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Girth, minimum degree, independence, and broadcast independence

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Dedicated to Prof. Dr. Lutz Volkmann, on the occasion of his 75th birthday.

Abstract: An independent broadcast on a connected graph G is a function $f : V(G) \to \mathbb{N}_0$ such that, for every vertex x of G, the value $f(x)$ is at most the eccentricity of x in G, and $f(x) > 0$ implies that $f(y) = 0$ for every vertex y of G within distance at most $f(x)$ from x. The broadcast independence number $\alpha_b(G)$ of G is the largest weight $\sum_{x\in V(G)} f(x)$ of an independent broadcast f on G.

It is known that $\alpha(G) \leq \alpha_b(G) \leq 4\alpha(G)$ for every connected graph G, where $\alpha(G)$ is the independence number of G. If G has girth g and minimum degree δ, we show that $\alpha_b(G) \leq 2\alpha(G)$ provided that $g \geq 6$ and $\delta \geq 3$ or that $g \geq 4$ and $\delta \geq 5$. Furthermore, we show that, for every positive integer k, there is a connected graph G of girth at least k and minimum degree at least k such that $\alpha_b(G) \geq 2\left(1 - \frac{1}{k}\right)\alpha(G)$. Our results imply that lower bounds on the girth and the minimum degree of a connected graph G can lower the fraction $\frac{\alpha_b(G)}{\alpha(G)}$ from 4 below 2, but not any further.

Keywords: broadcast independence, independence, packing

AMS Subject classification: 05C69, 05C12

1. Introduction

In the present paper, we relate broadcast independence to independence and packings in graphs of large girth and minimum degree. We consider finite, simple, and undirected graphs, and use standard terminology and notation. A set I of pairwise

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nonadjacent vertices of a graph G is an *independent set* in G, and the maximum cardinality of an independent set in G is the *independence number* $\alpha(G)$ of G. Similarly, a set P of vertices of G is a *packing* if $\text{dist}_G(x, y) \geq 3$ for every two distinct vertices x and y in P, where $\text{dist}_G(x, y)$ is the distance of x and y in G. The maximum cardinality of a packing in G is the *packing number* $\rho(G)$ of G. The independence number and the packing number are among the most fundamental and well studied graph parameters [10]. Broadcast independence was introduced by Erwin [8], cf. also [6], and was studied in [1–4]. Let $\mathbb{N}_0$ be the set of nonnegative integers. For a connected graph G, a function $f : V(G) \to \mathbb{N}_0$ is an *independent broadcast on G* if

(B1) $f(x) \leq \text{ecc}_G(x)$ for every vertex x of G, where $\text{ecc}_G(x)$ is the eccentricity of x in G, and

(B2) $\text{dist}_G(x, y) > \max\{f(x), f(y)\}$ for every two distinct vertices x and y of G with $f(x), f(y) > 0$.

The *weight* of f is $\sum\limits_{x \in V(G)} f(x)$. The *broadcast independence number* $\alpha_b(G)$ of G is the maximum weight of an independent broadcast on G, and an independent broadcast on G of weight $\alpha_b(G)$ is *optimal*. For an integer k, let $[k]$ be the set of all positive integers at most k.

Let G be a connected graph. A function f that assigns 1 to every vertex in some independent set in G, and 0 to every other vertex of G, is an independent broadcast on G, which implies $\alpha_b(G) \geq \alpha(G)$. Our main result in [3] implies $\alpha_b(G) \leq 4\alpha(G)$, and, hence,

$$1 \leq \frac{\alpha_b(G)}{\alpha(G)} \leq 4 \text{ for every connected graph } G.$$

The existing results and proofs suggest that $\frac{\alpha_b(G)}{\alpha(G)}$ should be smaller than 4 for connected graphs G of sufficiently large local expansion and sparsity. Natural hypotheses ensuring these properties are lower bounds on the girth and the minimum degree. In the present paper, we explore how much the upper bound on $\frac{\alpha_b(G)}{\alpha(G)}$ can be improved for connected graphs G of large girth and minimum degree. Our two main results are the following.

Theorem 1. *If G is a connected graph of girth at least 6 and minimum degree at least 3, then*

$$\alpha_b(G) < 2\alpha(G).$$

Theorem 2. *For every positive integer k, there is a connected graph G of girth at least k and minimum degree at least k such that*

$$\alpha_b(G) \geq 2\left(1 - \frac{1}{k}\right)\alpha(G).$$

Together, these two results imply that lower bounds on the girth and the minimum degree of a connected graph G can lower the fraction $\frac{\alpha_b(G)}{\alpha(G)}$ from 4 below 2, but not any further. The proof of Theorem 2 is an adaptation of Erdős's [7] famous probabilistic proof of the existence of graphs of arbitrarily large girth and chromatic number, and it actually implies the existence, for every positive integer k, of a connected graph G of girth at least k and minimum degree at least k such that $\rho(G) \geq \left(1 - \frac{1}{k}\right)\alpha(G)$. The method used in the proof of Theorem 1 also yields the following.

Theorem 3. *Let G be a connected graph of girth at least g and minimum degree at least δ.*

(i) If $g = 6$ and $\delta = 5$, then $\alpha_b(G) \leq \alpha(G) + \rho(G)$.

(ii) If ξ is a real number with $2 \leq \xi < 4$, $g = 4$, and $\delta \geq \frac{10}{\xi}$, then $\alpha_b(G) \leq \xi\alpha(G)$.

All proofs are given in the next section.

2. Proofs

Proof of Theorem 1. Let G be as in the statement. Let $f : V(G) \to \mathbb{N}_0$ be an optimal independent broadcast on G. Let $X = \{x \in V(G) : f(x) > 0\}$. To every vertex x in X, we assign a set $I(x)$ as follows:

- If $1 \leq f(x) \leq 2$, then let $I(x) = \{x\}$.
- If $3 \leq f(x) \leq 5$, then let $I(x) = N_G(x)$.
- If $6 \leq f(x) \leq 13$, then let $I(x) = \big\{y \in V(G) : \mathrm{dist}_G(x,y) \in \{0,2\}\big\}$.
- If $f(x) \geq 14$, then, by (B1), there is a shortest path $P(x) : xx_1 \ldots x_{2\ell+4}$ in G with $\ell = \left\lfloor \frac{f(x)-9}{4} \right\rfloor$. Let

$$I(x) = \big\{y \in V(G) : \mathrm{dist}_G(x,y) \in \{0,2\}\big\} \cup \bigcup_{i=1}^{\ell} \big(N_G(x_{2i+3}) \setminus \{x_{2i+2}\}\big).$$

See Figure 1 for an illustration.

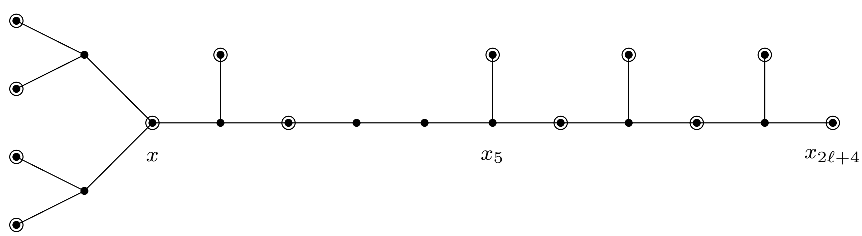

Figure 1. The set $I(x)$ for a vertex x with $f(x) \in \{21, 22, 23, 24\}$, where we assume that certain vertices have degree exactly 3.

By the girth condition and the choice of $P(x)$ as a shortest path, the set $I(x)$ is an independent set for every x in X.
Suppose, for a contradiction, that there are distinct vertices x and x' in X such that the sets $I(x)$ and $I(x')$ intersect or are joined by an edge. Let $f(x) \geq f(x')$. If $1 \leq f(x) \leq 2$, then $\text{dist}_G(x,x') = 1$, if $3 \leq f(x) \leq 5$, then $\text{dist}_G(x,x') \leq 3$, and if $6 \leq f(x) \leq 13$, then $\text{dist}_G(x,x') \leq 5$, which contradicts (B2) in each case. Now, let $f(x) \geq 14$. If $f(x') \leq 13$, then

$$\text{dist}_G(x,x') \leq \left(2\left\lfloor\frac{f(x)-9}{4}\right\rfloor + 4\right) + 3 \leq \frac{f(x)-9}{2} + 7 \leq f(x),$$

and, if $f(x') \geq 14$, then

$$\begin{aligned}\text{dist}_G(x,x') &\leq \left(2\left\lfloor\frac{f(x)-9}{4}\right\rfloor + 4\right) + 1 + \left(2\left\lfloor\frac{f(x')-9}{4}\right\rfloor + 4\right)\\ &\leq \frac{f(x)}{2} + \frac{f(x')}{2}\\ &\leq \max\{f(x), f(x')\},\end{aligned}$$

again contradicting (B2) in each case. Therefore, $I = \bigcup\limits_{x\in X} I(x)$ is an independent set in G.
Let x be a vertex in X. If either $f(x) = 1$ or $3 \leq f(x) \leq 13$, then the girth and degree conditions imply $|I(x)| > \frac{f(x)}{2}$. Similarly, if $f(x) \geq 14$, then, by the girth and degree conditions, and the choice of $P(x)$ as a shortest path, we obtain

$$|I(x)| \geq 7 + 2\left\lfloor\frac{f(x)-9}{4}\right\rfloor \geq 7 + \frac{f(x)-12}{2} > \frac{f(x)}{2}.$$

Finally, if $f(x) = 2$, then $|I(x)| = \frac{f(x)}{2}$, that is, only in this final case, equality holds. Altogether, we obtain

$$\alpha(G) \geq |I| \geq \sum_{x\in X} |I(x)| \geq \sum_{x\in X} \frac{f(x)}{2} \geq \frac{\alpha_b(G)}{2}.$$

Suppose, for a contradiction, that $\alpha(G) = \frac{\alpha_b(G)}{2}$, that is, the above inequality chain holds with equality throughout. This implies that $f(x) = 2$ for every x in X. By (B2), the set X is a packing in G, which implies

$$\alpha(G) \geq \rho(G) \geq |X| = \frac{\alpha_b(G)}{2} = \alpha(G),$$

that is, $\alpha(G) = \rho(G)$, and X is a maximum packing in G. Now, replacing x within X by two nonadjacent neighbors yields an independent set of order $|X|+1$, contradicting $\alpha(G) = \rho(G)$; cf. [9] for a structural characterization of the graphs that satisfy $\alpha(G) = \rho(G)$. This completes the proof. □

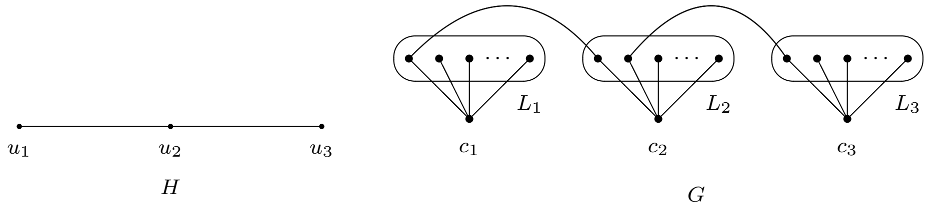

Figure 2. **Some H and G.**

Proof of Theorem 2. Let k be a fixed integer at least 3. Let the real ϵ be such that $0 < \epsilon < \frac{1}{k^2}$. Let H be a random graph in $\mathcal{G}(n,p)$ for $p = n^{\epsilon-1}$. Let $V(H) = \{u_1, \ldots, u_n\}$. Let G arise from the disjoint union of n copies $S_1, \ldots, S_n$ of the star $K_{1,k}$ of order $k+1$, where S_i has center vertex c_i and set of endvertices L_i for i in $[n]$, as follows: For every edge u_iu_j of H, select one vertex x_i in L_i uniformly at random and one vertex x_j in L_j uniformly at random, and add the edge x_ix_j to G. See Figure 2 for an illustration.

If X denotes the number of cycles of length less than k in H, then it is known (cf. Theorem 11.2.2. in [5]) that

$$\lim_{n\to\infty} \mathbb{P}\left[X \geq \frac{n}{2}\right] = 0.$$

A set I of vertices of G is an *independent transversal* if

(i) I is an independent set in G,

(ii) $I \cap \{c_1, \ldots, c_n\} = \emptyset$, and

(iii) $|I \cap L_i| \leq 1$ for every i in $[n]$.

Note that if i and j are distinct indices in $[n]$, then a vertex in L_i is adjacent to a vertex in L_j with probability $\frac{p}{k^2}$. Note furthermore, that there are $\binom{n}{r}k^r$ sets I of order r that satisfy the conditions (ii) and (iii) above. Therefore, if β denotes the maximum order of an independent transversal, then, by the union bound, we obtain, for $r = \frac{n}{2k^2}$,

$$\begin{aligned}
\mathbb{P}\left[\beta \geq r\right] &\leq \binom{n}{r}k^r\left(1-\frac{p}{k^2}\right)^{\binom{r}{2}}\\
&\leq n^rk^r\left(1-\frac{p}{k^2}\right)^{r(r-1)/2}\\
&= \left(nk\left(1-\frac{p}{k^2}\right)^{(r-1)/2}\right)^r\\
&\leq \left(nke^{-\frac{p(r-1)}{2k^2}}\right)^r \quad \text{(using } 1-x \leq e^{-x}\text{)}.
\end{aligned}$$

For n sufficiently large, we have $p \geq \frac{6k^4 \ln n}{n}$, which implies (cf. Lemma 11.2.1. in [5])

$$nke^{-\frac{p(r-1)}{2k^2}} = nke^{\left(-\frac{pn}{4k^4}+\frac{p}{2k^2}\right)} \leq nke^{\left(-\frac{3}{2}\ln(n)+\frac{1}{2}\right)} = \frac{k\sqrt{e}}{\sqrt{n}} \to 0 \text{ for } n \to \infty,$$

and, hence,

$$\lim_{n\to\infty} \mathbb{P}\left[\beta \geq \frac{n}{2k^2}\right] = 0.$$

Therefore, if n is sufficiently large, then

$$\mathbb{P}\left[X \geq \frac{n}{2}\right] + \mathbb{P}\left[\beta \geq \frac{n}{2k^2}\right] < 1,$$

which implies the existence of a graph H in $\mathcal{G}(n,p)$, and a graph G as above such that $X < \frac{n}{2}$ and $\beta < \frac{n}{2k^2}$.

For an induced subgraph H' of H, let $G(H') = G\left[\bigcup_{u_i \in V(H')} V(S_i)\right]$.

Let F be a set of at most $\frac{n}{2}$ vertices of H such that $H_0 = H - F$ has no cycle of length less than k. By construction, the graph $G(H_0)$ has no cycle of length less than k. Note that H_0 has order at least $\frac{n}{2}$.

We construct a finite sequence $H_0, \ldots, H_\ell$ as follows: Let i be a nonnegative integer such that H_i is defined. If $G(H_i)$ has minimum degree at least k, then let $\ell = i$, and terminate the sequence. Otherwise, $G(H_i)$ has a vertex x_i of degree less than k. By construction, there is a vertex u_s of H_i with $x_i \in L_s$. Let N be the set of indices j in $[n]$ such that x_i has a neighbor in L_j, and let $H_{i+1} = H_i - (\{u_s\} \cup \{u_j : j \in N\})$. Note that $|N| < k$.

Since $\{x_1, \ldots, x_\ell\}$ is an independent transversal, we have $\ell \leq \frac{n}{2k^2}$, which implies that H_ℓ has order n_ℓ at least $\frac{n}{2} - \frac{nk}{2k^2} = \frac{n}{2}\left(1 - \frac{1}{k}\right)$. The graph $G(H_\ell)$ has girth at least k, minimum degree at least k, and no independent transversal of order $\frac{n}{2k^2}$. If $G(H_\ell)$ is disconnected, then adding some bridges to $G(H_\ell)$ between different sets L_i yields a connected graph G^* that has girth at least k, minimum degree at least k, and no independent transversal of order $\frac{n}{2k^2}$.

The function $f : V(G^*) \to \mathbb{N}_0$ that assigns 2 to every vertex in $\{c_i : u_i \in V(H_\ell)\}$, and 0 to every other vertex, is an independent broadcast on G^*, which implies $\alpha_b(G^*) \geq 2n_\ell$.

Now, let J be a maximum independent set in G^*. Since G^* has no independent transversal of order $\frac{n}{2k^2}$, there are less than $\frac{n}{2k^2}$ indices i in $[n]$ such that J intersects L_i, which implies $\alpha(G^*) = |J| \leq n_\ell + \frac{nk}{2k^2} = n_\ell + \frac{n}{2k}$. Now,

$$\frac{\alpha_b(G^*)}{\alpha(G^*)} \geq \frac{2n_\ell}{n_\ell + \frac{n}{2k}} \geq \frac{2\frac{n}{2}\left(1-\frac{1}{k}\right)}{\frac{n}{2}\left(1-\frac{1}{k}\right) + \frac{n}{2k}} = 2\left(1 - \frac{1}{k}\right),$$

which completes the proof. □

Proof of Theorem 3. Let G be a connected graph of girth at least g and minimum degree at least δ. Let $f : V(G) \to \mathbb{N}_0$ be an optimal independent broadcast on G. Let $X = \{x \in V(G) : f(x) > 0\}$.

(i) First, we assume that $g = 6$ and $\delta = 5$.
To every vertex x in X, we assign a set $I(x)$ as follows:

- If $1 \leq f(x) \leq 2$, then let $I(x) = \{x\}$.
- If $f(x) \geq 3$, then, by (B1), there is a shortest path $P(x) : xx_1 \dots x_{2\ell-1}$ in G with $\ell = \left\lfloor \frac{f(x)+1}{4} \right\rfloor$. Let

$$I(x) = N_G(x) \cup \bigcup_{i=2}^{\ell} \big(N_G(x_{2i-2}) \setminus \{x_{2i-3}\}\big).$$

See Figure 3 for an illustration.

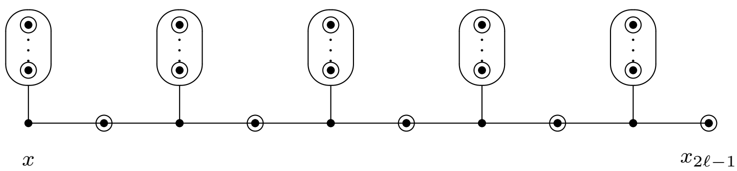

Figure 3. **The set $I(x)$ for a vertex x with $f(x) \in \{19, 20, 21, 22\}$.**

It follows similarly to the proof of Theorem 1 that the $I(x)$ are disjoint independent sets in G that are not joined by edges within G.
Let x be a vertex in X. If $f(x) = 1$, then $|I(x)| = f(x)$, if $f(x) = 2$, then $|I(x)| = f(x) - 1$, and, if $f(x) \geq 3$, then, by the girth and degree conditions and the choice of $P(x)$ as a shortest path,

$$|I(x)| \geq 5 + 4\left(\left\lfloor \frac{f(x)+1}{4} \right\rfloor - 1\right) \geq 5 + 4\left(\frac{f(x)-2}{4} - 1\right) = f(x) - 1.$$

Let $X_1 = \{x \in V(G) : f(x) = 1\}$. It follows that $I = \bigcup\limits_{x \in X} I(x)$ is an independent set in G of order at least $\alpha_b(G) - |X \setminus X_1| = \sum\limits_{x \in X_1} f(x) + \sum\limits_{x \in X \setminus X_1} (f(x) - 1)$. Since $X \setminus X_1$ is a packing in G, we obtain $\alpha(G) \geq \alpha_b(G) - |X \setminus X_1| \geq \alpha_b(G) - \rho(G)$, which completes the proof of (i).

(ii) Next, we assume that ξ is a real number with $2 \leq \xi < 4$, $g = 4$, and $\delta \geq \frac{10}{\xi}$.
To every vertex x in X, we assign a set $I(x)$ as follows:

- If $1 \leq f(x) \leq 2$, then let $I(x) = \{x\}$.
- If $f(x) \geq 3$, then, by (B1), there is a shortest path $P(x) : xx_1 \dots x_{4\ell-3}$ in G with $\ell = \left\lfloor \frac{f(x)+5}{8} \right\rfloor$. Let $x_0 = x$, and let

$$I(x) = \bigcup_{i=1}^{\ell} N_G(x_{4(i-1)}).$$

See Figure 4 for an illustration.

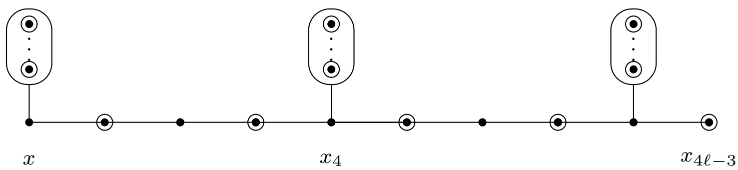

Figure 4. **The set $I(x)$ for a vertex x with $f(x) \in \{19, \dots, 26\}$.**

Again, the $I(x)$ are disjoint independent sets in G that are not joined by edges within G.
Let x be a vertex in X. If $1 \leq f(x) \leq 2$, then $|I(x)| \geq \frac{f(x)}{2} \geq \frac{f(x)}{\xi}$, if $3 \leq f(x) \leq \lfloor \xi\delta \rfloor$, then $|I(x)| \geq \delta \geq \frac{f(x)}{\xi}$, and, if $f(x) \geq \lfloor \xi\delta \rfloor + 1$ then, by the girth and degree conditions and the choice of $P(x)$ as a shortest path,

$$|I(x)| \geq \delta \left\lfloor \frac{f(x)+5}{8} \right\rfloor \geq \delta \frac{f(x)-2}{8} \geq \frac{f(x)}{\xi},$$

where we use $f(x) \geq \xi\delta$ and $\delta \geq \frac{10}{\xi}$. It follows that $\alpha(G) \geq \frac{\alpha_b(G)}{\xi}$, which completes the proof of (ii). □

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