Communications in Combinatorics and Optimization
Vol. 2 No. 1, 2017 pp.1-9
DOI: 10.22049/CCO.2017.13577

CCO
Commun. Comb. Optim.

The locating-chromatic number for Halin graphs

I.A. Purwasih[1], E.T. Baskoro[1*], H. Assiyatun[1], D. Suprijanto[1] and M. Bača[2]

[1]Combinatorial Mathematics Research Group, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Jalan Ganesa 10 Bandung 40132, Indonesia
ira_apni_p@students.itb.ac.id and {ebaskoro, hilda, djoko}@math.itb.ac.id

[2]Department of Applied Mathematics and Informatics, Faculty of Mechanical Engineering, Technical University in Košice, Slovakia
martin.baca@tuke.sk

Received: 22 September 2016; Accepted: 9 January 2017;
Available Online: 17 January 2017

Communicated by ***Seyed Mahmoud Sheikholeslami***

Abstract: Let G be a connected graph. Let f be a proper k-coloring of G and $\Pi = \{R_1, R_2, \dots, R_k\}$ be an ordered partition of $V(G)$ into color classes. For any vertex v of G, define the *color code* $c_\Pi(v)$ of v with respect to Π to be a k-tuple $(d(v,R_1), d(v,R_2), \dots, d(v,R_k))$, where $d(v,R_i) = \min\{d(v,x)|x \in R_i\}$. If distinct vertices have distinct color codes, then we call f a *locating coloring* of G. The *locating-chromatic number* of G is the minimum number k such that G admits a locating coloring with k colors. In this paper, we determine a lower bound of the locating-chromatic number of Halin graphs. We also give the locating-chromatic number of a Halin graph of a double star.

Keywords: locating-chromatic number, Halin graph, double star.

2010 Mathematics Subject Classification: 05C15, 05C12

* *Corresponding Author*

1. Introduction

Let G be a connected graph. Let f be a proper k-coloring of G and $\Pi = \{R_1, R_2, \ldots, R_k\}$ be an ordered partition of $V(G)$ into color classes. For any vertex v of G, define the *color code* $c_\Pi(v)$ of v with respect to Π to be a k-tuple $(d(v, R_1), d(v, R_2), \ldots, d(v, R_k))$, where $d(v, R_i) = \min\{d(v, x) \mid x \in R_i\}$. If distinct vertices have distinct color codes, then we call f a *locating coloring* of G. The *locating-chromatic number* of G, denoted by $\chi_L(G)$, is the minimum number k such that G admits a locating coloring with k colors. Two distinct vertices x and y in G are *resolved* by a vertex z if $d(x, z) \neq d(y, z)$. Without loss of generality, if $f(x) = t \neq f(y)$, then the t^{th}-element of color code of x is 0 but the t^{th}-element of color code of y is at least 1. So, in this case, we only need to consider two distinct vertices in G which have same color.

The locating-chromatic number of graphs was initially studied by Chartrand *et al.* in [9]. They established some bounds on the locating-chromatic number of a graph and determined values of this parameter for some well known classes of graphs such as path, cycle, and double star. They also studied the locating-chromatic number of trees. However, the locating-chromatic number of all trees are not completely solved. Furthermore, regarding trees, the locating-chromatic number for the amalgamation of stars is given in [1], and the value for a firecracker is given in [4]. Baskoro and Asmiati [3] was able to characterize all trees with locating-chromatic number 3.

Chartrand *et al.* [10] characterized all non-tree graphs of order n with locating-chromatic number $n-1$. Asmiati and Baskoro [2] gave a characterization of all non-tree graphs with locating-chromatic number 3. One well-known kind of non-tree graphs is a Halin graph.

Let T be a tree with no vertices of degree two and having at least four vertices. A *Halin* graph $H(T)$ is a planar graph constructed from a plane embedding of tree T by connecting all the leaves of the tree (the vertices of degree 1) with a cycle that passes around the tree in the natural cyclic order defined by the embedding of the tree [12]. Halin graphs play a vital role in understanding the inherent complexity of the problem and a good candidate graph class between the class of trees and planar graphs. Therefore, there are many researchers, both from theoretical and application perspective, interested in studying about Halin graph. The first result was given by Rudolf Halin [12]. He studied minimally n-connected graphs and introduced the Halin graphs as a class of minimally 3-vertex-connected graphs. Some researchers studied problems dealing with Halin graphs, including travelling salesman problem [11], hamiltonian properties [5], coloring [13].

Some researchers also considered on the locating-chromatic number of graphs produced by a graph operation, i.e., the join of graphs [7], Cartesian product

[8], and corona product [6].
Based on the results given by Asmiati and Baskoro [2], we know that there is no Halin graph having locating-chromatic number 3. So, we can make a conclusion as in the following corollary.

Corollary 1. *Let T be a tree with no vertices of degree two and having at least 4 vertices. Then, $\chi_L(H(T)) \geq 4$.*$\square$

In [7], Behtoei and Anbarloei defined another new parameter which is useful in finding the locating-chromatic number of some graphs. Let f be a proper $k-$coloring of a graph G. If $f(N_G(u)) \neq f(N_G(v))$ for each two distinct vertices u and v with the same color, then we say that f is a *neighbor locating coloring* of G. The *neighbor locating-chromatic number* $\chi_{L2}(G)$ is the minimum positive integer k such that G has a neighbor locating $k-$coloring. They also gave the following theorem for the neighbor locating-chromatic number of paths.

Theorem 1. *[7] For $n \geq 2$, $\chi_{L2}(P_n) = m$, where $m = min\{k|k \in \mathbb{N}, n \leq \frac{1}{2}(k^3 - k^2)\}$. In particular, there exists a neighbor locating $m-$coloring f of the path $P_n = u_1u_2...u_n$ such that $f(u_{n-1}) = 2$ and $f(u_n) = 1$. Additionally, for $n \geq 9$, $f(u_{n-2}) = m$. Moreover, for $n \geq 9$ and $n \neq \frac{1}{2}(m^3-m^2)-1$, $f(u_1) = 2$ and $f(u_2) = 1$.*

For $n \geq 2$, let F_n be a fan on $n+1$ vertices with the hub u, namely $F_n = K_1+P_n$ where $K_1 = \{u\}$ and P_n is a path $u_1u_2...u_n$. By using the neighbor locating coloring of a path, we can construct a neighbor locating coloring of a fan. By Theorem 1, there exists a neighbor locating $m-$coloring f of the path $P_n = u_1u_2...u_n$ where $m = \chi_{L2}(P_n)$ such that $f(u_{n-1}) = 2$, $f(u_n) = 1$. For $n \geq 9$, $f(u_{n-2}) = m$. Moreover, for $n \geq 9$ and $n \neq \frac{1}{2}(m^3 - m^2) - 1$, $f(u_1) = 2$, and $f(u_2) = 1$. Now define a new coloring f' of F_n as $f'(u_i) = f(u_i)$, for $1 \leq i \leq n$, and $f'(u) = m+1$. Since $f'(u) = m+1 \neq f'(u_i)$ for each $1 \leq i \leq n$ then f' is a proper coloring of F_n. Note that $E(F_n) = E(P_n) \cup \{u_iu|1 \leq i \leq n\}$. Hence, for each integer i with $1 \leq i \leq n$, we have $f'(N_{F_n}(u_i)) = f(N_{P_n}(u_i)) \cup \{m + 1\}$. Therefore, f' is a neighbor locating $(m+1)-$coloring of F_n where $f'(u_{n-1}) = 2$, $f'(u_n) = 1$, $f'(u) = m + 1$. For $n \geq 9$, $f'(u_{n-2}) = m$. Moreover, for $n \geq 9$ and $n \neq \frac{1}{2}(m^3 - m^2) - 1$, $f'(u_1) = 2$ and $f'(u_2) = 1$. Hence, we have the following observation as a combination of two results in [7].

Observation 1. *For $n \geq 2$, let $F_n = K_1+P_n$ where $K_1 = \{u\}$ and $P_n = u_1u_2...u_n$. Let $m = \min\{k \in \mathbb{N} \mid n \leq \frac{1}{2}(k^3 - k^2)\}$. Then, $\chi_L(F_n) = m + 1$. In particular, there exists a neighbor locating $(m + 1)-$coloring f' of F_n such that $f'(u_{n-1}) = 2$, $f'(u_n) = 1$, $f'(u) = m + 1$. Additionally, for $n \geq 9$, $f'(u_{n-2}) = m$. Moreover, for $n \geq 9$ and $n \neq \frac{1}{2}(m^3 - m^2) - 1$, $f'(u_1) = 2$ and $f'(u_2) = 1$.*$\square$

By using the neighbor locating coloring concept, Behtoei and Anbarloei were able to obtain the locating-chromatic number of wheels in the following theorem.

Theorem 2. *[7] For $n \geq 3$, let W_n be a wheel of order $n+1$. Let $m = \min\{k \in \mathbb{N} \mid n \leq \frac{1}{2}(k^3 - k^2)\}$. Then,*

$$\chi_L(W_n) = \begin{cases} 1 + \chi_L(C_n), & \text{if } n < 9; \\ 1 + m, & \text{if } n \geq 9 \text{ and } n \neq \frac{1}{2}(m^3 - m^2) - 1; \\ 2 + m, & \text{if } n \geq 9 \text{ and } n = \frac{1}{2}(m^3 - m^2) - 1. \end{cases}$$

Now, we will derive a lower bound of the locating-chromatic number of a Halin graph in general. In particular, we determine the locating-chromatic number of a Halin graph of a double star.

2. Main Results

Let m be an integer with $m \geq 11$. In this section, consider a Halin graph H of order m. Let F_n be a largest fan on $n+1$ vertices in H. Obviously, $n < m$. Let $A := \{u, u_1, u_2, \ldots, u_n\}$ be the vertex set of F_n and f be a locating coloring on H. Now, let $B = \{u_3, \ldots, u_{n-2}\}$. By the definition of H, every two distinct vertices x and y in B will have the same distance to each vertex z in $V(H) - B - \{u_2, u_{n-1}\}$, since $d(x,z) = d(x,u) + d(u,z) = 1 + d(u,z) = d(y,u) + d(u,z) = d(y,z)$. This means that any two vertices in B can be resolved only by their neighbors. The other fact that we can observe is two distinct vertices $x \in \{u_2, u_{n-1}\}$ and $y \in A$ can be resolved by a vertex $z \in V(H) - A$ if and only if z has a color different from all vertices in A. If $x \in \{u_1, u_n\}$ and $y \in A$, then two neighbors of x or two neighbors of y are the only vertices which can resolve x and y. Therefore, we have the following observation.

Observation 2. *For $n \geq 10$, let F_n be a largest fan of order $n+1$ in a Halin graph H. Let $A := \{u, u_1, u_2, \ldots, u_n\}$ be the vertex set of F_n and f be a locating coloring of H. Then, we have the following facts.*

1. *Every two distinct vertices in $\{u_3, \ldots, u_{n-2}\}$ can only be resolved by their neighbors.*
2. *If $x \in \{u_2, u_{n-1}\}$ and $y \in A$ are resolved only by a vertex $z \in V(H) - A$, then z has different color with all vertices in A.*
3. *If $x \in \{u_1, u_n\}$ and $y \in A$ are resolved by a vertex $z \in V(H) - A$, and $f(z) \in f(A)$, then z is adjacent to x.* $\square$

Now, we are ready to prove the following theorem.

Theorem 3. *Let H be a Halin graph. If F_n is a largest fan in H for some $n \geq 2$, then $\chi_L(H) \geq \chi_L(F_n)$.*

Proof. If $n \leq 9$, then it is obvious that $\chi_L(H) \geq 4 \geq \chi_L(F_n)$ by the Corollary 1 and Observation 1.
Now, assume $n \geq 10$. Let $A = \{u, u_1, u_2, \ldots, u_n\}$ be the vertex set of the largest fan subgraph F_n of H and u be the hub (the vertex of degree n) of F_n. Let $\chi_L(F_n) = t$. By Observation 1, we have $t = m + 1$ where $m = \min\{k \in \mathbb{N} \mid n \leq \frac{1}{2}(k^3 - k^2)\}$. Therefore, $n \leq \frac{1}{2}(m^3 - m^2)$ but $n > \frac{1}{2}((m-1)^3 - (m-1)^2)$. So, $n \geq \frac{1}{2}((t-2)^3 - (t-2)^2) + 1$. Suppose on the contrary that $\chi_L(H) = t - 1$. Then, we have a locating $(t-1)-$coloring on H, say f.
Case 1. $|f(A)| = t - 1$.
Since u is adjacent to all u_i, then $f(u) \neq f(u_i)$ for all $1 \leq i \leq n$. So, $|f(A')| = t-2$ where $A' = \{u_1, \ldots, u_n\}$. Since all colors are in $f(A)$, then by Observation 2 every two distinct vertices in A' can be only distinguished by their adjacent vertices which are not u. Hence, there are at most $(t-2)(t-3)$ possible color codes of $x \in A'$ having exactly two ordinates 1 and there are at most $(t-2)\begin{pmatrix} t-3 \\ 2 \end{pmatrix}$ possible color codes of $x \in A'$ having exactly three ordinates 1. This means that there are at most $(t-2)\left((t-3) + \begin{pmatrix} t-3 \\ 2 \end{pmatrix}\right) = \frac{1}{2}((t-2)^3 - (t-2)^2)$ distinct color codes. It contradicts the fact $n \geq \frac{1}{2}((t-2)^3 - (t-2)^2) + 1$.
Case 2. $|f(A)| < t - 1$.
Since u is adjacent to all u_i, then $f(u) \neq f(u_i)$ for all $1 \leq i \leq n$. So, $|f(A')| < t-2$ where $A' = \{u_1, \ldots, u_n\}$. Since all colors, except one color, are in $f(A)$, then by Observation 2 every two distinct vertices in $A'' = \{u_3, u_4, \ldots, u_{n-2}\}$ can be only distinguished by their adjacent vertices which are not u. Hence, there are at most $(t-3)(t-4)$ possible color codes of $x \in A''$ having exactly two ordinates 1 and there are at most $(t-3)\begin{pmatrix} t-4 \\ 2 \end{pmatrix}$ possible color codes of $x \in A'$ having exactly three ordinates 1. This means that there are at most $(t-3)\left((t-4) + \begin{pmatrix} t-4 \\ 2 \end{pmatrix}\right) = \frac{1}{2}((t-3)^3 - (t-3)^2)$ distinct color codes. It contradicts the fact $n - 4 \geq \frac{1}{2}((t-2)^3 - (t-2)^2) - 3$.
Therefore, $\chi_L(H) \geq t = \chi_L(F_n)$. □

The lower bound in Theorem 3 is sharp for some wheels W_n, i.e. for $n \geq 9$ and $n \neq \frac{1}{2}(m^3 - m^2) - 1$ where $m = \min\{k \in \mathbb{N} \mid n \leq \frac{1}{2}(k^3 - k^2)\}$. Next, we will give the another graph that has the locating-chromatic number in this bound, i.e. a Halin graph of a double star.
Let $S_{a,b}$ be a double star with $V(S_{a,b}) = \{u, u_1, u_2, \ldots, u_a, v, v_1, v_2, \ldots, v_b\}$ and $E(S_{a,b}) = \{uu_i | 1 \leq i \leq a\} \cup \{vv_i | 1 \leq i \leq b\} \cup \{uv\}$. Let W_n be a wheel of order

$n+1$.

Theorem 4. *Let a, b be integers such that $a \geq b$ and $a+b \geq 10$. Let $H(S_{a,b})$ be a Halin graph of a double star $S_{a,b}$ and $m = \min\{k \in \mathbb{N} \mid a \leq \frac{1}{2}(k^3 - k^2)\}$. Then,*

$$\chi_L(H(S_{a,b})) = \begin{cases} m+2, & \text{if } \chi_L(F_a) < \chi_L(W_{a+b}) \text{ and } \chi_L(F_a) = \chi_L(F_b); \\ m+1, & \text{otherwise.} \end{cases}$$

Proof. Let $V(H(S_{a,b})) = V(S_{a,b})$ and $E(H(S_{a,b})) = E(S_{a,b}) \cup \{u_i u_{i+1} | 1 \leq i \leq a-1\} \cup \{v_i v_{i+1} | 1 \leq i \leq b-1\} \cup \{u_1 v_b\} \cup \{u_a v_1\}$.
Case 1. $\chi_L(F_a) = \chi_L(W_{a+b})$.
Since $a \geq b$ and $a+b \geq 10$ and $m = \min\{k \in \mathbb{N} \mid a \leq \frac{1}{2}(k^3 - k^2)\}$, by Theorem 3, $\chi_L(H(S_{a,b})) \geq \chi_L(F_a) = m+1$. Now, we need to construct a locating $(m+1)-$coloring of $H(S_{a,b})$.

Subcase 1.1. $\chi_L(F_a) > \chi_L(F_b)$.
Let f be a locating $(m+1)-$coloring of F_a and f_1 be a locating coloring of F_b. Without loss of generality, by Observation 1 we can choose the colorings f and f_1 such that $f(u) = m+1$, $f_1(v_1) \neq f(u_{a-1})$ and $f_1(v_b) \neq f(u_2)$. Now, construct a locating coloring f_2 of $H(S_{a,b})$ such that

$$f_2(x) = \begin{cases} f(u_i), & \text{if } x = u_i \text{ for } 1 \leq i \leq a; \\ f_1(v_i), & \text{if } x = v_i \text{ for } 1 \leq i \leq b; \\ f(u), & \text{if } x = u; \\ f_1(v), & \text{if } x = v. \end{cases}$$

Let Π, Π_1, and Π_2 be the resulting partitions by the colorings f, f_1, and f_2, respectively and R_i be the i^{th} color class induced by f_2. Let $x, y \in V(H(S_{a,b}))$ with $f_2(x) = f_2(y)$. Since $\chi_L(F_a) = \chi_L(W_{a+b})$, by coloring f_2, $c_{\Pi_2}(u) = c_{\Pi}(u)$ and $c_{\Pi_2}(u_i) = c_{\Pi}(u_i)$ for all $1 \leq i \leq a$. For each $x \in \{v, v_1, v_2, \ldots, v_b\}$, the $i^{th}-$element of $c_{\Pi_2}(x)$ is the same as the $i^{\text{th}}-$element of $c_{\Pi_1}(x)$ for every i, $1 \leq i \leq \chi_L(F_b)$ and the $(m+1)^{\text{th}}-$element of $c_{\Pi_2}(x)$ is 1 if $x = v$ and 2 otherwise. So, the remaining case to be considered is $x = u_i$ and $y = v_j$ for $1 \leq i \leq a$ and $1 \leq j \leq b$. In this case, we have $d(x, R_{m+1}) = 1 \neq 2 = d(y, R_{m+1})$. So, f_2 is a locating coloring on $H(S_{a,b})$.

Subcase 1.2. $\chi_L(F_a) = \chi_L(F_b)$.
Let $V(W_{a+b}) = \{u; u_1, u_2, \cdots, u_a, v_1, v_2, \cdots, v_b\}$ where u is the hub of W_{a+b}. Let f be a locating $(m+1)-$coloring of W_{a+b} such that $f(u) = m+1$. Let $A = \{v_i \in V(W_{a+b}) | f(v_i) = m)\}$. Now, construct a locating coloring f_1 for

$H(S_{a,b})$ such that

$$f_1(x) = \begin{cases} f(u_i), & \text{if } x = u_i \text{ for } 1 \leq i \leq a; \\ f(v_i), & \text{if } x = v_i \text{ for } 1 \leq i \leq b \text{ and } x \notin A; \\ m, & \text{if } x = v; \\ m+1, & \text{otherwise.} \end{cases}$$

Let Π and Π_1 be the resulting partitions by the colorings f and f_1, respectively. By the coloring f_1, vertex v becomes a dominant vertex. Now, consider vertex v_i with $f_1(v_i) \neq m$. For each $j \in \{1, 2, \ldots, m-1\}$, the $j^{\text{th}}-$element of the color code of v_i is the same as before. The $m^{\text{th}}-$element of the color code of v_i under coloring f_1 becomes the $(m+1)^{\text{th}}-$element of the color code of v_i under coloring f, and vice versa. Then all color codes of u_i and v_i are different. So, f_1 is a locating coloring on $H(S_{a,b})$.

Case 2. $\chi_L(F_a) < \chi_L(W_{a+b})$.

Subcase 2.1. $\chi_L(F_a) > \chi_L(F_b)$.

Since $a \geq b$ and $a + b \geq 10$ and $m = \min\{k \in \mathbb{N} \mid a \leq \frac{1}{2}(k^3 - k^2)\}$, by Theorem 3, $\chi_L(H(S_{a,b})) \geq \chi_L(F_a) = m + 1$. Now, we need to construct a locating $(m+1)-$coloring of $H(S_{a,b})$.

Let f be a locating $(m+1)-$coloring of F_a and f_1 be a locating coloring of F_b. Without loss of generality, by Observation 1 we can choose the colorings f and f_1 such that $f(u) = m + 1$, $f_1(v_1) \neq f(u_{a-1})$ and $f_1(v_b) \neq f(u_2)$. Now, construct a locating coloring f_2 of $H(S_{a,b})$ such that

$$f_2(x) = \begin{cases} f(u_i), & \text{if } x = u_i \text{ for } 1 \leq i \leq a; \\ f_1(v_i), & \text{if } x = v_i \text{ for } 1 \leq i \leq b; \\ f(u), & \text{if } x = u; \\ f_1(v), & \text{if } x = v. \end{cases}$$

Let Π, Π_1, and Π_2 be the resulting partitions by the colorings f, f_1, and f_2, respectively and R_i be the i^{th} color class induced by f_2. Let $x, y \in V(H(S_{a,b}))$ with $f_2(x) = f_2(y)$. Since $\chi_L(F_a) > \chi_L(F_b)$, by coloring f_2, $c_{\Pi_2}(u) = c_{\Pi}(u)$ and $c_{\Pi_2}(u_i) = c_{\Pi}(u_i)$ for $1 \leq i \leq a$. For each $x \in \{v, v_1, v_2, \ldots, v_b\}$, the $i^{th}-$element of $c_{\Pi_2}(x)$ is the same as the $i^{th}-$element of $c_{\Pi_1}(x)$ for every i, $1 \leq i \leq \chi_L(F_b)$ and the $(m+1)^{th}-$element of $c_{\Pi_2}(x)$ is 1 if $x = v$ and 2 otherwise. So, the remaining case to be considered is $x = u_i$ and $y = v_j$ for $1 \leq i \leq a$ and $1 \leq j \leq b$. In this case, we have $d(x, R_{m+1}) = 1 \neq 2 = d(y, R_{m+1})$. So, f_2 is a locating coloring on $H(S_{a,b})$.

Subcase 2.2. $\chi_L(F_a) = \chi_L(F_b)$.

For the lower bound, suppose that $\chi_L(H(S_{a,b})) = m + 1$. Then we have a locating $(m+1)-$coloring f of $H(S_{a,b})$ such that $f(u) = m + 1$ and $f(v) = m$.

Let $A = \{v_i \in V(H(S_{a,b})) | f(v_i) = m+1\}$. Now, construct a coloring f_1 on W_{a+b} such that

$$f_1(x) = \begin{cases} f(u_i), & \text{if } x = u_i \text{ for } 1 \le i \le a; \\ f(v_i), & \text{if } x = v_i \text{ for } 1 \le i \le b \text{ and } x \notin A; \\ m+1, & \text{if } x = u; \\ m, & \text{otherwise.} \end{cases}$$

It is easy to check that f_1 is a locating $(m+1)-$coloring on W_{a+b}. This is a contradiction with the fact that $\chi_L(F_a) < \chi_L(W_{a+b})$.
For the upper bound, let f be a locating $(m+1)-$coloring of F_a and f_1 be a locating $(m+1)-$coloring of F_b. Without loss of generality, by Observation 1 we can choose the colorings f and f_1 such that $f(u) = m+1$, $f_1(v_1) \neq f(u_{a-1})$ and $f_1(v_b) \neq f(u_2)$. Now, construct a locating coloring f_2 of $H(S_{a,b})$ such that

$$f_2(x) = \begin{cases} f(u_i), & \text{if } x = u_i \text{ for } 1 \le i \le a; \\ f_1(v_i), & \text{if } x = v_i \text{ for } 1 \le i \le b; \\ f(u), & \text{if } x = u; \\ m+2, & \text{if } x = v. \end{cases}$$

Let Π, Π_1, and Π_2 be the resulting partitions by the colorings f, f_1, and f_2, respectively and R_i be a i^{th} color class induced by f_2. Let $x, y \in V(H(S_{a,b}))$ with $f_2(x) = f_2(y)$. By coloring f_2, for every vertex in $V(H(S_{a,b}))$, the $i^{\text{th}}-$element of $c_{\Pi_2}(x)$ is same as the $i^{\text{th}}-$element of $c_\Pi(x)$ or $c_{\Pi_1}(x)$, except on $i = m+2$. So, if $x = u_i$ and $y = u_j$ or $x = v_i$ and $y = v_j$, then $c_{\Pi_2}(x) \neq c_{\Pi_2}(y)$. Therefore, the remaining case to be considered is $x = u_i$ and $y = v_j$ for $1 \le i \le a$ and $1 \le j \le b$. In this case, we have $d(x, R_{m+2}) = 2 \neq 1 = d(y, R_{m+2})$. So, f_2 is a locating coloring on $H(S_{a,b})$. □

Acknowledgment

This research was supported by the Ministry of Research, Technology and Higher Education, Indonesia with the scheme of "Program Penelitian Unggulan Perguruan Tinggi".

References

[1] Asmiati, H. Assiyatun, and E.T. Baskoro, *Locating-chromatic number of amalgamation of stars*, ITB J. Sci. **43A** (2011), no. 1, 1–8.

[2] Asmiati and E.T. Baskoro, *Characterizing all graphs containing cycles with locating-chromatic number 3*, AIP Conf. Proc. **1450** (2012), 351–357.

[3] ———, *Characterizing all trees with locating-chromatic number 3*, Electron. J. Graph Theory Appl. **1** (2013), no. 2, 109–117.

[4] Asmiati, E.T. Baskoro, H. Assiyatun, D. Suprijanto, R. Simanjuntak, and S. Uttunggadewa, *The locating-chromatic number of firecracker graphs*, Far East J. Math. Sci. **63** (2012), no. 1, 11–23.

[5] C.A. Barefoot, *Hamiltonian connectivity of the halin graphs*, Congr. Numer. **58** (1987), 93–102.

[6] E.T. Baskoro and I.A. Purwasih, *The locating-chromatic number for corona product of graphs*, Southeast Asian J. Sci. **1** (2012), no. 1, 126–136.

[7] A. Behtoei and M. Anbarloei, *The locating-chromatic number of the join of graphs*, Bull. Iranian Math. Soc. **40** (2014), no. 6, 1491–1504.

[8] A. Behtoei and B. Omoomi, *On the locating-chromatic number of cartesian product of graphs*, Ars. Combin. **126** (2016), 221–235.

[9] G. Chartrand, D. Erwin, M.A. Henning, P.J. Slater, and P. Zhang, *The locating-chromatic number of a graph*, Bull. Inst. Combin. Appl. **36** (2002), 89–101.

[10] ———, *Graphs of order n with locating-chromatic number $n-1$*, Discrete Math. **269** (2003), no. 1-3, 65–79.

[11] G. Cornuejols, D. Naddef, and W.R. Pulleyblank, *Halin graphs and the travelling salesman problem*, Math. Program. **26** (1983), no. 3, 287–294.

[12] R. Halin, *Studies on minimally n-connected graphs*, Combinatorial Mathematics and its applications (1971), 129–136.

[13] H.-H. Lai, K.-W. Lih, and P.-Y. Tsai, *The strong chromatic index of Halin graphs*, Discrete Math. **312** (2012), no. 9, 1536–1541.