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The Roman domination and domatic numbers of a digraph

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Abstract: Let D be a simple digraph with vertex set V. A Roman dominating function (RDF) on a digraph D is a function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex v with $f(v) = 0$ has an in-neighbor u with $f(u) = 2$. The weight of an RDF f is the value $\sum_{v \in V} f(v)$. The Roman domination number of a digraph D is the minimum weight of an RDF on D. A set $\{f_1, f_2, \dots, f_d\}$ of Roman dominating functions on D with the property that $\sum_{i=1}^{d} f_i(v) \leq 2$ for each $v \in V$, is called a Roman dominating family (of functions) on D. The maximum number of functions in a Roman dominating family on D is the Roman domatic number of D, denoted by $d_R(D)$. In this paper we continue the investigation of the Roman domination number, and we initiate the study of the Roman domatic number in digraphs. We present some bounds for $d_R(D)$. In addition, we determine the Roman domatic number of some digraphs.

Keywords: Roman dominating function, Roman domination number, Roman domatic number, digraph

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1. Introduction

Throughout this paper, $D = (V, A)$ denotes a finite simple digraph. If (u, v) is an arc of D, then v is an *out-neighbor* of u and u is an *in-neighbor* of v. For a vertex

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$v \in V(D)$, the *out-neighborhood* and *in-neighborhood* of v, denoted by $N^+(v) = N_D^+(v)$ and $N^-(v) = N_D^-(v)$, are the sets of out-neighbors and in-neighbors of v, respectively. The *closed out-neighborhood* and *closed in-neighborhood* of a vertex $v \in V(D)$ are the sets $N^+[v] = N_D^+[v] = N^+(v) \cup \{v\}$ and $N^-[v] = N_D^-[v] = N^-(v) \cup \{v\}$, respectively. The *out-degree* and *in-degree* of a vertex $v \in V(D)$ are defined by $d^+(v) = d_D^+(v) = |N^+(v)|$ and $d^-(v) = d_D^-(v) = |N^-(v)|$, respectively. The *maximum out-degree*, *minimum out-degree*, *maximum in-degree* and *minimum in-degree* among the vertices of D are denoted by $\Delta^+ = \Delta^+(D)$, $\delta^+ = \delta^+(D)$, $\Delta^- = \Delta^-(D)$ and $\delta^- = \delta^-(D)$, respectively.

A digraph D is *out-regular* or *r-out-regular* if $d^+(v) = r$ for each $v \in V(D)$. Likewise, a digraph D is *in-regular* or *r-in-regular* if $d^-(v) = r$ for each $v \in V(D)$. The *complement* $\overline{D}$ of a digraph D is the digraph defined on the vertex set $V(D)$, where $(u, v) \in A(\overline{D})$ if and only if $(u, v) \notin A(D)$. The *complete digraph* K_n^* is the digraph obtained from the complete graph K_n when each edge e of K_n is replaced by two oppositely oriented arcs with the same ends as e.

A *signed dominating function* (SDF) on D is a function $f : V(D) \to \{-1, 1\}$ such that $\sum_{x \in N^-[v]} f(x) \geq 1$ for each vertex $v \in V(D)$. The *weight* of an SDF f is $\omega(f) = \sum_{v \in V(D)} f(v)$. The *signed domination number* $\gamma_S(D)$ of a digraph D is the minimum weight of an SDF on D. An SDF on D with weight $\gamma_S(D)$ is called a *$\gamma_S(D)$-function*. The signed domination number of a digraph was introduced by Zelinka [16].

A *Roman dominating function* (RDF) on a digraph D is a function $f : V(D) \to \{0, 1, 2\}$ satisfying the condition that every vertex v with $f(v) = 0$ has an in-neighbor u with $f(u) = 2$. The *weight* of an RDF f is the value $\omega(f) = \sum_{v \in V(D)} f(v)$. The *Roman domination number* of a digraph D, denoted by $\gamma_R(D)$, is the minimum weight of an RDF on D. A *$\gamma_R(D)$-function* is an RDF on D with weight $\gamma_R(D)$. For a sake of simplicity, an RDF f on D will be represented by the ordered partition (V_0, V_1, V_2) of $V(D)$ induced by f, where $V_i = \{v \in V(D) : f(v) = i\}$ for $i \in \{0, 1, 2\}$. The Roman domination of a digraph was introduced by Kamaraj and Hemalatha [6], which has been studied by several authors [4, 5, 12].

The definition of the Roman domination number for undirected graphs was introduced multiplicity by Stewart [13] and ReVelle and Rosing [10], results on which could be found, for example, in [2, 3, 8, 9, 15, 17].

A set $\{f_1, f_2, \ldots, f_d\}$ of distinct Roman dominating functions on D with the property that $\sum_{i=1}^{d} f_i(v) \leq 2$ for each $v \in V(D)$, is called a *Roman dominating family* (of functions) on D. The maximum number of functions in a Roman dominating family (RD family) on D is the *Roman domatic number* of D, denoted by $d_R(D)$. The Roman domatic number is well-defined and $d_R(D) \geq 1$ for all digraphs D since the set consisting of any RDF forms an RD family on D.

The definition of the Roman domatic number for undirected graphs was given by Sheikholeslami and Volkmann [11] and has been studied in [7, 14].

Our purpose in this paper is to continue the investigations of the Roman domination number and to initiate the study of the Roman domatic number in digraphs. We start with some bounds on the Roman domination number, and then we study basic properties for the Roman domatic number of a digraph.

2. Bounds on the Roman domination number

We will make use of the following result due to Sheikholeslami and Volkmann [12].

Proposition 1 ([12]). *(a) For any digraph D of order $n \geq 3$, $\gamma_R(D) = 3$ if and only if $\Delta^+ = n - 2$, or $n = 3$ and $\Delta^+ \leq 1$.*

(b) For any digraph D of order $n \geq 4$, $\gamma_R(D) = 4$ if and only if $\Delta^+ = n-3$, or $\Delta^+ \leq n-4$ and there exist two vertices $u, v \in V(D)$ such that $N^+[u] \cup N^+[v] = V(D)$, or $n = 4$ and $\Delta^+ \leq 1$.

Let $n = 2r + 1$ with an integer $r \geq 1$. We define the *circulant tournament* $CT(n)$ of order n with vertex set $\{u_0, u_1, \dots, u_{n-1}\}$ as follows. For each $i \in \{0, 1, \dots, n-1\}$, the arcs are going from u_i to the vertices $u_{i+1}, u_{i+2}, \dots, u_{i+r}$, where the indices are taken modulo n.

As a consequence of Proposition 1, we have the following corollary.

Corollary 1. *Let $n = 2r + 1$ with an integer $r \geq 1$. Then*

$$\gamma_R(CT(n)) = \begin{cases} 3, & \text{if } r = 1, \\ 4, & \text{otherwise.} \end{cases}$$

Proof. If $r = 1$, then $\Delta^+ = r = 1 = n - 2$ and hence by Proposition 1(a), $\gamma_R(CT(n)) = 3$. If $r = 2$, then $\Delta^+ = r = 2 = n - 3$ and hence by Proposition 1(b), $\gamma_R(CT(n)) = 4$. Hence we may assume that $r \geq 3$. Observe that $\Delta^+ = r \leq n - 4$ and $N^+[u_0] \cup N^+[u_r] = V(CT(n))$. Consequently, again by Proposition 1(b), we get $\gamma_R(CT(n)) = 4$, which completes our proof. □

Let $n \geq 5$. By Corollary 1, we have $\gamma_R(CT(n)) = 4$. However, we obtain $\gamma_R(CT(n)) \geq 5$ by Theorem 9 in [12], which is a contradiction. This demonstrates that Theorem 9 in [12] is not correct when $n \geq 5$.

Theorem 1. *For any digraph D of order n with $\Delta^+ \geq 1$,*

$$\gamma_R(D) \geq \lceil 2n/(\Delta^+ + 1) \rceil,$$

and this bound is sharp.

Proof. Let $f = (V_0, V_1, V_2)$ be a $\gamma_R(D)$-function. Since each $v \in V_0$ has an in-neighbor in V_2, we observe that $|V_0| \leq \Delta^+|V_2|$. Consequently, we obtain

$$\begin{aligned}(\Delta^+ + 1)\gamma_R(D) &= (\Delta^+ + 1)|V_1| + 2(\Delta^+ + 1)|V_2| \\ &\geq (\Delta^+ + 1)|V_1| + 2|V_2| + 2|V_0| \\ &\geq 2|V_1| + 2|V_2| + 2|V_0| \\ &= 2n,\end{aligned}$$

which implies the desired lower bound.
Let $n = 2r + 1$ with an integer $r \geq 1$. Then by Corollary 1, we have

$$\gamma_R(CT(n)) = \left\lceil \frac{3(r+1) + r - 1}{r+1} \right\rceil = \left\lceil \frac{2(2r+1)}{r+1} \right\rceil = \left\lceil \frac{2n}{\Delta^+ + 1} \right\rceil,$$

which implies that the bound of Theorem 1 is sharp. □

A set $S \subseteq V(D)$ is a 2-*packing* of the digraph D if $N^+[u] \cap N^+[v] = \emptyset$ for any two distinct vertices $u, v \in S$. The 2-*packing number* $\rho(D)$ of D is defined by

$$\rho(D) = \max\{|S| : S \text{ is a 2-packing of } D\}.$$

Theorem 2. *For any digraph D of order n with $\delta^+ \geq 1$,*

$$\gamma_R(D) \leq n - (\delta^+ - 1)\rho(D).$$

Proof. Let $S = \{v_1, v_2, \ldots, v_{\rho(D)}\}$ be a 2-packing of D and let $T = \bigcup_{i=1}^{\rho(D)} N^+(v_i)$. Then clearly $|T| = |\bigcup_{i=1}^{\rho(D)} N^+(v_i)| = \sum_{i=1}^{\rho(D)} d^+(v_i) \geq \rho(D)\delta^+$. Define the function f by $f(x) = 2$ for $x \in S$, $f(x) = 0$ for $x \in T$ and $f(x) = 1$ otherwise. We observe that f is an RDF on D and hence we get

$$\begin{aligned}\gamma_R(D) &\leq \sum_{x \in S} f(x) + \sum_{x \in T} f(x) + \sum_{x \in V(D)\setminus(S \cup T)} f(x) \\ &= 2|S| + (n - |S| - |T|) \\ &= n + |S| - |T| \\ &\leq n + \rho(D) - \rho(D)\delta^+ \\ &= n - (\delta^+ - 1)\rho(D),\end{aligned}$$

establishing the desired result. □

We shall relate the Roman domination number to signed domination number of digraphs. For this purpose, we need a result due to Ahangar et al. [1].

Let G be a bipartite (undirected) graph with bipartition $(\mathcal{L}, \mathcal{R})$ (standing for "left" and "right"). A subset S of vertices in $\mathcal{R}$ is a *left dominating set* of G if every vertex of $\mathcal{L}$ is adjacent to a vertex in S. The *left domination number*, denoted by $\gamma_{\mathcal{L}}(G)$, is the minimum cardinality of a left dominating set of G. A left dominating set of G of cardinality $\gamma_{\mathcal{L}}(G)$ is called a $\gamma_{\mathcal{L}}(G)$*-set*. Let $\delta_{\mathcal{L}}(G)$ denote the minimum degree of a vertex of $\mathcal{L}$ in G. Ahangar et al. [1] established the following upper bound on the left domination number of a bipartite (undirected) graph in terms of its order.

Theorem 3 ([1]). *Let G be a bipartite (undirected) graph of order n with bipartition $(\mathcal{L}, \mathcal{R})$. If $\delta_{\mathcal{L}}(G) \geq 2$, then $\gamma_{\mathcal{L}}(G) \leq n/3$.*

Using Theorem 3, we shall obtain the following result.

Theorem 4. *For any digraph D of order n,*

$$\gamma_R(D) \leq \gamma_S(D)/2 + 5n/6.$$

Proof. Let f be a $\gamma_S(D)$-function and let $\mathcal{L}$ and $\mathcal{R}$ denote the sets of those vertices in D which are assigned under f the values -1 and 1, respectively. Then $|\mathcal{L}|+|\mathcal{R}| = n$ and $\gamma_S(D) = \omega(f) = |\mathcal{R}| - |\mathcal{L}|$, which implies that $2|\mathcal{R}| = n + \gamma_S(D)$.
If $\mathcal{L} = \emptyset$, that is, if $\mathcal{R} = V(D)$, then we set $g(x) = 1$ for each $x \in V(D)$. Observe that g is an RDF on D, implying that

$$\gamma_R(D) \leq \omega(g) = n = |\mathcal{R}| = \gamma_S(D) < \gamma_S(D)/2 + 5n/6.$$

So in the following we may assume that $\mathcal{L} \neq \emptyset$. Let D' be the bipartite spanning subdigraph of D with bipartition $(\mathcal{L}, \mathcal{R})$, where $A(D') = \{(u, v) \in A(D) : u \in \mathcal{R} \text{ and } v \in \mathcal{L}\}$. Since f is a $\gamma_S(D)$-function, each vertex of $\mathcal{L}$ has at least 2 in-neighbors in $\mathcal{R}$ in D' and hence $\delta^-_{\mathcal{L}}(D') \geq 2$, where $\delta^-_{\mathcal{L}}(D') = \min\{d^-_{D'}(v) : v \in \mathcal{L}\}$. Let H be the (undirected) graph obtained from D' by replacing any arc with an edge. It is easy to see that H is a bipartite (undirected) graph of order n with bipartition $(\mathcal{L}, \mathcal{R})$. Let $\mathcal{R}_2$ be a $\gamma_{\mathcal{L}}(H)$-set. Observe that $\delta_{\mathcal{L}}(H) = \delta^-_{\mathcal{L}}(D') \geq 2$ and hence by Theorem 3, $|\mathcal{R}_2| = \gamma_{\mathcal{L}}(H) \leq n/3$. Moreover, since $\mathcal{R}_2$ is a $\gamma_{\mathcal{L}}(H)$-set, any vertex in $\mathcal{L}$ is adjacent to some vertex in $\mathcal{R}_2$ in H and hence any vertex in $\mathcal{L}$ has at least one in-neighbor in $\mathcal{R}_2$ in D' and so in D. Let $\mathcal{R}_1 = \mathcal{R} \backslash \mathcal{R}_2$. Set

$$h(x) = \begin{cases} 0, & \text{if } x \in \mathcal{L}, \\ 1, & \text{if } x \in \mathcal{R}_1, \\ 2, & \text{if } x \in \mathcal{R}_2. \end{cases}$$

It is easy to see that h is an RDF on D and hence

$$\begin{aligned}\gamma_R(D) &\leq \omega(h) = |\mathcal{R}_1| + 2|\mathcal{R}_2| \\ &= (|\mathcal{R}_1| + |\mathcal{R}_2|) + |\mathcal{R}_2| = |\mathcal{R}| + |\mathcal{R}_2| \\ &\leq (n + \gamma_S(D))/2 + n/3 \\ &= \gamma_S(D)/2 + 5n/6,\end{aligned}$$

which completes our proof. □

3. Properties of the Roman domatic number

In this section we turn our attention to the Roman domatic number of digraphs. We begin with the following result, which will be useful in many of the results of this paper.

Theorem 5. *For any digraph D, $d_R(D) \geq 1$ with equality if and only if D has no directed even cycle.*

Proof. The lower bound is trivial. We proceed to show the sufficiency. Let D has no directed even cycle. Suppose, to the contrary, that $d_R(D) \geq 2$. Let $\{f_1, f_2, \ldots, f_d\}$ be an RD family on D such that $d = d_R(D)$.
We now claim that there exist some RDF, say f_1, of $\{f_1, f_2, \ldots, f_d\}$ and some vertex, say v_0, of D such that $f_1(v_0) = 0$. By contradiction. Suppose that $f_i(x) \geq 1$ for all $i \in \{1, 2, \ldots, d\}$ and $x \in V(D)$. If $f_i(x) = 1$ for all $i \in \{1, 2, \ldots, d\}$ and $x \in V(D)$, then $d = d_R(D) = 1$, a contradiction. Hence there exist some RDF, say f_1, of $\{f_1, f_2, \ldots, f_d\}$ and some vertex, say v, of D such that $f_1(v) = 2$. Then by the definition of RD family, we have $f_2(v) = 0$, which is a contradiction. So, this claim is true.
It follows from the definition of RDF that there exists some vertex, say v_1, of D such that $f_1(v_1) = 2$. Hence by the definition of RD family, we get $f_2(v_1) = 0$. Then again by the definition of RDF, there exists some vertex, say v_2, of D such that $f_2(v_2) = 2$. So again by the definition of RD family, we get $f_1(v_2) = 0$. Repeating this process we can obtain a sequence $v_1, v_2, \ldots$ of vertices of D satisfying the following properties: For each $i \in \{1, 2, \ldots\}$,

(*a*) v_{i+1} is an in-neighbor of v_i.

(*b*) $f_1(v_{2i}) = f_2(v_{2i-1}) = 0$ and $f_1(v_{2i-1}) = f_2(v_{2i}) = 2$.

Note that D is finite. Therefore, we may assume that l is minimum such that the vertex v_l of D coincides with the vertex v_k of D for some $k < l$. By (a) and (b), we get $f_1(v_{l-1}) = 0$ and $f_1(v_l) = f_1(v_k) = 2$, or $f_2(v_{l-1}) = 0$ and $f_2(v_l) = f_2(v_k) = 2$. If $f_1(v_{l-1}) = 0$ and $f_1(v_l) = f_1(v_k) = 2$, then by (b), we have that both k and l are odd.

This implies that $v_l v_{l-1} \cdots v_k$ is a directed even cycle of length $l-k$, a contradiction. The discussion for the case when $f_2(v_{l-1}) = 0$ and $f_2(v_l) = f_2(v_k) = 2$ is analogous. Consequently, we have $d_R(D) = 1$.
Conversely, assume that $d_R(D) = 1$. Suppose, to the contrary, that D has a directed even cycle $u_1u_2 \cdots u_{2k}$, where $k \geq 1$ is an integer. Define the functions f and g as follows:
$f(u_{2i-1}) = 0$ and $f(u_{2i}) = 2$ for $i = 1, 2, \ldots, k$, and $f(x) = 1$ otherwise, and
$g(u_{2i-1}) = 2$ and $g(u_{2i}) = 0$ for $i = 1, 2, \ldots, k$, and $g(x) = 1$ otherwise.
We observe that $\{f, g\}$ is an RD family on D and hence $d_R(D) \geq 2$, a contradiction. Consequently, D has no directed even cycle.
The proof is completed. □

Theorem 6. *For any digraph D of order n,*

$$\gamma_R(D) \cdot d_R(D) \leq 2n.$$

Moreover, if $\gamma_R(D) \cdot d_R(D) = 2n$, then for each RD family $\{f_1, f_2, \ldots, f_d\}$ on D with $d = d_R(D)$, each function f_i is a $\gamma_R(D)$-function, and $\sum_{i=1}^{d} f_i(v) = 2$ for all $v \in V(D)$.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be an RD family on D such that $d = d_R(D)$. Then

$$\begin{aligned}\gamma_R(D) \cdot d_R(D) &= \sum_{i=1}^{d} \gamma_R(D) \leq \sum_{i=1}^{d} \sum_{v \in V(D)} f_i(v) \\ &= \sum_{v \in V(D)} \sum_{i=1}^{d} f_i(v) \leq \sum_{v \in V(D)} 2 = 2n.\end{aligned}$$

If $\gamma_R(D) \cdot d_R(D) = 2n$, then the two inequalities occurring in the proof become equalities. Hence for the RD family $\{f_1, f_2, \ldots, f_d\}$ on D and for each i, we get $\sum_{v \in V(D)} f_i(v) = \gamma_R(D)$, implying that each function f_i is a $\gamma_R(D)$-function, and $\sum_{i=1}^{d} f_i(v) = 2$ for all $v \in V(D)$. This completes the proof. □

As a consequence of Theorem 6, we have the following result on the Roman domatic number.

Corollary 2. *For any digraph D of order $n \geq 2$, $d_R(D) \leq n$ with equality if and only if D is a complete digraph.*

Proof. It is easy to see that $\gamma_R(D) \geq 2$ since $n \geq 2$. Therefore, it follows from Theorem 6 that $d_R(D) \leq 2n/\gamma_R(D) \leq n$, establishing the desired upper bound.
Let $V(D) = \{v_1, v_2, \ldots, v_n\}$. Suppose that D is a complete digraph. For each $i, j \in \{1, 2, \ldots, n\}$, set $f_i(v_j) = 2$ if $i = j$ and $f_i(v_j) = 0$ otherwise. We observe

that $\{f_1, f_2, \ldots, f_n\}$ is an RD family on D and hence $d_R(D) \geq n$. As shown earlier, $d_R(D) \leq n$. Consequently, we have $d_R(D) = n$.
Conversely, suppose that $d_R(D) = n$. If $n = 2$, then by Theorem 5, the result is trivial. Hence we may assume that $n \geq 3$. Observe that $\gamma_R(D) \geq 2$. Moreover, we get $\gamma_R(D) \leq 2n/d_R(D) = 2$ by Theorem 6. Thus we have $\gamma_R(D) = 2$. Let $\{f_1, f_2, \ldots, f_n\}$ be an RD family on D. Note that $\gamma_R(D) \cdot d_R(D) = 2n$. Therefore, again by Theorem 6, we obtain that for $i \in \{1, 2, \ldots, n\}$, each function f_i is a $\gamma_R(D)$-function and hence $\omega(f_i) = \gamma_R(D) = 2$, implying that there exists one vertex, say v_i, assigned 2 under f_i and the others assigned 0. By the definition of RDF, we have that for each $i \in \{1, 2, \ldots, n\}$, v_i is an in-neighbor of the other vertices of D and hence $d^+(v_i) \geq n - 1$. Note that D is simple. Therefore, $d^+(v_i) = n - 1$ for each $i \in \{1, 2, \ldots, n\}$. Consequently, we obtain that D is a complete digraph. This completes the proof. □

Sheikholeslami and Volkmann [12] established the following upper bound on the Roman domination number of a digraph.

Proposition 2 ([12]). *Let D be a digraph of order $n \geq 2$. Then $\gamma_R(D) < n$ if and only if $\Delta^+ \geq 2$.*

Using Proposition 2, Theorem 5 and Theorem 6, we may derive the following corollary.

Corollary 3. *Let D be a digraph of order $n \geq 2$. Then $\gamma_R(D) = n$ and $d_R(D) = 2$ if and only if $\Delta^+ \leq 1$ and D has a directed even cycle.*

Proof. Suppose that $\gamma_R(D) = n$ and $d_R(D) = 2$. It is easy to see that $\Delta^+ \leq 1$ by Proposition 2 and D has a directed even cycle by Theorem 5.
Conversely, suppose that $\Delta^+ \leq 1$ and D has a directed even cycle. Then by Proposition 2, $\gamma_R(D) = n$ since $\Delta^+ \leq 1$. Moreover, it follows from Theorem 5 that $d_R(D) \geq 2$ since D has a directed even cycle. Note that $d_R(D) \leq 2n/\gamma_R(D) = 2$ by Theorem 6. Therefore, we have $d_R(D) = 2$. □

Corollary 4. *Let D be a digraph of order $n \geq 2$. Then*

$$\gamma_R(D) + d_R(D) \leq n + 2$$

with equality if and only if D is a complete digraph, or $\Delta^+ \leq 1$ and D has a directed even cycle.

Proof. If $d_R(D) = 1$, then obviously $\gamma_R(D) + d_R(D) \leq n + 1$. Let now $d_R(D) \geq 2$. Note that $d_R(D) \leq n$ by Corollary 2. Using these inequalities, and the fact that the function $g(x) = x + (2n)/x$ is decreasing for $2 \le x \le \sqrt{2n}$ and increasing for $\sqrt{2n} \le x \le n$, we obtain that by Theorem 6,

$$\gamma_R(D) + d_R(D) \le \frac{2n}{d_R(D)} + d_R(D) \le \max\left\{\frac{2n}{2} + 2, \frac{2n}{n} + n\right\} = n + 2, \qquad (1)$$

establishing the desired upper bound.

If D is a complete digraph, then clearly $\gamma_R(D) = 2$ and by Corollary 2, $d_R(D) = n$. This implies that $\gamma_R(D) + d_R(D) = n + 2$. If $\Delta^+ \le 1$ and D has a directed even cycle, then by Corollary 3, $\gamma_R(D) = n$ and $d_R(D) = 2$. This also implies that $\gamma_R(D) + d_R(D) = n + 2$.

Conversely, let $\gamma_R(D) + d_R(D) = n + 2$. Then we have equality throughout the inequality chain (1), implying that $\gamma_R(D) = 2$ and $d_R(D) = n$, or $\gamma_R(D) = n$ and $d_R(D) = 2$. If $\gamma_R(D) = 2$ and $d_R(D) = n$, then by Corollary 2, D is a complete digraph. If $\gamma_R(D) = n$ and $d_R(D) = 2$, then by Corollary 3, $\Delta^+ \le 1$ and D has a directed even cycle. □

Sheikholeslami and Volkmann [12] determined the exact value of the Roman domination number of directed cycles.

Proposition 3 ([12]). *For any directed cycle C_n of length n, $\gamma_R(C_n) = n$.*

We now derive the exact value of the Roman domatic number of directed cycles.

Proposition 4. *For any directed cycle C_n of length n,*

$$d_R(C_n) = \begin{cases} 1, & \text{if } n \text{ is odd,} \\ 2, & \text{if } n \text{ is even.} \end{cases}$$

Proof. If n is odd, then by Theorem 5, $d_R(C_n) = 1$. Now let n be even. Then by Theorem 6 and Proposition 3, we have $d_R(C_n) \le 2n/\gamma_R(C_n) = 2$. Let $C_n = v_1v_2\cdots v_nv_1$. Define the Roman dominating functions f_1 and f_2 as follows: $f_1(v_{2i-1}) = f_2(v_{2i}) = 2$ and $f_1(v_{2i}) = f_2(v_{2i-1}) = 0$ for $i = 1, 2, \ldots, n/2$. Then $\{f_1, f_2\}$ is an RD family on C_n, implying that $d_R(C_n) \ge 2$. Consequently, we obtain $d_R(C_n) = 2$. □

Theorem 7. *For any digraph D,*

$$d_R(D) \le \delta^- + 2.$$

Moreover, this upper bound is sharp.

Proof. If $d_R(D) \le 2$, then the result is immediate. Hence we may assume that $d_R(D) \ge 3$. Let $\{f_1, f_2, \dots, f_d\}$ be an RD family on D such that $d = d_R(D)$ and let v be a vertex of in-degree δ^-. Note that $\sum_{u \in N^-[v]} f_i(u) \ge 1$ and the equality holds for at most two indices $i \in \{1, 2, \dots, d\}$. Therefore, we get

$$\begin{aligned} 2 + 2(d-2) &\le \sum_{i=1}^{d} \sum_{u \in N^-[v]} f_i(u) = \sum_{u \in N^-[v]} \sum_{i=1}^{d} f_i(u) \\ &\le \sum_{u \in N^-[v]} 2 = 2(\delta^- + 1), \end{aligned}$$

which implies the desired upper bound.
To prove the sharpness, let D_i be a copy of the complete digraph K^*_{k+3} with vertex set $V(D_i) = \{v^i_1, v^i_2, \dots, v^i_{k+3}\}$ for $1 \le i \le k$ and let D be the digraph obtained from $\bigcup_{i=1}^{k} D_i$ by adding a new vertex v and two new arcs (v, v^i_1) and (v^i_1, v). Define the functions $f_1, f_2, \dots, f_{k+2}$ as follows: For $1 \le i \le k$,

$f_i(v^i_1) = 2$, $f_i(v^j_{i+1}) = 2$ if $j \in \{1, 2, \dots, k\} \backslash \{i\}$ and $f_i(x) = 0$ otherwise,
$f_{k+1}(v) = 1$, $f_{k+1}(v^j_{k+2}) = 2$ if $j \in \{1, 2, \dots, k\}$ and $f_{k+1}(x) = 0$ otherwise,
and
$f_{k+2}(v) = 1$, $f_{k+2}(v^j_{k+3}) = 2$ if $j \in \{1, 2, \dots, k\}$ and $f_{k+2}(x) = 0$ otherwise.

We obverse that f_i is an RDF on D for each $i = 1, 2, \dots, k+2$ and hence $\{f_1, f_2, \dots, f_{k+2}\}$ is an RD family on D, which implies that $d_R(D) \ge k + 2 = \delta^- + 2$. As shown earlier, $d_R(D) \le \delta^- + 2$. Therefore, we have $d_R(D) = \delta^- + 2$. □

Theorem 8. *If D is a k-out-regular digraph of order n, where $n = p(k+1) + r$ with integers $p \ge 1$ and $0 \le r \le k$, then*

$$d_R(D) \le k + \epsilon$$

with $\epsilon = 1$ when $k = 0$, or $r = 0$, or $2r = k + 1$, and $\epsilon = 0$ otherwise.

Proof. If $k = 0$, then Theorem 5 implies the desired result. Hence we may assume that $k \ge 1$. Let $\{f_1, f_2, \dots, f_d\}$ be an RD family on D such that $d = d_R(D)$. Consequently, we obtain

$$\sum_{i=1}^{d} \omega(f_i) = \sum_{i=1}^{d} \sum_{v \in V(D)} f_i(v) = \sum_{v \in V(D)} \sum_{i=1}^{d} f_i(v) \le \sum_{v \in V(D)} 2 = 2n. \qquad (2)$$

Assume first that $r = 0$. By Theorem 1, we have $\omega(f_i) \ge \gamma_R(D) \ge \lceil 2n/(k+1) \rceil = 2p$ for each $i \in \{1, 2, \dots, d\}$. If $d \ge k + 2$, then

$$\sum_{i=1}^{d} \omega(f_i) \ge \sum_{i=1}^{d} (2p) = 2pd \ge 2p(k+2) > 2n,$$

which is a contradiction to (2). Therefore, we get $d_R(D) \le k+1$.
Assume now that $2 \le 2r < k+1$. By Theorem 1, we have $\omega(f_i) \ge \gamma_R(D) \ge \lceil 2n/(k+1) \rceil = 2p+1$ for each $i \in \{1, 2, \dots, d\}$. If $d \ge k+1$, then

$$\sum_{i=1}^{d} \omega(f_i) \ge \sum_{i=1}^{d} (2p+1) = (2p+1)d \ge (2p+1)(k+1) > 2n,$$

which is a contradiction to (2). Therefore, we get $d_R(D) \le k$.
Assume next that $2r = k+1$. By Theorem 1, we have $\omega(f_i) \ge \gamma_R(D) \ge \lceil 2n/(k+1) \rceil = 2p+1$ for each $i \in \{1, 2, \dots, d\}$. If $d \ge k+2$, then

$$\sum_{i=1}^{d} \omega(f_i) \ge \sum_{i=1}^{d} (2p+1) = (2p+1)d \ge (2p+1)(k+2) > 2n,$$

which is a contradiction to (2). Therefore, we get $d_R(D) \le k+1$.
Finally assume that $2r > k+1$. By Theorem 1, we have $\omega(f_i) \ge \gamma_R(D) \ge \lceil 2n/(k+1) \rceil = 2p+2$ for each $i \in \{1, 2, \dots, d\}$. If $d \ge k+1$, then

$$\sum_{i=1}^{d} \omega(f_i) \ge \sum_{i=1}^{d} (2p+2) = (2p+2)d \ge (2p+2)(k+1) > 2n$$

since $k+1 > r$, which is a contradiction to (2). Therefore, we get $d_R(D) \le k$.
The proof is completed. □

Theorem 9. *If D is a digraph of order $n \ge 2$, then*

$$d_R(D) + d_R(\overline{D}) \le n + \epsilon$$

with $\epsilon = 1$ when D is out-regular, $\epsilon = 2$ when D is not in-regular and $\epsilon = 3$ otherwise.

Proof. If D is k-out-regular, then $\overline{D}$ is $(n-k-1)$-out-regular and hence by Theorem 8, we obtain

$$d_R(D) + d_R(\overline{D}) \le (k+1) + (n-k-1+1) = n+1.$$

If D is not in-regular, then $\Delta^-(D) - \delta^-(D) \ge 1$ and hence by Theorem 7, we have

$$\begin{aligned} d_R(D) + d_R(\overline{D}) &\le (\delta^-(D)+2) + (\delta^-(\overline{D})+2) \\ &= (\delta^-(D)+2) + (n - \Delta^-(D) - 1 + 2) \\ &\le n+2. \end{aligned}$$

If D is in-regular but not out-regular, then again by Theorem 7, we get

$$\begin{aligned} d_R(D) + d_R(\overline{D}) &\leq (\delta^-(D) + 2) + (\delta^-(\overline{D}) + 2) \\ &= (\delta^-(D) + 2) + (n - \Delta^-(D) - 1 + 2) \\ &= n + 3, \end{aligned}$$

and this completes the proof. □

Corollary 5. *If D is a digraph of order $n \geq 2$, then*

$$d_R(D) \cdot d_R(\overline{D}) \leq (n + \epsilon)^2/4$$

with $\epsilon = 1$ when D is out-regular, $\epsilon = 2$ when D is not in-regular and $\epsilon = 3$ otherwise.

Proof. It follows from Theorem 9 that

$$\begin{aligned} (n + \epsilon)^2 &\geq (d_R(D) + d_R(\overline{D}))^2 \\ &= (d_R(D) - d_R(\overline{D}))^2 + 4d_R(D) \cdot d_R(\overline{D}) \\ &\geq 4d_R(D) \cdot d_R(\overline{D}), \end{aligned}$$

implying the desired upper bound. □

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