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Mixed Roman domination and 2-independence in trees

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Abstract: Let $G = (V, E)$ be a simple graph with vertex set V and edge set E. A *mixed Roman dominating function* (MRDF) of G is a function $f : V \cup E \to \{0, 1, 2\}$ satisfying the condition that every element $x \in V \cup E$ for which $f(x) = 0$ is adjacent or incident to at least one element $y \in V \cup E$ for which $f(y) = 2$. The weight of an MRDF f is $\sum_{x\in V\cup E} f(x)$. The mixed Roman domination number $\gamma_R^*(G)$ of G is the minimum weight among all mixed Roman dominating functions of G. A subset S of V is a 2-independent set of G if every vertex of S has at most one neighbor in S. The maximum cardinality of a 2-independent set of G is the 2-independence number $\beta_2(G)$. These two parameters are incomparable in general, however, we show that if T is a tree, then $\frac{4}{3}\beta_2(T) \geq \gamma_R^*(T)$. Moreover, we characterize all trees attaining the equality.

Keywords: mixed Roman dominating function, mixed Roman domination number, 2-independent set, 2-independence number

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1. Terminology and introduction

For terminology and notation on graph theory not given here, the reader is referred to [12]. In this paper, G is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* $|V|$ of G is denoted by $n = n(G)$. For every vertex $v \in V$, the *open neighborhood* of v is the set $N(v) = \{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = |N(v)|$. A *leaf* of G is a vertex of degree 1, a *support vertex* of G is a vertex adjacent to a leaf and a *strong support vertex* is a support vertex adjacent to at least two leaves. For a vertex v in a (rooted) tree T, let $C(v)$ and $D(v)$ denote the set of children and descendants of v, respectively and let $D[v] = D(v) \cup \{v\}$. Also the *depth of* v, depth(v), is the largest distance from v to a vertex in $D(v)$. The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted by T_v. We denote the set of leaves adjacent to a vertex v by L_v. A *pendant path* P of a graph G is an induced path such that one of its end points has degree one in G, and its other end point has degree at least 3. The *diameter* of a graph G, denoted by diam(G), is the greatest distance between two vertices of G. We write P_n for the *path* of order n and $K_{1,n-1}$ for the *star* of order n. A *double star* $DS_{p,q}$ is a tree containing exactly two non-pendant vertices which one is adjacent to p leaves and the other is adjacent to q leaves. The *corona* of two graphs G_1 and G_2, is the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2 where the ith vertex of G_1 is adjacent to every vertex in the ith copy of G_2.

In [5], Fink and Jacobson generalized the concept of independent sets as follows. Let k be a positive integer. A subset X of V is *k-independent* if the maximum degree of the subgraph induced by X is at most $k-1$. The *k-independence number* $\beta_k(G)$ is the maximum cardinality among all k-independent sets of G. A k-independent set of a graph G with maximum cardinality, is called a $\beta_k(G)$-set. For additional information on k-independence see the survey by Chellali, Favaron, Hansberg and Volkmann [2].

A *Roman dominating function* (RDF) of a graph G is a function f from the vertex set $V(G)$ to the set {0,1,2}, such that each vertex $v \in V(G)$ with $f(v) = 0$ is adjacent to at least one vertex u with $f(u) = 2$. The *weight* of an RDF f is the value $\omega(f) = \sum_{v\in V} f(v)$. The *Roman domination number* of a graph G, denoted by $\gamma_R(G)$, is the minimum weight of an RDF of G. A $\gamma_R(G)$-*function* is a Roman dominating function of G with weight $\gamma_R(G)$. The concept of Roman dominating function was defined by Cockayne, Dreyer, Hedetniemi and Hedetniemi [3] and was motivated by Ian Stewart [11]. Roman domination in graphs is now well studied [4, 6–10, 13].

A *mixed Roman dominating function* (MRDF) of G is a function $f : V \cup E \to \{0, 1, 2\}$ satisfying the condition that every element $x \in V \cup E$ for which $f(x) = 0$ is adjacent or incident to at least one element $y \in V \cup E$ for which $f(y) = 2$. The weight of an MRDF f is $\omega(f) = \sum_{x\in V\cup E} f(x)$. The *mixed Roman domination number* of G, denoted by $\gamma^*_R(G)$, is the minimum weight of a mixed Roman dominating function of G. A $\gamma^*_R(G)$-function is an MRDF of G with $\omega(f) = \gamma^*_R(G)$. Mixed Roman domination was introduced by Abdollahzadeh Ahangar, Haynes and Valenzuela-Tripodoro in [1] in 2015.

The next result shows that the two parameters $\gamma^*_R(G)$ and $\beta_2(G)$ are incomparable in general.

Proposition 1. For every positive integer t, there exist two graphs G_t and H_t such that $\beta_2(G_t) - \gamma^*_R(G_t) \geq t$ and $\gamma^*_R(H_t) - \beta_2(H_t) \geq t$.

Proof. Let $G_t = K_{1,t+2}$. Clearly, $\beta_2(G_t) = t + 2$ and $\gamma^*_R(G_t) = 2$ and so $\beta_2(G_t) - \gamma^*_R(G_t) \geq t$. Assume that $H_t = P_{2t+1} \circ K_2$. Since every $\beta_2(H_t)$-set contains at most two vertices from each triangle, we have $\beta_2(H_t) \leq 2(2t + 1) = 4t + 2$. On the other hand, $V(H_t) - V(P_{2t+1})$ is clearly a 2-independent set of H_t yielding $\beta_2(H_t) = 4t + 2$.

It is easy to see that $\gamma_R^*(H_t) \geq 5t+2$ and so $\gamma_R^*(H_t) - \beta_2(H_t) \geq t$. □

Motivated by Proposition 1, in this paper, we focus on trees and prove that for any tree T, $\gamma_R^*(T) \leq \frac{4}{3}\beta_2(T)$. Moreover, we provide a constructive characterization of all trees T with $\gamma_R^*(T) = \frac{4}{3}\beta_2(T)$.

2. Preliminaries

In this section, we give some useful definitions and results.

Definition 1. For a graph G, let

$$W_G^1 = \{v \in V(G) | \text{ every } \beta_2(G)\text{-set contains } v\}.$$

Definition 2. Let $u \in V(G) \cup E(G)$ and $f : V(G) \cup E(G) \to \{0,1,2\}$ be a function. The element u is said to be *mixed Roman dominated* by f if $f(u) \geq 1$, or if $f(u) = 0$, then u is adjacent or incident to one element assigned 2 under f. Let v be a vertex of the graph G and $E(v)$ be the set of all edges incident to v. A function $f : V(G) \cup E(G) \to \{0,1,2\}$ is said to be an *almost mixed Roman dominating function* (almost MRDF) with respect to v, if each element $u \in (V(G) \cup E(G)) - (E(v) \cup \{v\})$ is mixed Roman dominated under f. Let

$$\gamma_R^*(G; v) = \min\{\omega(f) \mid f \text{ is an almost MRDF with respect to } v\}.$$

Clearly, any mixed Roman dominating function on G is an almost MRDF with respect to any vertex of G and hence $\gamma_R^*(G; v)$ is well defined and $\gamma_R^*(G; v) \leq \gamma_R^*(G)$ for each $v \in V(G)$. Define $W_G^2 = \{v \in V(G) | \gamma_R^*(G; v) \geq \gamma_R^*(G) - 1$ and there is no $\gamma_R^*(G)-$function such that $f(v) = 1\}$ and $W_G^3 = \{v \in V(G) | \gamma_R^*(G; v) = \gamma_R^*(G)\}$.

Definition 3. For a graph G, we define

$$W_G^4 = \{v \in V(G) | \text{there is no } \gamma_R^*(G)\text{-function } f \text{ such that } f(v) = 2\}.$$

Definition 4. For a graph G, we define

$$W_G^5 = \{v \in V(G) | \text{there is no } \gamma_R^*(G)\text{-function } f \text{ such that } f(v) = 1\}.$$

Lemma 1. Let T' be a tree and let $u \in V(T')$. If T is a tree obtained from T' by adding a path $P_5 = x_5x_4x_3x_2x_1$ and joining u to x_5, then $\beta_2(T) \geq \beta_2(T') + 3$ and $\gamma_R^*(T) = \gamma_R^*(T') + 4$.

Proof. Clearly, any $\beta_2(T')$-set can be extended to a 2-independent set of T by adding x_1, x_2, x_4 and so $\beta_2(T) \geq \beta_2(T') + 3$. Also any $\gamma_R^*(T')$-function can be extended to a an MRDF of T by assigning a 2 to x_2, x_4x_5 and a 0 to the remaining elements, and this implies that $\gamma_R^*(T) \leq \gamma_R^*(T') + 4$. Now let f be a $\gamma_R^*(T)$-function and let $\ell = f(ux_5) + \sum_{i=1}^{5} f(x_i) + \sum_{i=1}^{4} f(x_ix_{i+1})$. It is easy to see that $\ell \geq 4$. If $\ell \geq 6$

or $f(u) \geq 1$ and $\ell = 5$, then the function $g : V(T') \cup E(T') \to \{0, 1, 2\}$ defined by $g(u) = 2$ and $g(x) = f(x)$ for $x \in V(T') \cup E(T') - \{u\}$, is an MRDF of T' and so $\gamma^*_R(T') \leq \gamma^*_R(T) - 4$. If $f(u) = 0$ and $\ell = 5$, then $f(ux_5) \neq 2$ and the function $g : V(T') \cup E(T') \to \{0, 1, 2\}$ defined by $g(u) = 1$ and $g(x) = f(x)$ for $x \in V(T') \cup E(T') - \{u\}$, is an MRDF of T' and so $\gamma^*_R(T') \leq \gamma^*_R(T) - 4$. If $f(u) \geq 1$ and $\ell = 4$, then $f(ux_5) \neq 2$ and the function f restricted to T' is an MRDF of T' and so $\gamma^*_R(T') \leq \gamma^*_R(T) - 4$. Henceforth, we assume that $f(u) = 0$ and $\ell = 4$. This implies that $f(ux_5) = f(x_5) = 0$. Hence, the function f restricted to T' is an MRDF of T' yielding $\gamma^*_R(T') \leq \gamma^*_R(T) - 4$. Thus $\gamma^*_R(T) \geq \gamma^*_R(T') + 4$ that leads to the equality, as desired. □

Lemma 2. Let T' be a tree and let $u \in V(T')$. If T is a tree obtained from T' by adding a path $P_4 = x_4x_3x_2x_1$ and joining u to x_3, then $\beta_2(T) = \beta_2(T') + 3$ and $\gamma^*_R(T) \leq \gamma^*_R(T') + 4$.

Proof. Clearly, any $\beta_2(T')$-set can be extended to a 2-independent set of T by adding x_1, x_2, x_4 and so $\beta_2(T) \geq \beta_2(T') + 3$. On the other hand, for any $\beta_2(T)$-set S, we have $|S \cap \{x_1, x_2, x_3, x_4\}| \leq 3$ and since $S \cap V(T')$ is a 2-independent set of T', we deduce that $\beta_2(T') \geq \beta_2(T) - 3$. Thus $\beta_2(T) = \beta_2(T') + 3$.
Let f be a $\gamma^*_R(T')$-function. Then the function $g : V(T) \cup E(T) \to \{0, 1, 2\}$ defined by $g(x_2) = g(x_3) = 2$ and $g(x_1) = g(x_4) = g(x_1x_2) = g(x_2x_3) = g(x_3x_4) = g(ux_3) = 0$ and $g(u') = f(u')$ for $u' \in V(T') \cup E(T')$, is an MRDF of T of weight $\gamma^*_R(T') + 4$ and so $\gamma^*_R(T) \leq \gamma^*_R(T') + 4$. □

3. Main Result

In this section we prove that for any tree T, $\gamma^*_R(T) \leq \frac{4}{3}\beta_2(T)$ and we provide a constructive characterization of all trees T with $\gamma^*_R(T) = \frac{4}{3}\beta_2(T)$. In order to do this, let $\mathcal{T}$ be the family of unlabeled trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_m$ ($m \geq 1$) of trees such that T_1 is a path P_4, and if $m \geq 2$, T_{i+1} can be obtained recursively from T_i by one of the following operations for $1 \leq i \leq m - 1$.

Operation $\mathcal{O}_1$. If $u \in W^1_{T_i}$, then Operation $\mathcal{O}_1$ adds a path $P_5 = x_5x_4x_3x_2x_1$ and joins u to x_5 to obtain T_{i+1}.

Operation $\mathcal{O}_2$. If $u \in W^2_{T_i}$, then Operation $\mathcal{O}_2$ adds a path $P_4 = x_4x_3x_2x_1$ and joins u to x_3 to obtain T_{i+1}.

Operation $\mathcal{O}_3$. If $u \in W^3_{T_i} \cap W^4_{T_i}$ and there is a pendant path uz_2z_1 in T_i, then Operation $\mathcal{O}_3$ adds a path $P_4 = x_4x_3x_2x_1$ and joins u to x_4 to obtain T_{i+1}.

Operation $\mathcal{O}_4$. If $u \in W^5_{T_i}$, then Operation $\mathcal{O}_4$ adds a path $P_9 = x_9x_8x_7x_6x_5x_4x_3x_2x_1$ and joins u to x_5 to obtain T_{i+1}.

Lemma 3. If T_i is a tree with $\gamma^*_R(T_i) = \frac{4}{3}\beta_2(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_1$, then $\gamma^*_R(T_{i+1}) = \frac{4}{3}\beta_2(T_{i+1})$.

Proof. Let $u \in W^1_{T_i}$, and let Operation $\mathcal{O}_1$ add a path $P_5 = x_5x_4x_3x_2x_1$ and the edge ux_5 to obtain T_{i+1}. By Lemma 1, $\beta_2(T_{i+1}) \geq \beta_2(T_i) + 3$ and $\gamma^*_R(T_{i+1}) = \gamma^*_R(T_i) + 4$. We show that $\beta_2(T_{i+1}) \leq \beta_2(T_i) + 3$. Suppose, to the contrary, that $\beta_2(T_i) \leq \beta_2(T_{i+1}) - 4$ and let S be a $\beta_2(T_{i+1})$-set. Since $S \cap V(T')$ is a 2-independent set of T', we deduce from $\beta_2(T_i) \leq \beta_2(T_{i+1}) - 4$ that $\{x_1, x_2, x_4, x_5\} \subseteq S$. Hence, $S' = S - \{x_1, x_2, x_4, x_5\}$ is a 2-independent set of T' not containing u. Since $u \in W^1_{T_i}$, we have $\beta_2(T_i) \geq |S'| + 1 = \beta_2(T_{i+1}) - 3$ which is a contradiction. Thus $\beta_2(T_{i+1}) \leq \beta_2(T_i) + 3$ yielding $\beta_2(T_{i+1}) = \beta_2(T_i) + 3$. We now deduce from $\gamma^*_R(T_i) = \frac{4}{3}\beta_2(T_i)$ that

$$\frac{4}{3}\beta_2(T_{i+1}) = \frac{4}{3}(\beta_2(T_i) + 3) = \frac{4}{3}\beta_2(T_i) + 4 = \gamma^*_R(T_i) + 4 = \gamma^*_R(T_{i+1}).$$

□

Lemma 4. If T_i is a tree with $\gamma^*_R(T_i) = \frac{4}{3}\beta_2(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_2$, then $\gamma^*_R(T_{i+1}) = \frac{4}{3}\beta_2(T_{i+1})$.

Proof. Let Operation $\mathcal{O}_2$ add a path $P_4 = x_4x_3x_2x_1$ and the edge ux_3 to obtain T_{i+1}. By Lemma 2, we have $\beta_2(T_{i+1}) = \beta_2(T_i)+3$ and $\gamma^*_R(T_{i+1}) \leq \gamma^*_R(T_i)+4$. We now prove that $\gamma^*_R(T_{i+1}) \geq \gamma^*_R(T_i)+4$. Let f be a $\gamma^*_R(T_{i+1})$-function and let $\ell = f(ux_3) + \sum_{i=1}^4 f(x_i)+\sum_{i=1}^3 f(x_ix_{i+1})$. If $\ell \geq 6$, then the function $g_1 : V(T_i)\cup E(T_i) \to \{0, 1, 2\}$ defined by $g_1(u) = 2$ and $g_1(x) = f(x)$ for $x \in V(T_i) \cup E(T_i) - \{u\}$, is an MRDF of T_i of weight at most $\omega(f) - 4$ implying that $\gamma^*_R(T_{i+1}) \geq \gamma^*_R(T_i) + 4$. If $\ell = 5$, then the function f, restricted to T_i is an almost MRDF of T_i of weight $\omega(f)-5$. Since $u \in W^2_{T_i}$, we have $\omega(f|_{T_i}) \geq \gamma^*_R(v; T_i) \geq \gamma^*_R(T_i) - 1$ and so $\gamma^*_R(T_{i+1}) \geq \gamma^*_R(T_i) + 4$. Now let $\ell = 4$. Then we must have $f(ux_3) = 0$. Then the function $g : V(T_i)\cup E(T_i) \to \{0, 1, 2\}$ defined by $g(u) = \max\{1, f(u)\}$ and $g(x) = f(x)$ for $x \in V(T_i) \cup E(T_i) - \{u\}$, is an MRDF of T_i of weight at most $\omega(f) - 3$. Since $u \in W^2_{T_i}$, we deduce that $\omega(g) \geq \gamma^*_R(T_i) + 1$ yielding $\gamma^*_R(T_i) \geq \gamma^*_R(T_i) + 4$. Thus $\gamma^*_R(T_{i+1}) = \gamma^*_R(T_{i+1}) + 4$. Now, the assumption $\gamma^*_R(T_i) = \frac{4}{3}\beta_2(T_i)$ implies that $\frac{4}{3}\beta_2(T_{i+1}) = \frac{4}{3}(\beta_2(T_i) + 3) = \frac{4}{3}\beta_2(T_i) + 4 = \gamma^*_R(T_i) + 4 = \gamma^*_R(T_{i+1})$. □

Lemma 5. If T_i is a tree with $\gamma^*_R(T_i) = \frac{4}{3}\beta_2(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_3$, then $\gamma^*_R(T_{i+1}) = \frac{4}{3}\beta_2(T_{i+1})$.

Proof. Let Operation $\mathcal{O}_3$ add a path $P_4 = x_4x_3x_2x_1$ and the edge ux_4 to obtain T_{i+1}. First we show that $\beta_2(T_{i+1}) = \beta_2(T_i) + 3$. Let S be a $\beta_2(T_i)$-set and let $S' = S\cup\{x_4, x_2, x_1\}$ if $u \notin S$ and $S' = (S-\{u\})\cup\{x_4, x_2, x_1, z_1, z_2\}$ if $u \in S$. Clearly, S' is a 2-independent set of T_{i+1} and so $\beta_2(T_{i+1}) \geq \beta_2(T_i) + 3$. On the other hand, if S is a $\beta_2(T_{i+1})$-set, then clearly $|S \cap \{x_1, x_2, x_3, x_4\}| \leq 3$ and $S - \{x_1, x_2, x_3, x_4\}$ is a 2-independent set of T_i. This implies that $\beta_2(T_i) \geq \beta_2(T_{i+1}) - 3$ and so $\beta_2(T_{i+1}) = \beta_2(T_i) + 3$.

Next we show that $\gamma_R^*(T_{i+1}) = \gamma_R^*(T_i) + 4$. Clearly, any $\gamma_R^*(T_i)$-function can be extended to an MRDF by assigning a 2 to x_3x_4 and x_2, and a 0 to the remaining elements yielding $\gamma_R^*(T_{i+1}) \leq \gamma_R^*(T_i) + 4$. Suppose now that f is a $\gamma_R^*(T_{i+1})$-function and let $\ell = f(ux_4)+\sum_{i=1}^4 f(x_i)+\sum_{i=1}^3 f(x_ix_{i+1})$. Clearly $\ell \geq 3$. If $f(u) = 2$, then the function f, restricted to T_i is an MRDF of T_i of weight at most $\omega(f)-3$, and we deduce from $u \in W_{T_i}^4$ that $\omega(f|_{T_i}) \geq \gamma_R^*(T_i) + 1$. This implies that $\gamma_R^*(T_{i+1}) \geq \gamma_R^*(T_i) + 4$. Let $f(u) \leq 1$. Since there is a pendant path uz_2z_1, we must have $f(z_2) = 2$ and $f(u) = 0$ and this implies that $\ell \geq 4$.
Then the function f, restricted to T_i is an almost MRDF of T_i with respect to u and we deduce from the assumption $u \in W_{T_i}^3$ that $\gamma_R^*(T_{i+1}) \geq 4 + \omega(f|_{T_i}) \geq \gamma_R^*(T_i) + 4$. Hence $\gamma_R^*(T_{i+1}) \geq \gamma_R^*(T_i) + 4$ yielding $\gamma_R^*(T_{i+1}) = \gamma_R^*(T_i) + 4$. By the assumption $\gamma_R^*(T_i) = \frac{4}{3}\beta_2(T_i)$, we obtain $\frac{4}{3}\beta_2(T_{i+1}) = \frac{4}{3}(\beta_2(T_i)+3) = \frac{4}{3}\beta_2(T_i)+4 = \gamma_R^*(T_i)+4 = \gamma_R^*(T_{i+1})$. □

Lemma 6. If T_i is a tree with $\gamma_R^*(T_i) = \frac{4}{3}\beta_2(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_4$, then $\gamma_R^*(T_{i+1}) = \frac{4}{3}\beta_2(T_{i+1})$.

Proof. Let Operation $\mathcal{O}_4$ add a path $P_9 = x_9x_8x_7x_6x_5x_4x_3x_2x_1$ and the edge ux_5 to obtain T_{i+1}. First we show that $\beta_2(T_{i+1}) = \beta_2(T_i) + 6$. Clearly, any $\beta_2(T_i)$-set can be extended to a 2-independent set by adding $x_1, x_2, x_4, x_6, x_8, x_9$ and so $\beta_2(T_{i+1}) \geq \beta_2(T_i) + 6$. On the other hand, if S is a $\beta_2(T_{i+1})$-set, then clearly $|S \cap V(P_9)| \leq 6$ and $S - V(P_9)$ is a 2-independent set of T_i yielding $\beta_2(T_i) \geq \beta_2(T_{i+1}) - 6$. Hence $\beta_2(T_{i+1}) = \beta_2(T_i) + 6$.
Next we show that $\gamma_R^*(T_{i+1}) = \gamma_R^*(T_i) + 8$. Clearly, any $\gamma_R^*(T_i)$-function can be extended to an MRDF by assigning a 2 to x_2, x_5, x_8, a 1 to x_3x_4, x_6x_7 and a 0 to the remaining elements and so $\gamma_R^*(T_{i+1}) \leq \gamma_R^*(T_i) + 8$. Suppose now that f is a $\gamma_R^*(T_{i+1})$-function and let $\ell = \sum_{i=1}^9 f(x_i) + \sum_{i=1}^8 f(x_ix_{i+1})$. Clearly $\ell \geq 8$. If $\ell \geq 10$, then the function $h : V(T_i) \cup E(T_i) \to \{0, 1, 2\}$ defined by $h(u) = 2$ and $h(x) = f(x)$ for $x \in V(T_i) \cup E(T_i) - \{u\}$ is an MRDF of T' of weight at most $\omega(f) - 8$ and we have $\gamma_R^*(T_{i+1}) \geq \gamma_R^*(T_i) + 8$. If $8 \leq \ell \leq 9$, then clearly $f(ux_5) \leq 1$ and the function $h : V(T_i) \cup E(T_i) \to \{0, 1, 2\}$ defined by $h(u) = 1$ and $h(x) = f(x)$ for $x \in V(T_i) \cup E(T_i) - \{u\}$ is an MRDF of T' of weight at most $\omega(f) - 7$ and we conclude from $u \in W_{T_i}^5$ that $\omega(h) \geq \gamma_R^*(T_i)+1$. It follows that $\gamma_R^*(T_{i+1}) \geq \gamma_R^*(T_i)+8$. Thus $\gamma_R^*(T_{i+1}) \geq \gamma_R^*(T_i) + 8$. It follows from $\gamma_R^*(T_i) = \frac{4}{3}\beta_2(T_i)$ that

$$\frac{4}{3}\beta_2(T_{i+1}) = \frac{4}{3}(\beta_2(T_i) + 6) = \frac{4}{3}\beta_2(T_i) + 8 = \gamma_R^*(T_i) + 8 = \gamma_R^*(T_{i+1}).$$

□

Theorem 1. *If $T \in \mathcal{T}$, then $\gamma_R^*(T) = \frac{4}{3}\beta_2(T)$.*

Proof. Let $T \in \mathcal{T}$. Then there exists a sequence of trees $T_1, T_2, \ldots, T_k$ $(k \geq 1)$ such that T_1 is P_4, and if $k \geq 2$, then T_{i+1} can be obtained recursively from T_i by one of the aforementioned operations for $i = 1, 2, \ldots, k-1$.
We proceed by induction on the number of operations applied to construct T. If $k = 1$, then $T = P_4 \in \mathcal{T}$. Suppose that the result is true for each tree $T \in \mathcal{T}$ which can be obtained from a sequence of operations of length $k-1$ and let $T' = T_{k-1}$. By the induction hypothesis, we have $\gamma^*_R(T') = \frac{4}{3}\beta_2(T')$. Since $T = T_k$ is obtained from T' by one of the operations $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4$, we conclude from Lemmas 3, 4, 5, 6 that $\gamma^*_R(T) = \frac{4}{3}\beta_2(T)$. □

Now we are ready to prove the main result of this section.

Theorem 2. Let T be a tree. Then $\gamma^*_R(T) \leq \frac{4}{3}\beta_2(T)$ with equality if and only if $T \in \mathcal{T}$.

Proof. If $T \in \mathcal{T}$, then by Theorem 1 we have $\gamma^*_R(T) = \frac{4}{3}\beta_2(T)$. Hence, we only need to prove that for any tree T, $\gamma^*_R(T) \leq \frac{4}{3}\beta_2(T)$ with equality only if $T \in \mathcal{T}$. Let T be a tree. The proof is by induction on $n(T)$. If $n(T) \leq 3$, then clearly $\gamma^*_R(T) < \frac{4}{3}\beta_2(T)$. Let $n(T) \geq 4$ and the statements be true for any tree of order less than $n(T)$. If $\mathrm{diam}(T) = 2$, then T is a star and we have $\gamma^*_R(T) = 2 < \frac{4(n(T)-1)}{3} = \frac{4\beta_2(T)}{3}$. If $\mathrm{diam}(T) = 3$, then T is a double star $DS_{p,q}$ $(q \geq p \geq 1)$ and we have $\gamma^*_R(DS_{p,q}) = 4$, $\beta_2(DS_{p,q}) = p + q$ if $p + q \geq 3$ and $\beta_2(DS_{1,1}) = 3$. Hence $\gamma^*_R(DS_{p,q}) \leq \frac{4}{3}\beta_2(DS_{p,q})$ and the equality holds if $p = q = 1$, that is $T = P_4$. Assume that $\mathrm{diam}(T) \geq 4$ and let $v_1v_2 \ldots v_{d+1}$ be a diametrical path in T such that $\deg(v_2)$ is as large as possible. Among these diametrical paths, choose one such that $\deg(v_2)$ is as large as possible and root T at v_{d+1}.
Assume first that $k = \deg(v_2) \geq 3$. Let $T' = T - T_{v_2}$. Clearly, every $\beta_2(T')$-set can be extended to a 2-independent set of T by adding all leaves adjacent to v_2 and this implies that $\beta_2(T) \geq \beta_2(T') + k - 1$. On the other hand, any $\gamma^*_R(T')$-function can be extended to an MRDF of T by assigning a to v_2 and a 0 to the leaves adjacent to v_2 and this implies that $\gamma^*_R(T) \leq \gamma^*_R(T') + 2$. It follows from the induction hypothesis that

$$\frac{4}{3}\beta_2(T) \geq \frac{4}{3}\beta_2(T') + \frac{4k}{3} - \frac{4}{3} \geq \gamma^*_R(T') + \frac{4k}{3} - \frac{4}{3} \geq \gamma^*_R(T) + \frac{4k}{3} - \frac{10}{3} > \gamma^*_R(T).$$

Now let $\deg(v_2) = 2$. By the choice of Diametrical path, we may assume that any child of v_3 with depth 1 is of degree 2. Consider the following cases.

Case 1. $\deg(v_3) \geq 3$.
Let v_3 have s children with depth one and r children with depth 0. Let $T' = T - T_{v_3}$. Clearly, any $\beta_2(T')$-set can be extended to a 2-independent set of T by adding all vertices of T_{v_3} but v_3 and this implies that $\beta_2(T) \geq 2s + r + \beta_2(T')$. Let g be a $\gamma^*_R(T')$-function. If $r \geq 1$, then define $h : V(T) \cup E(T) \to \{0, 1, 2\}$ by $h(x) = 2$ for $x \in N(v_3) - (L_{v_3} \cup \{v_4\})$, $h(v_3) = 2$, $h(x) = g(x)$ for $x \in V(T') \cup E(T')$ and $h(x) = 0$ otherwise. If $r = 0$, then define $h : V(T) \cup E(T) \to \{0, 1, 2\}$ by $h(x) = 2$ for

$x \in N(v_3) - (L_{v_3} \cup \{v_4\})$, $h(v_3) = 0$, $h(v_4v_3) = 1$, $h(x) = g(x)$ for $x \in V(T') \cup E(T')$ and $h(x) = 0$ otherwise. In both cases, h is an MRDF of T yielding $\gamma_R^*(T) \leq \gamma_R^*(T') + 2s + 2$ if $r \geq 1$ and $\gamma_R^*(T) \leq \gamma_R^*(T') + 2s + 1$ if $r = 0$. If $r = 0$, then $s \geq 2$ and we deduce from the induction hypothesis that $\frac{4}{3}\beta_2(T) \geq \frac{8s}{3} + \frac{4}{3}\beta_2(T') \geq \frac{8s}{3} + \gamma_R^*(T') \geq \frac{8s}{3} + \gamma_R^*(T) - 2s - 1 > \gamma_R^*(T)$.
If $r \geq 1$, then it follows from $s \geq 1$ and the induction hypothesis that

$$\begin{aligned}
\tfrac{4}{3}\beta_2(T) &\geq \tfrac{8s}{3} + \tfrac{4r}{3} + \tfrac{4}{3}\beta_2(T') \\
&\geq \tfrac{8s}{3} + \tfrac{4r}{3} + \gamma_R^*(T') \\
&\geq \tfrac{8s}{3} + \tfrac{4r}{3} + \gamma_R^*(T) - 2s - 2 \\
&= \gamma_R^*(T) + \tfrac{2s}{3} + \tfrac{4r}{3} - 2 \\
&\geq \gamma_R^*(T).
\end{aligned}$$

Moreover, if $\gamma_R^*(T) = \frac{4}{3}\beta_2(T)$, then we have equalities throughout the above inequality chain. In particular, $r = s = 1$, $\gamma_R^*(T') = \frac{4}{3}\beta_2(T')$ and $\gamma_R^*(T) = \gamma_R^*(T') + 4$. Hence, by the inductive hypothesis we have $T' \in \mathcal{T}$. Now we show that $v_4 \in W^2_{T'}$. If $\gamma_R^*(T', v_4) \leq \gamma_R^*(T') - 2$, then let g be an almost MRDF of T' with respect to v_4 of weight at most $\gamma_R^*(T') - 2$ and define $h : V(T) \cup E(T) \to \{0, 1, 2\}$ by $h(u') = g(u')$ for $u \in V(T') \cup E(T')$, $h(v_4v_3) = h(v_2) = 2, h(w) = 1$ where w is the leaf adjacent to v_3, and $h(x) = 0$ otherwise. Clearly, h is an MRDF of T that leads to the contradiction $\gamma_R^*(T) < \gamma_R^*(T') + 3$. If there is a $\gamma_R^*(T')$-function with $f(v_4) = 1$, then define $h : V(T) \cup E(T) \to \{0, 1, 2\}$ by $h(u') = g(u')$ for $u \in V(T') \cup E(T') - \{v_4\}$, $h(v_3) = h(v_2) = 2$ and $h(x) = 0$ otherwise. Clearly, h is an MRDF of T that leads to a contradiction again. Thus $v_4 \in W^2_{T'}$. Now T can be obtained from T' by Operation $\mathcal{O}_2$ and so $T \in \mathcal{T}$.

Case 2. $\deg(v_3) = 2$ and $\deg(v_4) \geq 3$.
Considering Case 1 and the choice of diametrical path, we may assume that any child of v_4 with depth 2, is of degree 2. We consider the following subcases.

Subcase 2.1 v_4 is adjacent to a leaf w.
Let $T' = T - T_{v_3}$. Any $\beta_2(T')$-set can be extended to a 2-independent set of T by adding v_1, v_2 and so $\beta_2(T) \geq \beta_2(T') + 2$. On the other hand, if f is a $\gamma_R^*(T')$-function such that $f(w)$ is as small as possible, then to Roman dominate w and v_4w, we must have $f(v_4) = 2$ or $f(v_4u) = 2$ for some $u \in N(v_4) - \{v_3, w\}$, and the function $h : V(T) \cup E(T) \to \{0, 1, 2\}$ defined by $h(v_2) = 2$, $h(v_3v_2) = h(v_2v_1) = h(v_3v_4) = h(v_1) = h(v_3) = 0$ and $h(u) = f(u)$ for $u \in V(T') \cup E(T')$, is an MRDF of T. Hence $\gamma_R^*(T) \leq \gamma_R^*(T') + 2$ and we deduce from the induction hypothesis that $\frac{4}{3}\beta_2(T) \geq \frac{4}{3}(\beta_2(T') + 2) \geq \gamma_R^*(T') + \frac{8}{3} \geq \gamma_R^*(T) + \frac{2}{3} > \gamma_R^*(T)$.

Subcase 2.2 v_4 is not a support vertex and v_4 has r children with depth 2 and s children with depth 1.
Let $z_1, \ldots, z_s$ be the children of v_4 with depth 1, if any, and let z_i' be a leaf adjacent to z_i for $1 \leq i \leq s$. Also assume that $y_1, \ldots, y_r$ are the children of v_4 with depth 2 where $y_1 = v_3$ and let $v_4y_jy_j'y_j''$ be a pendant path in T for each j. Note that $\deg(y_j) = \deg(y_j') = 2$ for each j. If $\deg(z_i) \geq 3$ for some i, then as in the second

paragraph of proof, we can see that $\frac{4}{3}\beta_2(T) > \gamma_R^*(T)$. Henceforth, we assume that $\deg(z_i) = 2$ for each i, if any. Consider the following.

- $s \geq 1$.
  Let $T' = T - T_{v_4}$. Clearly, any $\beta_2(T')$-set can be extended to a 2-independent set of T by adding z_i, z_i' for $1 \leq i \leq s$ and y_j', y_j'' for $1 \leq j \leq r$ and this implies that $\beta_2(T) \geq \beta_2(T') + 2r + 2s$. On the other hand, if f is a $\gamma_R^*(T')$-function, then the function $h : V(T) \cup E(T) \to \{0, 1, 2\}$ defined by $h(v_4z_1) = h(z_i) = 2$ for $2 \leq i \leq s$, $h(z_1') = 1$, $h(v_4z_i) = h(z_i') = 0$ for $2 \leq i \leq s$, $h(z_iz_i') = 0$ for $1 \leq i \leq s$, $h(y_j') = 2$ for $1 \leq j \leq r$, $h(u) = f(u)$ for $u \in V(T') \cup E(T')$ and $h(x) = 0$ otherwise, is an MRDF of T implying that $\gamma_R^*(T) \leq \gamma_R^*(T') + 2r + 2s + 1$. We conclude from the induction hypothesis that

$$\begin{array}{rcl} \frac{4}{3}\beta_2(T) & \geq & \frac{4}{3}(\beta_2(T') + 2r + 2s) \\ & = & \frac{4}{3}\beta_2(T') + \frac{8r}{3} + \frac{8s}{3} \\ & \geq & \gamma_R^*(T') + \frac{8r}{3} + \frac{8s}{3} \\ & \geq & \gamma_R^*(T) - 2r - 2s - 1 + \frac{8r}{3} + \frac{8s}{3} \\ & = & \gamma_R^*(T) + \frac{2(r+s)}{3} - 1 \\ & > & \gamma_R^*(T) \end{array}.$$

- $r \geq 3$.
  Let $T' = T - T_{v_3}$. Clearly, every $\beta_2(T')$-set can be extended to a 2-independent set of T by adding v_1, v_2 and so $\beta_2(T) \geq \beta_2(T') + 2$. Let f be a $\gamma_R^*(T')$-function. To Roman dominate the edges joining v_4 to its children with depth 2, we may assume that some edge incident to v_4 assigned 2 by f. Now the function $h : V(T) \cup E(T) \to \{0, 1, 2\}$ defined by $h(v_4v_3) = h(v_3) = h(v_3v_2) = h(v_2v_1) = h(v_1) = 0, h(v_2) = 2$ and $h(u) = f(u)$ for $u \in V(T') \cup E(T')$, is an MRDF of T and so $\gamma_R^*(T) \leq \gamma_R^*(T') + 2$. It follows from the induction hypothesis that

$$\frac{4}{3}\beta_2(T) \geq \frac{4}{3}(\beta_2(T') + 2) \geq \gamma_R^*(T') + \frac{8}{3} \geq \gamma_R^*(T) + \frac{2}{3} > \gamma_R^*(T).$$

- $s = 0$ and $r = 2$.
  We consider the followings.

  (a) v_5 is a strong support vertex.
  Let w_1, w_2 be two leaves adjacent to v_5 and let $T' = T - T_{v_4}$. Assume S' is a $\beta_2(T')$-set. If $v_5 \in S'$, then we may assume that $w_1 \notin S'$. Let $S = S' \cup \{v_1, v_2, v_4, y_2', y_2''\}$ if $v_5 \notin S'$ and $S = (S' - \{v_5\}) \cup \{w_1, v_1, v_2, v_4, y_2', y_2''\}$ if $v_5 \in S'$. Clearly, S is a 2-independent set of T and so $\beta_2(T) \geq \beta_2(T') + 5$. On the other hand, any $\gamma_R^*(T')$-function f, can be extended to an MRDF of T by assigning a 2 to v_4, y_2', v_2 and a 0 to the remaining elements. This implies that $\gamma_R^*(T) \leq \gamma_R^*(T') + 6$. We deduce from the induction hypothesis that

$$\frac{4}{3}\beta_2(T) \geq \frac{4}{3}(\beta_2(T') + 5) \geq \gamma_R^*(T') + \frac{20}{3} \geq \gamma_R^*(T) - 6 + \frac{20}{3} > \gamma_R^*(T).$$

(b) v_5 has a child with depth 1.
Let w be a child of v_5 with depth 1. If $\deg(w) \geq 3$, then as in the second paragraph of proof, we can see that $\frac{4}{3}\beta_2(T) > \gamma_R^*(T)$. Assume $\deg(w) = 2$ and let $T' = T - T_{v_4}$. As in (a), we can see that $\frac{4}{3}\beta_2(T) > \gamma_R^*(T)$.

(c) v_5 has ℓ children with depth 2, r children with depth 3 and degree 2, and v_5 is adjacent to the central vertex of $t \geq 1$ copies of P_7.
Let $x_1^1, x_1^2, \ldots, x_1^\ell$ ($\ell \geq 0$) be the children of v_5 with depth 2 and $v_5x_1^ix_2^ix_3^i$ be a path in T for $1 \leq i \leq \ell$. As above we may assume that $\deg(x_2^i) = 2$ for each i and by Case 1, we may assume that $\deg(x_1^i) = 2$ for each i. Assume $y_1^1, y_1^2, \ldots, y_1^r$ ($r \geq 0$) be the children of v_5 with depth 3 and degree 2, and $v_5y_1^jy_2^jy_3^jy_4^j$ be a path in T for $1 \leq j \leq r$. As above we may assume that $\deg(x_2^i) = \deg(y_3^j) = 2$ for each i, j and by Case 2, we may assume that $\deg(x_1^i) = \deg(y_2^j) = 2$ for each i, j. Suppose $v_4 = z_4^1, z_4^2, \ldots, z_4^t$ ($t \geq 1$) are the children of v_5 where z_4^i is the central vertex of an induced path $P_7 = z_1^kz_2^k \ldots z_7^k$ for $1 \leq k \leq t$. Let $T' = T - T_{v_5}$. Clearly any $\beta_2(T')$-set can be extended to a 2-independent set of T by adding x_2^i, x_3^i for $1 \leq i \leq \ell$ and y_1^j, y_3^j, y_4^j for $1 \leq j \leq r$ and $z_1^k, z_2^k, z_4^k, z_6^k, z_7^k$ for $1 \leq k \leq t$ and this implies that $\beta_2(T) \geq \beta_2(T') + 2\ell + 3r + 5t$. Now we show that $\gamma_R^*(T) \leq \gamma_R^*(T') + 6t + 4r + 2\ell$.
Let f be a $\gamma_R^*(T')$-function. Define the function $h : V(T) \cup E(T) \to \{0, 1, 2\}$ by $h(x_2^i) = 2$ for $1 \leq i \leq \ell$ and $h(v_5y_1^j) = h(y_3^j) = 2$ for $1 \leq j \leq r$ and $h(z_2^k) = h(z_6^k) = h(v_5z_4^1) = 2$ for $1 \leq k \leq t$, $h(w) = f(w)$ for $w \in V(T') \cup E(T')$ and $h(w) = 0$ otherwise. Clearly, h is an MRDF of T and so $\gamma_R^*(T) \leq \gamma_R^*(T') + 6t + 4r + 2\ell$. It follows from the induction hypothesis that

$$\begin{aligned}
\tfrac{4}{3}\beta_2(T) &\geq \tfrac{4}{3}(\beta_2(T') + 2\ell + 3r + 5t) \\
&\geq \gamma_R^*(T') + \tfrac{8\ell}{3} + 4r + \tfrac{20t}{3} \\
&\geq \gamma_R^*(T) - 6t - 4r - 2\ell + \tfrac{8\ell}{3} + 4r + \tfrac{20t}{3}\ . \\
&= \gamma_R^*(T) + \tfrac{2(t+\ell)}{3} \\
&> \gamma_R^*(T).
\end{aligned}$$

(d) v_5 is a support vertex and v_5 has the children described in (c).
Let w be the leaf adjacent to v_5 and let $T = T - T_{v_5}$. Clearly, any $\beta_2(T')$-set can be extended to a 2-independent set of T by adding w and x_2^i, x_3^i for $1 \leq i \leq \ell$ and y_1^j, y_3^j, y_4^j for $1 \leq j \leq r$ and $z_1^k, z_2^k, z_4^k, z_6^k, z_7^k$ for $1 \leq k \leq t$ and this implies that $\beta_2(T) \geq \beta_2(T') + 2\ell + 3k + 5t + 1$. On the other hand, the function h defined in (c) can be extended to an MRDF of T by assigning 1 to w and 0 to v_5w and so $\gamma_R^*(T) \leq \gamma_R^*(T') + 6t + 4r + 2\ell + 1$.

As above, we obtain

$$\begin{array}{rcl}\frac{4}{3}\beta_2(T) & \geq & \frac{4}{3}(\beta_2(T')+2\ell+3r+5t+1)\\ & \geq & \gamma_R^*(T')+\frac{8\ell}{3}+4r+\frac{20t}{3}+\frac{4}{3}\\ & \geq & \gamma_R^*(T)-6t-4r-2\ell-1+\frac{8\ell}{3}+4r+\frac{20t}{3}+\frac{4}{3}\\ & = & \gamma_R^*(T)+\frac{2(t+\ell)}{3}+\frac{1}{3}\\ & > & \gamma_R^*(T).\end{array}$$

Case 3. $\deg(v_3)=\deg(v_4)=2$ and $\deg(v_5)\geq 3$.
We consider the following subcases.

Subcase 3.1 v_5 is a strong support vertex.
Let w_1, w_2 be two leaves adjacent to v_5 and let $T'=T-T_{v_4}$. As in Case 2 (a), we can see that $\beta_2(T)\geq\beta_2(T')+3$. Assume that f is a $\gamma_R^*(T')$-function. To Roman dominate w_1, w_2, we may assume that $f(v_5)=2$. Then the function $h: V(T)\cup E(T)\to\{0,1,2\}$ defined by $h(v_2)=2, h(v_3v_4)=1$, $h(v_1)=h(v_3)=h(v_4)=h(v_5v_4)=h(v_3v_2)=h(v_2v_1)=0$ and $h(u)=f(u)$ for $u\in V(T')\cup E(T')$, is an MRDF of T and so $\gamma_R^*(T)\leq\gamma_R^*(T')+3$. We conclude from the induction hypothesis that

$$\frac{4}{3}\beta_2(T)>\frac{4}{3}(\beta_2(T)-3)+3\geq\frac{4}{3}\beta_2(T')+3\geq\gamma_R^*(T')+3\geq\gamma_R^*(T).$$

Subcase 3.2 v_5 has a children z_2 with depth 1.
Applying the argument used in the second paragraph of proof, we may assume that $\deg(z_2)=2$. Let z_1 be the leaf adjacent to z_2 and let $T'=T-T_{v_4}$. Assume S' is a $\beta_2(T')$-set. If $v_5\in S'$, then $|S'\cap\{z_1,z_2\}|=1$. Let $S=S'\cup\{v_1,v_2,v_4\}$ if $v_5\notin S'$ and $S=(S'-\{v_5\})\cup\{z_1,z_2,v_1,v_2,v_4\}$ if $v_5\in S'$. Obviously, S is a 2-independent set of T and so $\beta_2(T)\geq\beta_2(T')+3$. On the other hand, for any $\gamma_R^*(T')$-function f, the function $h: V(T)\cup E(T)\to\{0,1,2\}$ defined by $h(v_4)=h(v_2)=2$, $h(v_1)=h(v_3)=0$, $h(v_iv_{i+1})=0$ for $i=1,2,3,4$ and $h(u)=f(u)$ for $u\in V(T')\cup E(T')$, is an MRDF of T and so $\gamma_R^*(T)\leq\gamma_R^*(T')+4$. By the induction hypothesis we have

$$\frac{4}{3}\beta_2(T)\geq\frac{4}{3}(\beta_2(T')+3)\geq\gamma_R^*(T')+4\geq\gamma_R^*(T).$$

If the equality $\frac{4}{3}\beta_2(T)=\gamma_R^*(T)$ holds, then all inequalities occurring in above chain become equalities. In particular, $\beta_2(T)=\beta_2(T')+3$, $\gamma_R^*(T)=\gamma_R^*(T')+4$ and $\gamma_R^*(T')=\frac{4}{3}\beta_2(T')$. We conclude from the induction hypothesis that $T'\in\mathcal{T}$.
If $v_5\notin W_{T'}^3$ and g is a mixed Roman dominating function of T' with $g(v_5)=2$, then the function $h: V(T)\cup E(T)\to\{0,1,2\}$ defined by $h(v_4v_3)=1, h(v_2)=2$, $h(v_1)=h(v_3)=h(v_4)=0$, $h(v_iv_{i+1})=0$ for $i=1,2,4$ and $h(u)=g(u)$ for $u\in V(T')\cup E(T')$, is an MRDF of T yielding $\gamma_R^*(T)\leq\gamma_R^*(T')+3$ which is a contradiction. Hence $v_5\in W_{T'}^3$. If $v_5\notin W_{T'}^4$ and g is an almost mixed Roman dominating function of weight less than $\gamma_R^*(T')$, then the function $h: V(T)\cup E(T)\to\{0,1,2\}$ defined by

$h(v_4v_5) = h(v_2) = 2$, $h(v_1) = h(v_3) = h(v_4) = 0$, $h(v_iv_{i+1}) = 0$ for $i = 1, 2, 3$ and $h(u) = g(u)$ for $u \in V(T') \cup E(T')$, is an MRDF of T yielding $\gamma_R^*(T) \leq \gamma_R^*(T') + 3$, a contradiction again. Thus $v_5 \in W^4_{T'}$. Now T can be obtained from T' by Operation $\mathcal{O}_3$ and so $T \in \mathcal{T}$.

Subcase 3.3 v_5 has a children x with depth 2.
Assume that v_5xyz is a path in T. As above, we may assume that $\deg(x) = \deg(y) = 2$. Let $T' = T - \{x, y, z\}$. Clearly, any $\beta_2(T')$-set can be extended to a 2-independent set of T by adding y, z and so $\beta_2(T) \geq \beta_2(T') + 2$. Let now f be a $\gamma_R^*(T')$-function. Then clearly, $\sum_{i=1}^4 f(v_i) + \sum_{j=1}^4 f(v_jv_{j+1}) \geq 4$ or $\sum_{i=1}^5 f(v_i) + \sum_{j=1}^4 f(v_jv_{j+1}) \geq 5$. If $\sum_{i=1}^4 f(v_i) + \sum_{j=1}^4 \geq 4$, then we can assume that $f(v_2) = f(v_5v_4) = 2$ and if $\sum_{i=1}^5 f(v_i) + \sum_{j=1}^4 \geq 5$, then we may assume that $f(v_2) = f(v_5) = 2$ and $f(v_3v_4) = 1$. Now the function $h : V(T) \cup E(T) \to \{0, 1, 2\}$ defined by $h(y) = 2$, $h(x) = h(z) = h(xy) + h(yz) = h(v_5x) = 0$ and, $h(u) = f(u)$ for $u \in V(T')$, is an MRDF of T and so $\gamma_R^*(T) \leq \gamma_R^*(T') + 2$. It follows from the induction hypothesis that $\frac{4}{3}\beta_2(T) > \gamma_R^*(T)$.

Subcase 3.4 v_5 has a children different from v_4 with depth 3.
Let $v_4 = z_4^1, \ldots, z_4^r$ be the children of v_5 with depth 3 and let $v_5z_4^iz_3^iz_2^iz_1^i$ be a path in T for $i = 1, 2, \ldots, r$. Considering above Cases and subcases, we may assume that $\deg(z_4^i) = \deg(z_3^i) = \deg(z_2^i) = 2$ for each $1 \leq i \leq r$. Let s be the number of leaves adjacent to v_5 and let $T' = T - T_{v_5}$. If $s \geq 2$, then the result follows as Subcase 3.1. Assume that $s \leq 1$. Clearly, any $\beta_2(T')$-set can be extended to a 2-independent set of T by adding z_4^i, z_2^i, z_1^i $(1 \leq i \leq r)$ and the leaf adjacent to v_5, if any, and so $\beta_2(T) \geq \beta_2(T') + 3r + s$. Also, any $\gamma_R^*(T')$-function can be extended to an MRDF of T by assigning a 2 to v_5, z_2^i $(1 \leq i \leq r)$ and a 1 to $z_4^iz_3^i$ $(1 \leq i \leq r)$ and a 0 to the remaining elements. Hence $\gamma_R^*(T) \leq \gamma_R^*(T') + 3r + 2$. By the induction hypothesis we obtain

$$\frac{4}{3}\beta_2(T) \geq \frac{4}{3}(\beta_2(T') + 3r + s) \geq \gamma_R^*(T') + 4r + \frac{4s}{3} \geq \gamma_R^*(T) + r - 2 + \frac{4s}{3}.$$

If the equality $\frac{4}{3}\beta_2(T) = \gamma_R^*(T)$ holds, then all inequalities occurring in above chain become equalities and so $\frac{4}{3}\beta_2(T') = \gamma_R^*(T')$, $s = 0$, $r = 2$ and $\beta_2(T) = \beta_2(T') + 6$, $\gamma_R^*(T) = \gamma_R^*(T') + 8$. We deduce from the induction hypothesis that $T' \in \mathcal{T}$. We now show that there is no $\gamma_R^*(T')$-function f such that $f(v_6) = 1$. Suppose, to the contrary, that there is a $\gamma_R^*(T')$-function f such that $f(v_6) = 1$. Then f can be extended to an MRDF of T by assigning a 2 to v_5, z_2^i $(1 \leq i \leq r)$ and a 1 to $z_4^iz_3^i$ $(1 \leq i \leq r)$ and a 0 to the remaining elements and v_6 and so $\gamma_R^*(T) \leq \gamma_R^*(T') + 7$, a contradiction. Now T can be obtained from T' by Operation $\mathcal{O}_4$ and so $T \in \mathcal{T}$.

Subcase 3.4. $\deg(v_5) = 3$ and v_5 has a child w with depth 0.
Let $T' = T - T_{v_5}$. Clearly, any $\beta_2(T')$-set can be extended to a 2-independent set of T by adding w, v_4, v_2, v_1 and so $\beta_2(T) \geq \beta_2(T') + 3r + s$. On the other hand, any $\gamma_R^*(T')$-function can be extended to an MRDF of T by assigning a 2 to v_2 and v_5v_4, a 1 to w and a 0 to the remaining elements and this implies that $\gamma_R^*(T) \leq \gamma_R^*(T') + 5$. It follows from the induction hypothesis that $\frac{4}{3}\beta_2(T) > \gamma_R^*(T)$.

Case 4. $\deg(v_3) = \deg(v_4) = \deg(v_5) = 2$.
Let $T' = T - T_{v_5}$. By Lemma 1 and the induction hypothesis we have

$$\frac{4}{3}\beta_2(T) \geq \frac{4}{3}(\beta_2(T') + 3) \geq \gamma_R^*(T') + 4 = \gamma_R^*(T).$$

If the equality $\frac{4}{3}\beta_2(T) = \gamma_R^*(T)$ holds, then all inequalities occurring in above chain become equalities and so $\frac{4}{3}\beta_2(T') = \gamma_R^*(T')$ and $\beta_2(T) = \beta_2(T') + 3$. It follows from the induction hypothesis that $T' \in \mathcal{T}$. If there exists a $\beta_2(T')$-set S' such that $v_6 \notin S'$, Then $S' \cup \{v_5, v_4, v_2, v_1\}$ is a 2-independent set of T yielding $\beta_2(T) \geq \beta_2(T') + 4$, a contradiction. Thus each $\beta_2(T')$-set contains v_6, i.e. $v_6 \in W^1_{T'}$. Now T can be obtained from T' by Operation $\mathcal{O}_1$ and so $T \in \mathcal{T}$. This completes the proof. □

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