$, and therefore $m_1 = m_2$. Then we obtain that $\lambda_1 = 1$ has multiplicity $\frac{n}{2}$, and $\lambda_i = -1 \quad (2 \leqslant i \leqslant n)$ has multiplicity $\frac{n}{2}$. Therefore G is the direct sum of $m_1 = \frac{n}{2}$ complete graphs of order $\lambda_1 + 1 = 2$. Consequently, G is $\frac{n}{2}K_2$.

Case 2. The absolute value of all eigenvalues of G are not equal. Then G has two distinct eigenvalues with different absolute values. By Proposition 3, $\mid \lambda_i \mid= 1(2 \leqslant i \leqslant n)$. Since, $\sum_{i=1}^{n} \lambda_i = 0$ and $\lambda_2 = \lambda_3 = \cdots = \lambda_n = -1$, we have, $\lambda_1 = n-1$. Hence λ_1 has multiplicity 1 and $\lambda_i = -1$ has multiplicity $n - 1$. By Proposition 3, G is the direct sum of a complete graph of order $\lambda_1 + 1 = n$. Consequently, G is K_n. □

The next upper bound involves the number of closed walks of lengths 4 and 6 of a graph.

Theorem 3. *Let G be a non-empty graph with n vertices and m edges. Then*

$$\mathcal{E}(G) \leq \frac{nN_5 + 4m^2}{2N_4}, \tag{9}$$

equality holds if and only if $G \cong \frac{n}{2}(K_2), (n = 2m)$.

Proof. Let a_i, b_i are sequences of real numbers and c_i, d_i are non-negative for $i = 1, 2, \ldots, n$. Then the following inequality is valid (see [6] p.8)

$$\sum_{i=1}^{n} d_i \sum_{i=1}^{n} c_i a_i^2 + \sum_{i=1}^{n} c_i \sum_{i=1}^{n} d_i b_i^2 \geqslant 2 \sum_{i=1}^{n} c_i a_i \sum_{i=1}^{n} d_i b_i. \tag{10}$$

For $a_i = c_i := |\lambda_i|^2$, $b_i := |\lambda_i|$ and $d_i := 1$, $i = 1, 2, \ldots, n$, inequality (10) becomes

$$\sum_{i=1}^{n} 1 \sum_{i=1}^{n} |\lambda_i|^5 + \sum_{i=1}^{n} |\lambda_i|^2 \sum_{i=1}^{n} |\lambda_i|^2 \geqslant 2 \sum_{i=1}^{n} |\lambda_i|^4 \sum_{i=1}^{n} |\lambda_i|.$$

Hence,

$$n \sum_{i=1}^{n} |\lambda_i|^5 + \Big(\sum_{i=1}^{n} |\lambda_i|^2 \Big)^2 \geqslant 2 \sum_{i=1}^{n} |\lambda_i|^4 \sum_{i=1}^{n} |\lambda_i|.$$

Then, considering the Proposition 1, we demonstrate the assertion of Theorem 3, i.e. the inequality (9).
If $G \cong \frac{n}{2}K_2$, then it is easy to check that the equality in (9) holds. Conversely, if the equality in (9) holds, since G is a non-empty graph, by Proposition 3, G has at least two distinct eigenvalues. We consider the following cases.

Case 1. The absolute value of all eigenvalues of G are equal.
Then clearly $\lambda_1 =\mid \lambda_i \mid \quad (2 \leqslant i \leqslant n)$, since G has at least two distinct eigenvalues. By Proposition 3, $\mid \lambda_i \mid= 1 (2 \leqslant i \leqslant n)$. Hence $2m = n$ and also, $\lambda_1 =\mid \lambda_2 \mid= \cdots =\mid \lambda_n \mid= 1$. By applying Proposition 3 again, we obtain that $m_2 = m_1\lambda_1, \lambda_1 = 1$, and therefore $m_1 = m_2$. Then we obtain that $\lambda_1 = 1$ has multiplicity $\frac{n}{2}$, and $\lambda_i = -1 \quad (2 \leqslant i \leqslant n)$ has multiplicity $\frac{n}{2}$. Therefore G is the direct sum of $m_1 = \frac{n}{2}$ complete graphs of order $\lambda_1 + 1 = 2$. Consequently, G is $\frac{n}{2}K_2$.

Case 2. The absolute value of all eigenvalues of G are not equal. Then G has two distinct eigenvalues with different absolute values. By Proposition 3, $\mid \lambda_i \mid= 1 \quad (2 \leqslant i \leqslant n)$. Since, $\sum_{i=1}^{n} \lambda_i = 0$ and $\lambda_2 = \lambda_3 = \cdots = \lambda_n = -1$, we have, $\lambda_1 = n-1$. Hence λ_1 has multiplicity 1 and $\lambda_i = -1$ has multiplicity $n-1$. By Proposition 3, G is the direct sum of a complete graph of order $\lambda_1 + 1 = n$. Consequently, G is K_n. □

Theorem 4. *Let G be a non-empty graph with n vertices and m edges. Then*

$$\mathcal{E}(G) \leq \frac{2mN_4 + n^2}{4m}, \tag{11}$$

equality holds if and only if $G \cong \frac{n}{2}(K_2), (n = 2m)$.

Proof. Let a_i, b_i, c_i and d_i are sequences of real numbers and p_i, q_i are non-negative for $i = 1, 2, \ldots, n$. Then the following inequality is valid (see [6] p.7)

$$\sum_{i=1}^{n} p_i a_i^2 \sum_{i=1}^{n} q_i b_i^2 + \sum_{i=1}^{n} p_i c_i^2 \sum_{i=1}^{n} q_i d_i^2 \geqslant 2 \sum_{i=1}^{n} p_i a_i c_i \sum_{i=1}^{n} q_i b_i d_i. \tag{12}$$

For $a_i := |\lambda_i|^2$, $b_i := |\lambda_i|$ and $c_i = d_i = p_i = q_i := 1$, $i = 1, 2, \ldots, n$, inequality (12) becomes

$$\sum_{i=1}^{n} |\lambda_i|^4 \sum_{i=1}^{n} |\lambda_i|^2 + \sum_{i=1}^{n} 1 \sum_{i=1}^{n} 1 \geqslant 2 \sum_{i=1}^{n} |\lambda_i|^2 \sum_{i=1}^{n} |\lambda_i|.$$

So,

$$\sum_{i=1}^{n} |\lambda_i|^4 \sum_{i=1}^{n} |\lambda_i|^2 + n^2 \geqslant 2 \sum_{i=1}^{n} |\lambda_i|^2 \sum_{i=1}^{n} |\lambda_i|.$$

Considering the Proposition 1, we demonstrate the assertion of Theorem 4, i.e. the inequality (11).
If $G \cong \frac{n}{2}K_2$, then it is easy to check that the equality in (11) holds. Conversely, if

the equality in (11) holds, since G is a non-empty graph, by Proposition 3, G has at least two distinct eigenvalues. We consider the following cases.

Case 1. The absolute value of all eigenvalues of G are equal.
Then clearly $\lambda_1 =\mid \lambda_i \mid \quad (2 \leqslant i \leqslant n)$, since G has at least two distinct eigenvalues. By Proposition 3, $\mid \lambda_i \mid = 1 (2 \leqslant i \leqslant n)$. Hence $2m = n$ and also, $\lambda_1 =\mid \lambda_2 \mid = \cdots =\mid \lambda_n \mid = 1$. By applying Proposition 3 again, we obtain that $m_2 = m_1\lambda_1, \lambda_1 = 1$, and therefore $m_1 = m_2$. Then we obtain that $\lambda_1 = 1$ has multiplicity $\frac{n}{2}$, and $\lambda_i = -1 \quad (2 \leqslant i \leqslant n)$ has multiplicity $\frac{n}{2}$. Therefore G is the direct sum of $m_1 = \frac{n}{2}$ complete graphs of order $\lambda_1 + 1 = 2$. Consequently, G is $\frac{n}{2}K_2$.

Case 2. The absolute value of all eigenvalues of G are not equal. Then G has two distinct eigenvalues with different absolute values. By Proposition 3, $\mid \lambda_i \mid = 1 (2 \leqslant i \leqslant n)$. Since, $\sum_{i=1}^{n} \lambda_i = 0$ and $\lambda_2 = \lambda_3 = \cdots = \lambda_n = -1$, we have, $\lambda_1 = n-1$. Hence λ_1 has multiplicity 1 and $\lambda_i = -1$ has multiplicity $n-1$. By Proposition 3, G is the direct sum of a complete graph of order $\lambda_1 + 1 = n$. Consequently, G is K_n. □

The next bounds involve lower bounds for the energy of a bipartite graph.

Theorem 5. *Let G be a non-empty bipartite graph of order at least 2, with m edges and having the spectral radius λ_1. Then*

$$\mathcal{E}(G) \geq \frac{2m}{\lambda_1} \tag{13}$$

equality holds if and only if one of the following statements holds:
(1) $G \cong \frac{n}{2}K_2$, $(n = 2m)$;
(2) $K_{1,r-1} \bigcup (n-1-r_{n-1})K_1$.

Proof. Let a_i, b_i are decreasing non-negative sequences with $a_i, b_i \neq 0$ and w_i a non negative sequence, for $i = 1, 2, \ldots, n$. Then the following inequality is valid (see [6] p.85)

$$\sum_{i=1}^{n} w_i a_i^2 \sum_{i=1}^{n} w_i b_i^2 \leqslant max \left\{ b_1 \sum_{i=1}^{n} w_i a_i, a_1 \sum_{i=1}^{n} w_i b_i \right\} \sum_{i=1}^{n} w_i a_i b_i. \tag{14}$$

For $a_i = b_i := |\lambda_i|$, and $w_i := 1$, $i = 1, 2, \ldots, n$, inequality (14) becomes

$$\sum_{i=1}^{n} |\lambda_i|^2 \sum_{i=1}^{n} |\lambda_i|^2 \leqslant max \left\{ \lambda_1 \sum_{i=1}^{n} |\lambda_i|, \lambda_1 \sum_{i=1}^{n} |\lambda_i| \right\} \sum_{i=1}^{n} |\lambda_i|^2.$$

Then

$$\sum_{i=1}^{n} |\lambda_i|^2 \leq \lambda_1 \sum_{i=1}^{n} |\lambda_i|$$

from the above inequality directly follows the assertion of Theorem 5, i.e. the inequality (13).

If $G \cong \frac{n}{2}K_2$, then it is easy to check that the equality in (13) holds. Conversely, if the equality in (13) holds, since G is a non-empty graph, by Proposition 3, G has at least two distinct eigenvalues. We consider the following cases.

Case 1. The absolute value of all eigenvalues of G are equal.
Then clearly $\lambda_1 =\mid \lambda_i \mid \quad (2 \leqslant i \leqslant n)$, since G has at least two distinct eigenvalues. By Proposition 3, $\mid \lambda_i \mid= 1 (2 \leqslant i \leqslant n)$. Hence $2m = n$ and also, $\lambda_1 =\mid \lambda_2 \mid= \cdots =\mid \lambda_n \mid= 1$. By applying Proposition 3 again, we obtain that $m_2 = m_1\lambda_1, \lambda_1 = 1$, and therefore $m_1 = m_2$. Then we obtain that $\lambda_1 = 1$ has multiplicity $\frac{n}{2}$, and $\lambda_i = -1 \quad (2 \leqslant i \leqslant n)$ has multiplicity $\frac{n}{2}$. Therefore G is the direct sum of $m_1 = \frac{n}{2}$ complete graphs of order $\lambda_1 + 1 = 2$. Consequently, G is $\frac{n}{2}K_2$.

Case 2. The absolute value of all eigenvalues of G are not equal. If two eigenvalues of G have different absolute values, then by Proposition 3, $\mid \lambda_i \mid= -1 (2 \leqslant i \leqslant n)$. Noting that G is a bipartite graph, we have $\lambda_1 = -\lambda_n$, that is a contradiction, since the two eigenvalues of G have different absolute values. Thus assume that G has three distinct eigenvalues. Since G is a bipartite graph, we have that $\lambda_1 = -\lambda_n \neq 0$ and $\sum_{i=1}^{n} \lambda_i = 0$, and therefore, $\lambda_i = 0 (2 \leqslant i \leqslant n-1)$. Thus $\mathcal{E}(G) = 2\lambda_1$, and by Proposition 2, we have that $2\lambda_1 \geqslant 2\sqrt{m}$, and so $2\lambda_1^2 \geqslant 2m$. Notice that $2m = \sum_{i=1}^{n} \lambda_i^2 = 2\lambda_1^2$. Therefore $\lambda_1 = \sqrt{m}$ and $\mathcal{E}(G) = 2\sqrt{m}$. Hence by Proposition 2, G is a complete bipartite graph plus arbitrarily many isolated vertices. Thus, there exist integers $r_1 \geqslant 1$ and $r_2 \geqslant 2$ such that G is $K_{r_1,r_2} \cup (n - r_1 - r_2)K_1$. □

Finally, considering the Theorem 5, we present two new lower bounds for connected bipartite graphs and tree graph.

Corollary 1. *Let G be a connected bipartite graph with n vertices and m edges, then*

$$\mathcal{E}(G) \geq \frac{2m}{n-1}.$$

Corollary 2. *Let G be a tree with n vertices and m edges, then*

$$\mathcal{E}(G) \geq \frac{2m}{\sqrt{n-1}}.$$

Acknowledgments J. Rodríguez was supported by MINEDUC-UA project, code ANT-1899, Funded by the Initiation Program in Research - Universidad de Antofagasta, INI-19-06 and PROGRAMA REGIONAL MATHAMSUD, MATH2020003.

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