Communications in Combinatorics and Optimization
Vol. 2 No. 1, 2017 pp.11-19
DOI: 10.22049/CCO.2017.13578

CCO
Comm. Comb. Optim.

On net-Laplacian energy of signed graphs

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Received: 28 August 2016; Accepted: 15 February 2017;
Available Online:19 February 2017

*Communicated by **Ivan Gutman***

Abstract: A signed graph is a graph where the edges are assigned either positive or negative signs. Net degree of a signed graph is the difference between the number of positive and negative edges incident with a vertex. It is said to be net-regular if all its vertices have the same net-degree. Laplacian energy of a signed graph Σ is defined as $\varepsilon(L(\Sigma)) = \sum_{i=1}^{n} |\gamma_i - \frac{2m}{n}|$ where $\gamma_1, \gamma_2, \ldots, \gamma_n$ are the eigenvalues of $L(\Sigma)$ and $\frac{2m}{n}$ is the average degree of the vertices in Σ. In this paper, we define net-Laplacian matrix considering the edge signs of a signed graph and give bounds for signed net-Laplacian eigenvalues. Further, we introduce net-Laplacian energy of a signed graph and establish net-Laplacian energy bounds.

Keywords: Net-regular signed graph, net-Laplacian matrix, net-Laplacian energy.

2010 Mathematics Subject Classification: 05C22, 05C50

1. Introduction

A signed graph is an ordered pair $\Sigma = (G, \sigma)$, where G is the underlying graph of Σ and $\sigma : E \rightarrow \{+1, -1\}$, called signing (or a signature), is a function from the edge set $E(G)$ of G into the set $\{+1, -1\}$. It is said to be homogeneous if its edges are all positive or negative otherwise heterogeneous. Negation of a signed graph is the same graph with all signs reversed. The sign of a cycle in a signed graph is the product of the signs of its edges. Thus a cycle is positive if and only if it contains an even number of negative edges. Signed graph is balanced (or cycle balanced) if all of its cycles are positive otherwise unbalanced.

The adjacency matrix of a signed graph is the square matrix $A(\Sigma) = (a_{ij})$ where (i, j) entry is $+1$ if $\sigma(v_iv_j) = +1$ and -1 if $\sigma(v_iv_j) = -1$, 0 otherwise. The characteristic polynomial of a signed graph Σ is defined as $\Phi(\Sigma : \lambda) = \det(\lambda I - A(\Sigma))$, where I is an identity matrix of order n. The roots of the characteristic equation $\Phi(\Sigma : \lambda) = 0$, denoted by $\lambda_1, \lambda_2, \ldots, \lambda_n$ are called the eigenvalues of signed graph Σ. If the distinct eigenvalues of $A(\Sigma)$ are $\lambda_1 > \lambda_2 > \cdots > \lambda_t$ and their multiplicities are $m_1, m_2, \ldots, m_t$, then the spectrum of Σ is $Sp(\Sigma) = \{\lambda_1^{(m_1)}, \lambda_2^{(m_2)}, \ldots, \lambda_n^{(m_t)}\}$.

Two signed graphs are cospectral if they have the same spectrum. The spectral criterion for balance in signed graph is given by B.D.Acharya as follows:

Theorem 1. *[1] A signed graph is balanced if and only if it is cospectral with the underlying graph. i.e. $Sp(\Sigma) = Sp(G)$.*

Theorem 2. *[15] Two signed graphs $\Sigma_1 = (G, \sigma_1)$ and $\Sigma_2 = (G, \sigma_2)$ on the same underlying graph are switching equivalent if and only if they are cycle isomorphic.*

In signed graph Σ, the degree of a vertex v is defined as $sdeg(v) = d(v) = d^+(v) + d^-(v)$, where $d^+(v)((d^-(v))$ is the number of positive(negative) edges incident with v. The net degree of a vertex v of a signed graph Σ is $d^{\pm}(v) = d^+(v) - d^-(v)$. It is said to be net-regular of degree k if all its vertices have same net-degree equal to k. Hence net-regularity of a signed graph can be either positive, negative or zero. We denote d_i^{net} for the net degree of a vertex v_i of a signed graph Σ and the vector $d^{net} = (d_1^{net}, d_2^{net}, \ldots, d_n^{net})$.

Lemma 1. *[14] If Σ is a k net-regular signed graph, then k is an eigenvalue of Σ with j as an eigenvector with all 1's.*

Co-regularity pair of a signed graph is defined as follows:

Definition 1. [6] A signed graph $\Sigma = (G, \sigma)$ is said to be co-regular if the underlying graph G is regular for some positive integer r and Σ is net-regular with net-degree k for some integer k and the co-regularity pair is an ordered pair of (r, k).

Theorem 3. *[11] If Σ is a net-regular signed graph then its underlying graph is not necessarily a regular graph.*

Definition 2. [4] If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of Σ, then signed energy is defined as

$$\varepsilon(\Sigma) = \sum_{i=1}^{n} |\lambda_i|.$$

Definition 3. [3] If $\gamma_1, \gamma_2, \ldots, \gamma_n$ are the eigenvalues of $L(\Sigma)$, then Laplacian energy of a signed graph is defined as

$$\varepsilon(L(\Sigma)) = \sum_{i=1}^{n} |\gamma_i - \frac{2m}{n}|$$

where $\frac{2m}{n}$ is the average degree of the vertices in Σ.

Graph energy is well documented in [9] and signed graph energy is discussed in [2–4, 10, 12]. For standard terminology and notations in graph theory we follow D.B. West [13] and for signed graphs we follow T. Zaslavsky [14, 16].

In [8] Hou et. al. extended the concept of Laplacian matrices of a graph to signed graphs. But, Laplacian matrix in signed graph is defined without considering the weight of the edges. i.e. $L(\Sigma) = D(\Sigma) - A(\Sigma)$ where $D(\Sigma)$ is a diagonal matrix with degrees of vertices of underlying graph of Σ. Main aim of this paper is to consider the edge signs of a signed graph and define signed net-Laplacian matrix and also give bounds for signed graph eigenvalues as well as signed net-Laplacian eigenvalues. Further, we introduce net-Laplacian energy of a signed graph and establish some upper and lower bounds for net-Lapalcian energy of signed graphs.

2. Signed eigenvalues and net-Laplacian eigenvalues

In this section, we give bounds for signed graph eigenvalues and signed net-Laplacian eigenvalues. Rayleigh-Ritz Theorem is used to find upper and lower bounds for the eigenvalues of a symmetric matrix.

Lemma 2. *[7] Let $A \in R^{n\times n}$ be symmetric. Then*

$$\lambda_{max}(A) = \max_{x\in R^n \setminus \{0\}} \frac{x^T Ax}{x^T x}$$

and

$$\lambda_{min}(A) = \min_{x\in R^n \setminus \{0\}} \frac{x^T Ax}{x^T x}.$$

We denote total number of positive and negative edges in signed graph by m^+ and m^- respectively. We consider $m = m^+ + m^-$ and $M = m^+ - m^-$. Lemma 3 characterizes net-regular signed graphs.

Lemma 3. *[11] If $\Sigma = (G, \sigma)$ is a connected net-regular signed graph with net degree k, then $k = \frac{2M}{n}$ where $M = (m^+ - m^-)$.*

Following result gives the eigenvalue bounds for adjacency matrix of a signed graph.

Theorem 4. *Let $\Sigma = (G, \sigma)$ be a connected signed graph with n vertices and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of adjacency matrix $A(\Sigma)$. Then*

$$\lambda_n(A(\Sigma)) \leq \frac{2M}{n} \leq \lambda_1(A(\Sigma))$$

where $M = m^+ - m^-$.

Proof. Let $A(\Sigma)$ be an adjacency matrix of a signed graph Σ and let $j = (1, 1, \ldots, 1)^T$. Since each row sum of $A(\Sigma)$ gives net-degree of each vertex v_i for $i = 1, 2, \ldots, n$, we have

$$\begin{aligned} A(\Sigma)j &= (\Sigma_{k=1}^n a_{1k}, \Sigma_{k=1}^n a_{2k}, \ldots, \Sigma_{k=1}^n a_{nk}) \\ &= (d_1^{net}, d_2^{net}, \ldots, d_n^{net}) \\ &= d^{net}. \end{aligned}$$

Hence $j^T A(\Sigma) j = j^T d^{net} = \sum_{i=1}^n d_i^{net} = 2M$.
By Lemma 2, we have

$$\lambda_n(A(\Sigma)) \leq \frac{j^T A(\Sigma) j}{j^T j} \leq \lambda_1(A(\Sigma)).$$

Thus $\lambda_n(A(\Sigma)) \leq \frac{2M}{n} \leq \lambda_1(A(\Sigma))$ and the proof is complete. □

Now we consider the net degrees of vertices of a signed graph Σ and define signed net-Laplacian matrix and denote it as $L^{net}(\Sigma)$. We define as follows.

Definition 4. Let $\mu_1, \mu_2, \ldots, \mu_n$ be the eigenvalues of a signed net-Laplacian matrix then $L^{net}(\Sigma) = D^{net}(\Sigma) - A(\Sigma)$, where $D^{net}(\Sigma) = diag(d_1^{net}, d_2^{net}, \ldots, d_n^{net})$ is the diagonal matrix and $A(\Sigma) = [a_{ij}]$ is the adjacency matrix of Σ.

Observation 1. $\sum_{i=1}^n \mu_i = 2M$*, where* $M = m^+ - m^-$.

Proof. We have $\sum_{i=1}^{n} \mu_i = \text{trace}(L^{net}(\Sigma)) = \sum_{i=1}^{n} d_i^{net} = 2M$. □

Observation 2. *0 is an eigenvalue of $L^{net}(\Sigma)$ with multiplicity at least p, the number of components of Σ.*

Proof. The sum of each row in $L^{net}(\Sigma)$ is 0, thus 0 is an eigenvalue with eigenvector $(1, 1, \ldots, 1)$. □

It is well known that the signed graph eigenvalues satisfy the following relations:

$$\sum_{i=1}^{n} \lambda_i = 0 \quad \text{and} \quad \sum_{i=1}^{n} \lambda_i^2 = 2m.$$

Next result gives the relation of signed net-Laplacian eigenvalues.

Lemma 4. *If $\mu_1, \mu_2, \ldots, \mu_n$ are the eigenvalues of a signed net-Laplacian matrix, then*

$$\sum_{i=1}^{n} \mu_i^2 = 2m + \sum_{i=1}^{n} (d_i^{net})^2.$$

Proof. We note that $\sum_{i<j} \mu_i \mu_j$ is equal to the sum of the determinants of all 2×2 principal submatrices of $L^{net}(\Sigma)$. Hence

$$\begin{aligned}
\textstyle\sum_{i<j} \mu_i \mu_j &= \textstyle\sum_{i<j} \det \begin{pmatrix} d_i^{net} & -a_{ij} \\ -a_{ji} & d_j^{net} \end{pmatrix} \\
&= \textstyle\sum_{i<j} [(d_i^{net})(d_j^{net}) - (a_{ij})^2] \\
&= \textstyle\sum_{i<j} (d_i^{net})(d_j^{net}) - \sum_{i<j} (a_{ij})^2 \\
&= \textstyle\sum_{i<j} (d_i^{net})(d_j^{net}) - m.
\end{aligned}$$

Therefore

$$\sum_{i \neq j} \mu_i \mu_j = 2 \sum_{i<j} \mu_i \mu_j = \sum_{i \neq j} (d_i^{net})(d_j^{net}) - 2m.$$

Since

$$\begin{aligned}
\textstyle\sum_{i=1}^{n} \mu_i^2 &= (\textstyle\sum_{i=1}^{n} \mu_i)^2 - \sum_{i \neq j} \mu_i \mu_j \\
&= (\textstyle\sum_{i=1}^{n} \mu_i)^2 - [\sum_{i \neq j} (d_i^{net})(d_j^{net}) - 2m] \\
&= (\textstyle\sum_{i=1}^{n} d_i^{net})^2 - [\sum_{i \neq j} (d_i^{net})(d_j^{net}) - 2m],
\end{aligned}$$

we obtain $\sum_{i=1}^{n} \mu_i^2 = \sum_{i=1}^{n} (d_i^{net})^2 + 2m$, as desired. □

Next theorem gives the bounds for net-Laplacian eigenvalues of a signed graph.

Theorem 5. *Let $\Sigma = (G, \sigma)$ be a connected signed graph and $\mu_1, \mu_2, \ldots, \mu_n$ be the eigenvalues of a signed net-Laplacian matrix. Then*

$$\mu_n(L^{net}(\Sigma)) \leq 0 \leq \mu_1(L^{net}(\Sigma)).$$

Proof. Let $L^{net}(\Sigma)$ be a net-Laplacian matrix of a signed graph Σ and let $j = (1, 1, \ldots, 1)^T$. Then

$$A(\Sigma)j = (\Sigma_{k=1}^{n} a_{1k}, \Sigma_{k=1}^{n} a_{2k}, \ldots, \Sigma_{k=1}^{n} a_{nk}) = (d_1^{net}, d_2^{net}, \ldots, d_n^{net}) = d^{net}.$$

Hence

$$\begin{aligned} j^T(L^{net}(\Sigma))j &= j^T(D^{net}(\Sigma) - A(\Sigma))j \\ &= j^T(D^{net}(\Sigma))j - j^T(A(\Sigma))j \\ &= \textstyle\sum_{i=1}^{n} d_i^{net} - \sum_{i=1}^{n} d_i^{net} \\ &= 0. \end{aligned}$$

By Lemma 2, we have

$$\mu_n(L^{net}(\Sigma)) \leq \frac{j^T(D^{net}(\Sigma) - A(\Sigma))j}{j^T j} \leq \mu_1(L^{net}(\Sigma))$$

and so

$$\mu_n(L^{net}(\Sigma)) \leq \frac{j^T(L^{net}(\Sigma))j}{j^T j} \leq \mu_1(L^{net}(\Sigma)).$$

Thus $\mu_n(L^{net}(\Sigma)) \leq 0 \leq \mu_1(L^{net}(\Sigma))$ and the proof is complete. □

3. Bounds for the net-Laplacian energy

In this section, we give definition of net-Laplacian energy of a signed graph in analogous to graph energy and establish net-Laplacian energy bounds.

Definition 5. Let $\mu_1, \mu_2, \ldots, \mu_n$ be the net-Laplacian eigenvalues of a signed graph Σ. Then

$$\varepsilon(L^{net}(\Sigma)) = \sum_{i=1}^{n} |\mu_i - \frac{2M}{n}|$$

where $\frac{2M}{n}$ is the average net-degree of a signed graph.

Let $\beta_i = \mu_i - \frac{2M}{n}$ $(i = 1, 2, \ldots, n)$, be the auxiliary eigenvalues. Then

$$\sum_{i=1}^{n} \beta_i = 0 \quad \text{and} \quad \sum_{i=1}^{n} \beta_i^2 = 2M_1$$

where $M_1 = m + \frac{1}{2}\sum_{i=1}^{n}(d_i^{net} - \frac{2M}{n})^2$.
Hence, $M_1 = m$ if and only if Σ is net-regular, otherwise $M_1 > m$.

Lemma 5. *If the signed graph $\Sigma = (G, \sigma)$ is a net-regular signed graph, then $\varepsilon(L^{net}(\Sigma)) = \varepsilon(\Sigma)$.*

Proof. The proof is similar to unsigned graphs [5]. Let $\Sigma = (G, \sigma)$ be a net-regular signed graph and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of Σ. Then by Lemma 3, $k = \frac{2M}{n}$ where $M = (m^+ - m^-)$.
Let $\mu_1, \mu_2, \ldots, \mu_n$ be the Laplace eigenvalues of Σ. Then $\mu_i - \frac{2M}{n} = -\lambda_{n-i+1}$ for all $i = 1, 2, \ldots, n$ and the proof follows from the above definition. □

Following result characterizes all signed graphs which have the same energy for their adjacency, Laplacian and net-Laplacian matrices.

Theorem 6. *If Σ is a co-regular signed graph, then $\varepsilon(L(\Sigma)) = \varepsilon(L^{net}(\Sigma)) = \varepsilon(\Sigma)$.*

Proof. Theorem follows from Definition 3 and Lemma 5. □

Following inequality that establishes an upper bound for signed energy in terms of parameters n and m is generalised in [12].

Theorem 7. *Let Σ be a signed graph with n vertices and m edges, then*

$$\varepsilon(\Sigma) \leq \sqrt{2mn}.$$

Using Theorem 7, we give the upper bound for the net-Laplacian energy of a signed graph.

Theorem 8. *Let Σ be a signed graph and let d_i^{net} be the net-degree of the i^{th} vertex of Σ, $i = 1, 2, \ldots, n$. If $\mu_1, \mu_2, \ldots, \mu_n$ are the eigenvalues of the net-Laplacian matrix $L^{net}(\Sigma) = D^{net}(\Sigma) - A(\Sigma)$, where $D^{net}(\Sigma) = diag(d_1^{net}, d_2^{net}, \ldots, d_n^{net})$ is the diagonal matrix and $A(\Sigma) = (a_{ij})$ is the adjacency matrix of Σ, then*

$$\varepsilon(L^{net}(\Sigma)) \leq \sqrt{2M_1 n}$$

where $M_1 = m + \frac{1}{2}\sum_{i=1}^{n}(d_i^{net} - \frac{2M}{n})^2$.

Proof. Consider the sum $S = \sum_{i=1}^{n} \sum_{j=1}^{n} (\mid \beta_i \mid - \mid \beta_j \mid)^2$. Then we have

$$S = 2n \sum_{i=1}^{n} \beta_i^2 - 2(\sum_{i=1}^{n} \mid \beta_i \mid)(\sum_{j=1}^{n} \mid \beta_j \mid) \leq 4nM_1 - 2\varepsilon(L^{net}(\Sigma))^2.$$

Since $S \geq 0$, we obtain $\varepsilon(L^{net}(\Sigma)) \leq \sqrt{2M_1 n}$. □

Theorem 9. *If $\mu_1, \mu_2, \ldots, \mu_n$ are the eigenvalues of the net-Laplacian matrix $L^{net}(\Sigma)$ then*

$$2\sqrt{|M|} \leq \varepsilon(L^{net}(\Sigma)) \leq 2M_1$$

where $M = (m^+ - m^-)$ and $M_1 = m + \frac{1}{2}\sum_{i=1}^{n}(d_i^{net} - \frac{2M}{n})^2$

Proof. Since $\sum_{i=1}^{n} \beta_i = 0$, we have $\sum_{i=1}^{n} \beta_i^2 + 2\sum_{i<j}^{n} \beta_i\beta_j = 0$. It follows from $\sum_{i=1}^{n} \beta_i^2 = 2M$ that $2M = -2\sum_{i<j}^{n} \beta_i\beta_j$. Hence

$$2|M| = 2|\sum_{i<j}^{n} \beta_i\beta_j| \leq 2\sum_{i<j}^{n} |\beta_i||\beta_j|.$$

Thus

$$\begin{aligned} (\varepsilon(L^{net}(\Sigma))^2 &= (\textstyle\sum_{i=1}^{n} |\beta_i|)^2 \\ &= \textstyle\sum_{i=1}^{n} |\beta_i|^2 + 2\sum_{i<j}^{n} |\beta_i||\beta_j| \\ &\geq 2|M| + 2\textstyle\sum_{i<j}^{n} |\beta_i||\beta_j| \\ &\geq 4|M| \end{aligned},$$

and this leads to $2\sqrt{|M|} \leq \varepsilon(L^{net}(\Sigma))$.
To prove the right-hand inequality, note that for a signed graph with m edges and no isolated vertex, $n \leq 2m$. By Theorem 8,

$$\varepsilon(L^{net}(\Sigma)) \leq \sqrt{2M_1 n} \leq \sqrt{2M_1(2m)} = 2\sqrt{M_1(m)}.$$

Since $M_1 \geq m$, we obtain $\varepsilon(L^{net}(\Sigma)) \leq 2M_1$. □

Acknowledgements The author thanks the University Grants Commission, India for providing grants under minor research project No.47-902/14 during XII plan.

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