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Research Article

On the powers of signed graphs

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Abstract: A signed graph is an ordered pair $\Sigma = (G, \sigma)$, where $G = (V, E)$ is the underlying graph of Σ with a signature function $\sigma : E \to \{1, -1\}$. In this article, we define the n^{th} power of a signed graph and discuss some properties of these powers of signed graphs. As we can define two types of signed graphs as the power of a signed graph, necessary and sufficient conditions are given for an n^{th} power of a signed graph to be unique. Also, we characterize balanced power signed graphs.

Keywords: Signed graph, signed distance, distance compatibility, power of signed graphs

AMS Subject classification: 05C12, 05C22, 05C50

1. Introduction

In this paper, we will treat only simple, finite and connected signed graphs. A signature on a graph $G = (V, E)$ is a function $\sigma : E \to \{1, -1\}$. A signed graph is a graph $G = (V, E)$ with a signature σ and is denoted as $\Sigma = (G, \sigma)$, where G is called the underlying graph of Σ. The sign of a cycle in a signed graph is the product of the signs of its edges. A signed graph is said to be balanced if no negative cycle exists and Σ is unbalanced, otherwise [2].
To begin with, we recall and adopt some definitions and notations from [3], in which the concept of signed distance in signed graphs and distance compatible signed graphs are introduced.

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Let u and v be any two vertices in a connected graph G. As usual, $d(u,v)$ denotes the distance (the lenght of the shortest path) between u and v. If u and v are adjacent vertices, then $d(u,v) = 1$. Let $\mathcal{P}_{(u,v)}$ denote the collection of all shortest paths $P_{(u,v)}$ between them. Then, the distance between u and v in a signed graph is defined as:
$d_{\max}(u,v) = \sigma_{\max}(u,v)d(u,v) = \max\{\sigma(P_{(u,v)}) : P_{(u,v)} \in \mathcal{P}_{(u,v)}\}d(u,v)$ and
$d_{\min}(u,v) = \sigma_{\min}(u,v)d(u,v) = \min\{\sigma(P_{(u,v)}) : P_{(u,v)} \in \mathcal{P}_{(u,v)}\}d(u,v)$, where the sign of a path P in Σ is defined as $\sigma(P) = \prod_{e\in E(P)} \sigma(e)$.
Two vertices u and v in Σ are said to be *distance-compatible* (briefly, *compatible*) if $d_{\min}(u,v) = d_{\max}(u,v)$. A signed graph Σ is said to be (distance)-compatible or simply compatible, if every pair of vertices is compatible and Σ is incompatible, otherwise.
Corresponding to the functions $d_{\max}$ and $d_{\min}$, there are two types of distance matrices in a signed graph called signed distance matrices [3] as given below.
(D1) $D^{\max}(\Sigma) = (d_{\max}(u,v))_{n\times n}$.
(D2) $D^{\min}(\Sigma) = (d_{\min}(u,v))_{n\times n}$.
Also, the concept of associated signed complete graphs associated with $D^{\max}(\Sigma)$ and $D^{\min}(\Sigma)$ are introduced in [3], as follows.

Definition 1. ([3]) The associated signed complete graph $K^{D^{\max}}(\Sigma)$ with respect to $D^{\max}(\Sigma)$ is obtained by joining the non-adjacent vertices of Σ with edges having signs

$$\sigma(uv) = \sigma_{\max}(uv)$$

The associated signed complete graph $K^{D^{\min}}(\Sigma)$ with respect to $D^{\min}(\Sigma)$ is obtained by joining the non-adjacent vertices of Σ with edges having signs

$$\sigma(uv) = \sigma_{\min}(uv)$$

whenever $D^{\max} = D^{\min} = D^{\pm}$, say, the associated signed complete graph of Σ is denoted by $K^{D^{\pm}}(\Sigma)$.
The concept of n^{th} power of graph is discussed in [1]. The n^{th} power of a graph $G = (V,E)$ is denoted as G^n and is defined as the graph having the same vertex set as that of G and any two vertices u and v are adjacent in G^n if their distance $d(u,v)$ is less than or equal to n.
In this paper, we define the n^{th} power of a signed graph by using the concept of signed distance in signed graphs and discuss some properties of n^{th} power of signed graphs. Also, we characterize the balanced power signed graphs.

2. Main Results

Corresponding to $\sigma_{\max}$ and $\sigma_{\min}$, two types of n^{th} powers of signed graph for a given signed graph Σ, can be defined as follows.

Definition 2. Let $\Sigma = (G, \sigma)$ be a signed graph.
(D1) The n^{th} power signed graph $\Sigma^n_{\max}$ is a signed graph $\Sigma^n_{\max} = (G^n, \sigma')$, where G^n is the n^{th} (unsigned) power of G and for any edge $e = uv \in G^n$, $\sigma'(uv) = \sigma_{\max}(u, v)$.
(D2) The n^{th} power signed graph $\Sigma^n_{\min}$ is a signed graph $\Sigma^n_{\min} = (G^n, \sigma'')$, where G^n is the n^{th} (unsigned) power of G and for any edge $e = uv \in G^n$, $\sigma''(uv) = \sigma_{\min}(u, v)$.

Remark 1. The n^{th} power of a signed graph Σ is said to be unique whenever $\Sigma^n_{\max} = \Sigma^n_{\min}$; if this is the case, it is denoted by Σ^n.

The following result immediately comes from the definitions.

Proposition 1. *Let $\Sigma = (G, \sigma)$ be a signed graph. Then, the n^{th} power of Σ is unique if and only if there exists no incompatible pair of vertices at a distance less than or equal to n.*

Proposition 2. *Let $\Sigma = (G, \sigma)$ be a signed graph with diameter less than or equal to n. Then, the n^{th} power signed graph $\Sigma^n_{\max} = (G^n, \sigma')$(or $\Sigma^n_{\min} = (G^n, \sigma'')$) is the associated signed complete graph $K^{D^{\max}}(\Sigma)$(or $K^{D^{\min}}(\Sigma)$). Moreover, if Σ is compatible then, $\Sigma^n = (G^n, \sigma')$ is the associated signed complete graph $K^{D^{\pm}}(\Sigma)$.*

Proof. Let $\Sigma = (G, \sigma)$ be a signed graph with diameter less than or equal to n. That is, the maximum distance between any two vertices in Σ is n. Therefore, while taking $\Sigma^n_{\max}$ all the non-adjacent vertices u and v will form an edge uv with sign $\sigma'(uv) = \sigma_{\max}(u, v)$. Hence, $\Sigma^n_{\max}$ is the associated signed complete graph $K^{D^{\max}}(\Sigma)$. Similarly, if we consider $\Sigma^n_{\min}$ all the non-adjacent vertices u and v will form an edge uv with sign $\sigma''(uv) = \sigma_{\min}(u, v)$. Hence, $\Sigma^n_{\min}$ is the associated signed complete graph $K^{D^{\min}}(\Sigma)$.
If Σ is compatible, then $\Sigma^n_{\max} = \Sigma^n_{\min} = \Sigma^n$. Since, the diameter is less than or equal to n, all the non-adjacent vertices in Σ will form an edge in Σ^n and sign of these edges are $\sigma'(uv) = \sigma_{\max}(u, v) = \sigma_{\min}(u, v)$. Hence, Σ^n is the associated signed complete graph $K^{D^{\pm}}(\Sigma)$. □

Lemma 1. *Let $\Sigma = (G, \sigma)$ be a signed graph and u, v be two vertices in Σ. If $\Sigma^n = (G^n, \sigma')$ exists, then for any u–v path P of length k in Σ, there exists a u–v path P' of length $\lceil \frac{k}{n} \rceil$ in Σ^n such that $\sigma(P) = \sigma'(P')$.*

Proof. Let $\Sigma = (G, \sigma)$ be a signed graph and let P be a u–v path of length k in Σ. In Σ^n all the vertices with distance at most n in Σ will form an edge. By considering division algorithm on k and n, we get $k = nq + r$, where $0 \leq r < n$.

Case 1: $r = 0$.
Then, $k = nq$. Hence, there will be q edges between u and v in Σ^n.

Case 2: $r \neq 0$.
Then, $k = nq + r$. Since, $r < n$ the path of length r will form an edge in Σ^n. Then,

there will be $q+1$ edges between u and v in Σ^n.
From the above cases, it can be concluded that, corresponding to the path P of length k in Σ there is a path P' from u to v in Σ^n of length $\lceil \frac{k}{n} \rceil$.
To prove $\sigma(P) = \sigma'(P')$. By the definition of Σ^n any path P of length $k \leq n$ will form an edge e in Σ^n and here $\sigma(P) = \sigma'(e)$.
Let $k > n$ and $k = nq + r$ and let $\{e_1, e_2, \dots e_{nq+r}\}$ be the edge set of P. Then, $\sigma(P) = \prod_{i=1}^{nq+r} \sigma(e_i)$. Suppose that $r \neq 0$. Then, P can be written as the union of edge disjoint paths, $P'_i = e_{(i-1)n+1}, e_{(i-1)n+2}, \dots, e_{in}$, $1 \leq i \leq q$ and $P'_{q+1} = e_{nq+1}, e_{nq+2}, \dots, e_{nq+r}$ whose length is at most n. Since, every path of length at most n will form an edge in Σ^n, $\{e'_1, e'_2, \dots, e'_{q+1}\}$ be the edges in Σ^n corresponding to paths $P'_1, \dots, P'_{q+1}$. Let P' be the path from u to v with edges $e'_1, e'_2, \dots, e'_{q+1}$ in Σ^n. Then, $\sigma'(P') = \prod_{i=1}^{q+1} \sigma'(e'_i) = \prod_{i=1}^{q+1} \sigma(P'_i) = \sigma(P)$. When $r = 0$, in a similar way we can see that $\sigma(P) = \sigma'(P')$. □

Lemma 2. *Let $\Sigma = (G, \sigma)$ be a signed graph and u, v be two vertices in Σ. If $\Sigma^n = (G^n, \sigma')$ exists, then for any u–$v$path P of length k in Σ^n, there exists a u–v path P' of length k', where $(k-1)n + 1 \leq k' \leq kn$ in Σ such that $\sigma'(P) = \sigma(P')$.*

Proof. Let $\Sigma^n = (G^n, \sigma')$ be the n^{th} power of $\Sigma = (G, \sigma)$ and P be a uv path of length k in Σ^n.
Case 1: If $k = 1$.
Then, uv is an edge in Σ^n. By the definition, each edge in Σ^n corresponds to a path P' of length $k' \leq n$ in Σ, where $\sigma'(uv) = \sigma(P')$.
Case 2: If $k > 1$.
Let $\{e_1, e_2, \dots, e_k\}$ be the edge set of P in Σ^n, where each edge e_i, $1 \leq i \leq k$ corresponds to a path P_i of length $l_i \leq n$ in Σ. Then, the concatenation $P' = \cup_i P_i$ will form a uv path of length $k' = \sum_{i=1}^{k} l_i \leq kn$ in Σ and the sign of P' is given by $\sigma(P') = \prod_{i=1}^{k} \sigma(P_i) = \prod_{i=1}^{k} \sigma'(e_i) = \sigma'(P)$.
Since, P is of length k by Lemma 1 k' should be greater than $(k-1)n$. □

Theorem 1. *Let $\Sigma = (G, \sigma)$ be a signed graph with diameter greater than n and the n^{th} power of Σ exists and unique. Then, $\Sigma^n = (G^n, \sigma')$ is compatible implies Σ is compatible.*

Proof. Suppose that Σ is incompatible. Since, the n^{th} power of Σ is unique, by Proposition 1, there exist no incompatible pair of vertices at a distance less than or equal to n. Let u and v be an incompatible pair of vertices at a distance $k > n$. Then, there exist two shortest path P and Q from u to v of length k with $\sigma(P)$ is positive and $\sigma(Q)$ is negative. Then, by Lemma 1, there exists two paths P' and Q' from u to v in Σ^n of length $\lceil \frac{k}{n} \rceil$ with $\sigma(P) = \sigma'(P')$ and $\sigma(Q) = \sigma'(Q')$. Thus, u and v will form an incompatible pair of vertices in Σ^n, a contradiction. Hence, the signed graph Σ is compatible. □

Remark 2. The converse of the above theorem is not generally true. For example, consider the cycle C_7^- and its square signed graph given in Figure 1. The cycle C_7^- is compatible, where its square signed graph contains incompatible vertices u_1 and u_4.

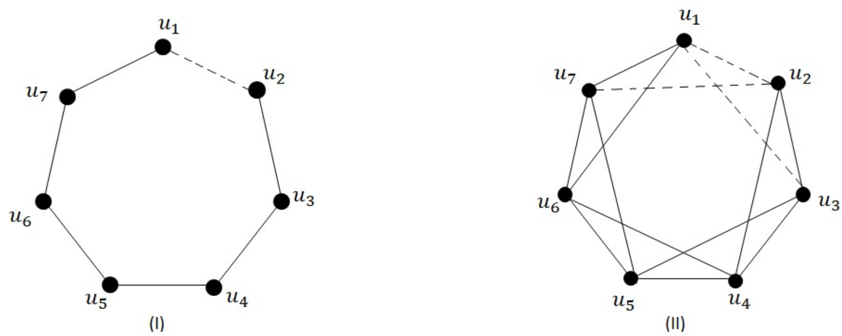

Figure 1. **The signed graph C_7^- and its square signed graph.**

2.1. Balance criterion in the power of a signed graph

In this section we give a characterization for balance in the n^{th} power of a signed graph. First we recall some results from [3, 4].

Theorem 2 ([3]). *For a signed graph Σ the following statements are equivalent:*

(1) *Σ is balanced*

(2) *The associated signed complete graph $K^{D^{\max}}(\Sigma)$ is balanced.*

(3) *The associated signed complete graph $K^{D^{min}}(\Sigma)$ is balanced.*

(4) *$D^{\max}(\Sigma) = D^{\min}(\Sigma)$ and the associated signed complete graph $K^{D^{\pm}}(\Sigma)$ is balanced.*

Theorem 3 ([3]). *A signed graph Σ is balanced if and only if the associated signed complete graph $K^{D^{\pm}}(\Sigma)$ has the spectrum* $\begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}$.

Lemma 3 ([4]). *Let u and v be incompatible pair of vertices with least distance in a 2-connected non-geodetic signed graph. Then, there will be two internally disjoint shortest paths from u to v of opposite signs.*

A connected graph G is called 2-*connected*, if for every vertex $x \in V(G)$, $G - x$ is connected. The following lemma discuss the compatibility of the n^{th} power of a 2-connected signed graph.

Lemma 4. *Let $\Sigma = (G, \sigma)$ be a 2-connected signed graph. If $\Sigma^n = (G^n, \sigma')$ exists, then Σ is balanced implies Σ^n is compatible.*

Proof. Let $\Sigma = (G, \sigma)$ be a 2-connected balanced signed graph. If possible, let u and v be an incompatible pair of vertices in Σ^n. Since, Σ^n is 2-connected by using Lemma 3, we get two internally disjoint shortest uv paths P and Q of opposite signs. Then, by Lemma 2, we can find two uv paths P' and Q' in Σ with $\sigma(P') = \sigma'(P)$ and $\sigma(Q') = \sigma'(Q)$. Therefore, P' and Q' can not be the same. If P' and Q' are internally disjoint, then the concatenation $(P') \cup (Q')^{-1}$ will be a negative cycle in Σ, a contradiction. Suppose that P' and Q' are not internally disjoint. Consider the cycles $C_1, C_2, \ldots, C_m$ formed by the common points of P' and Q'. Since, P' and Q' are not the same, there exists at least one such cycle. Also, since $\sigma(P') \neq \sigma(Q')$, we can find at least one cycle among $C_1, C_2, \ldots, C_m$, say C_i with common points u_i and v_i of P' and Q', where the sign of $u_i v_i$ path along P' and along Q' are distinct. Then, the cycle C_i will be a negative cycle in Σ, a contradiction. Hence, Σ^n should be compatible. □

Lemma 5. *Let $\Sigma = (G, \sigma)$ be a signed graph. Then, the associated signed complete graphs $K^{D^{\max}}(\Sigma) = K^{D^{\max}}(\Sigma^n_{\max})$ and $K^{D^{\min}}(\Sigma) = K^{D^{\min}}(\Sigma^n_{\min})$. Moreover, if Σ is balanced, then $K^{D^{\pm}}(\Sigma) = K^{D^{\pm}}(\Sigma^n)$.*

Proof. Let $\Sigma = (G, \sigma)$ be a signed graph and $\Sigma^n_{\max} = (G^n, \sigma')$ be the n^{th} power of Σ. Then, by Lemma 1, corresponding to the shortest uv path P in Σ, there is a shortest uv path P' in $\Sigma^n_{\max}$ where, $\sigma_{\max}(P_{(u,v)}) = \sigma'_{\max}(P'_{(u,v)})$. The associated signed complete graph $K^{D^{\max}}(\Sigma)$ is obtained by joining all the non-adjacent vertices in Σ, with edges having signs $\sigma(u, v) = \sigma_{\max}(u, v)$. Similarly, the associated signed complete graph $K^{D^{\max}}(\Sigma^n_{\max})$ is obtained by joining all the non-adjacent vertices in $\Sigma^n_{\max}$, with edges having signs $\sigma'(u, v) = \sigma'_{\max}(u, v) = \sigma_{\max}(u, v)$.
Also, if u and v are adjacent in Σ, we have $\sigma'(u, v) = \sigma_{\max}(u, v)$. Hence, $K^{D^{\max}}(\Sigma) = K^{D^{\max}}(\Sigma^n_{\max})$. Similarly, we get $K^{D^{\min}}(\Sigma) = K^{D^{\min}}(\Sigma^n_{\min})$.
If Σ is balanced, then $\Sigma^n_{\max} = \Sigma^n_{\min} = \Sigma^n$ and by Lemma 4 Σ^n is compatible. Therefore, $K^{D^{\max}}(\Sigma^n_{\max}) = K^{D^{\min}}(\Sigma^n_{\min}) = K^{D^{\pm}}(\Sigma^n)$. Also, Σ is balanced implies $\sigma(u, v) = \sigma_{\max}(u, v) = \sigma_{\min}(u, v)$ and $K^{D^{\max}}(\Sigma) = K^{D^{\min}}(\Sigma) = K^{D^{\pm}}(\Sigma)$. Hence, $K^{D^{\pm}}(\Sigma) = K^{D^{\pm}}(\Sigma^n)$. □

Theorem 4. *A 2-connected signed graph $\Sigma = (G, \sigma)$ is balanced if and only if $\Sigma^n = (G^n, \sigma')$ is balanced.*

Proof. Suppose that Σ is balanced. Then, by Theorem 2 we get $D^{\max}(\Sigma) = D^{\min}(\Sigma)$ and the associated signed complete graph $K^{D^{\pm}}(\Sigma)$ is balanced. Since, Σ is balanced by using Lemma 4 and Lemma 5 we get Σ^n is compatible and $K^{D^{\pm}}(\Sigma) = K^{D^{\pm}}(\Sigma^n)$. Which implies, $D^{\max}(\Sigma^n) = D^{\min}(\Sigma^n)$ and $K^{D^{\pm}}(\Sigma^n)$ is balanced. Again, by using Theorem 2, we get Σ^n is balanced.

Conversely, suppose that Σ^n is balanced. Being a subgraph of Σ^n, Σ should be balanced. □

The following Corollary is an immediate consequence of Theorem 3 and Theorem 4.

Corollary 1. *Let $\Sigma = (G, \sigma)$ be a 2-connected compatible signed graph of order m. Then, the n^{th} power signed graph Σ^n is balanced if and only if the associated signed complete graph $K^{D^{\pm}}(\Sigma)$ has the spectrum* $\begin{pmatrix} m-1 & -1 \\ 1 & m-1 \end{pmatrix}$.

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