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More skew-equienergetic digraphs

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Abstract: Two digraphs of same order are said to be skew-equienergetic if their skew energies are equal. One of the open problems proposed by Li and Lian was to construct non-cospectral skew-equienergetic digraphs on n vertices. Recently this problem was solved by Ramane et al. In this paper, we give some new methods to construct new skew-equienergetic digraphs.

Keywords: Energy of a graph, skew energy of a digraph, equienergetic graphs, skew-equienergetic digraphs.

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1. Introduction

Through out this paper we consider only simple graphs i.e, graphs with no multiple edges and loops. Let $G = (V, E)$ be a graph with vertex set $V(G) = V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = E$. The graph G together with an orientation σ which assigns to each edge of G a direction is called a digraph and is denoted by G^σ. Each directed edge joining the vertices v_i and v_j in G^σ with v_i and v_j being the initial and terminal vertex, respectively is known as an arc from v_i to v_j and is denoted by (v_i, v_j). The adjacency matrix of G, denoted by $A(G)$, is the $n \times n$ matrix $[a_{ij}]$, where $a_{ij} = 1$, if the vertices

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v_i and v_j are adjacent in G, otherwise $a_{ij} = 0$. We denote the adjacency spectrum of G by $(\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G))$, where $\lambda_i(G)$ $(i = 1, 2, \ldots, n)$ are the eigenvalues of $A(G)$. The energy of a graph G is denoted by $\varepsilon(G)$ and is defined to be the sum $\varepsilon(G) = \sum_{i=1}^{n} |\lambda_i(G)|$. The concept of energy of a graph was introduced by Gutman [10] with an application to chemistry (Huckel molecular orbital approximation for the total π-electron energy [12]). The energy of a graph G has been extensively studied by many mathematicians and their works can be found in [4, 5, 7, 8, 10, 11, 17] and therein references. Two graphs G_1 and G_2 of same order are said to be equienergetic graphs if $\varepsilon(G_1) = \varepsilon(G_2)$. More information about equienergetic graphs can be found in [8, 14, 18, 19, 21, 22] and therein references. Recently, Adiga et al. introduced skew-adjacency matrix and skew-energy of a digraph. The skew-adjacency matrix of a digraph G^σ of order n denoted by $S(G^\sigma)$, is the $n \times n$ matrix $[s_{ij}]$, where

$$s_{ij} = \begin{cases} 1 & \text{if there is an arc from } v_i \text{ to } v_j, \\ -1 & \text{if there is an arc from } v_j \text{ to } v_i, \\ 0 & \text{otherwise.} \end{cases}$$

$S(G^\sigma)$ is a skew-symmetric matrix and hence all its eigenvalues are purely imaginary. We denote the skew-adjacency spectrum of G^σ by $(\lambda_1(G^\sigma), \lambda_2(G^\sigma), \ldots, \lambda_n(G^\sigma))$, where $\lambda_i(G^\sigma)$ $(i = 1, 2, \ldots, n)$ are the eigenvalues of $S(G^\sigma)$. The skew-energy of a digraph G^σ, denoted by $\varepsilon_s(G)$ is the sum $\varepsilon_s(G) = \sum_{i=1}^{n} |\lambda_i(G^\sigma)|$. Works on skew-energy of a digraph can be found in [1–3, 6, 9, 13, 15, 16] and therein references. Two digraphs $G_1^{\sigma_1}$ and $G_2^{\sigma_2}$ of same order are said to be skew-equienergetic if $\varepsilon_s(G_1^{\sigma_1}) = \varepsilon_s(G_2^{\sigma_2})$.
Let G_1 and G_2 be two graphs. The join of G_1 and G_2 is the graph $G_1 \vee G_2$, obtained by joining each vertex of G_1 to every vertex of G_2. The join of $G_1^{\sigma_1}$ and $G_2^{\sigma_2}$ is the digraph $G_1^{\sigma_1} \to G_2^{\sigma_2}$, obtained by adding arcs from each vertex of $G_1^{\sigma_1}$ to every vertex of $G_2^{\sigma_2}$. Recently in [16] Li and Lian proposed the following problem.

Problem 1. [16] How to construct families of oriented graphs such that they have equal skew energy but they do not have the same spectra?

The above problem was addressed by Ramane et al. [20] and gave a method to construct skew-equienergetic digraphs. In fact they proved the following.

1. Let $G_1^{\sigma_1}$ and $G_2^{\sigma_2}$ be two digraphs of order n and m, respectively. Suppose that the in-vertex degree of each vertex v of $G_1^{\sigma_1}$ (respectively, $G_2^{\sigma_2}$) is same as the out-vertex degree of v, then

$$P_s(G_1^{\sigma_1} \to G_2^{\sigma_2}, x) = \frac{x^2 + nm}{x^2} P_s(G_1^{\sigma_1}, x) P_s(G_2^{\sigma_2}, x).$$

2. There exists a pair of skew-equienergetic digraphs of order n, for all $n \geq 6$.

Motivated by this we construct some new skew-equienergetic digraphs. The paper is organized as follows: In Section 2, we give a method to construct skew-equienegetic digraphs via equienergetic graphs, also we give an alternate proof of the Theorem 2.1 in [20]. Further we extend the class of digraphs $G_1^{\sigma_1} \to G_2^{\sigma_2}$ whose skew-adjacency spectrum is completely known, which helps us to construct new skew-equienergetic digraphs. In Section 3, we define some new join operations on digraphs and construct some new skew-equienergetic digraphs.

2. Construction of skew-equienergetic digraphs

The characteristic polynomial of a graph G is given by $P_a(G, x) := |xI - A(G)|$ and the characteristic polynomial of a digraph G^σ is given by $P_s(G^\sigma, x) := |xI - S(G^\sigma)|$. Let G be a graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$. The Duplication $D(G)$ of a graph G is a graph with vertex set $V(D(G)) = \{v_1, v_2, \dots, v_n, v_1', v_2', \dots, v_n'\}$ and edge set $E(D(G)) = \{v_i v_j' : v_i v_j \text{ is an edge in G}\}$. The Duplication $D(G^\sigma)$ of a digraph G^σ is a digraph with vertex set $V(D(G^\sigma)) = \{v_1, v_2, \dots, v_n, v_1', v_2', \dots, v_n'\}$ and arc set $E(D(G^\sigma)) = \{(v_i, v_j') : (v_i, v_j) \text{ is an arc in } G^\sigma\}$.
we need the following results to prove our main results.

Theorem 1. *[21] Let G_1 be an r_1-regular graph of order n, and G_2 be an r_2-regular graph of order m. Then the characteristic polynomial of their join $G_1 \vee G_2$ is given by*

$$P_a(G_1 \vee G_2, x) = \frac{(x - r_1)(x - r_2) - nm}{(x - r_1)(x - r_2)} P_a(G_1, x) P_a(G_2, x).$$

Theorem 2. *[21] There exists a pair of equienergetic graphs of order n for all $n \geq 9$.*

Lemma 1. *[13] Let G be a bipartite graph and G^σ be an orientation of G. If every even cycle is oriented uniformly then $Sp(G^\sigma) = iSp(G)$.*

Lemma 2. *[7] If $M, N, P,$ and Q are matrices with M being a non-singular matrix, then*

$$\begin{vmatrix} M & N \\ P & Q \end{vmatrix} = |M||Q - PM^{-1}N|.$$

An even cycle C of length $2k$ in G^σ is said to be oddly oriented (respectively, evenly oriented) if it has odd (respectively, even) number of arcs in the direction of routing. An even cycle C of length $2k$ in G^σ is said to be oriented uniformly if C is evenly oriented, when k is even and oddly oriented, when k is odd.

Theorem 3. *Let G_1 and G_2 be bipartite equienergetic graphs. If every cycle of $G_1^{\sigma_1}$ and $G_2^{\sigma_2}$ are oriented uniformly, then $G_1^{\sigma_1}$ and $G_2^{\sigma_2}$ are skew-equienergetic.*

Proof. Since $G_1^{\sigma_1}$ and $G_2^{\sigma_2}$ are oriented uniformly by Lemma 1, we have $Sp(G_1^{\sigma_1}) = iSp(G_1)$ and $Sp(G_2^{\sigma_2}) = iSp(G_2)$. Hence $\varepsilon(G_1) = \varepsilon_s(G_1^{\sigma_1})$ and $\varepsilon(G_2) = \varepsilon_s(G_2^{\sigma_2})$. Now, as G_1 and G_2 are equienergetic graphs we see that $\varepsilon_s(G_1^{\sigma_1}) = \varepsilon_s(G_2^{\sigma_2})$. □

Corollary 1. *There exists a pair of skew-equienergetic digraphs of order $2n$ for all $n \geq 9$.*

Proof. From Theorem 2, there exists a pair of graphs G_1 and G_2, both of order n $(n \geq 9)$, with $\varepsilon(G_1) = \varepsilon(G_2)$. It is easy to see that the duplication graph $D(G_i)$ (i=1,2) of G_i is bipartite and $\varepsilon(D(G_i)) = 2\varepsilon(G_i)$. So $D(G_1)$ and $D(G_2)$ are equienergetic bipartite graphs. Now let U_i, V_i be the partition sets of $D(G_i)$. Consider $D(G_i)^{\sigma_i}$, where σ_i is an orientation such that all arcs are from U_i to V_i. Clearly $D(G_1)^{\sigma_1}$ and $D(G_2)^{\sigma_2}$ are uniformly oriented. Hence by the above theorem $D(G_1)^{\sigma_1}$ and $D(G_2)^{\sigma_2}$ are skew-equienergetic. □

Ramane et al. proved the following theorem by using elementary row and column operations on determinants [20]. Here we use matrix theory and give an alternative proof (more compact) for the same.

Theorem 4. *[20] Let $G_1^{\sigma_1}$ and $G_2^{\sigma_2}$ be two digraphs of order n and m, respectively. Suppose that the in-vertex degree of each vertex v of $G_1^{\sigma_1}$ (respectively, $G_2^{\sigma_2}$) is same as the out-vertex degree of v, then*

$$P_s(G_1^{\sigma_1} \to G_2^{\sigma_2}, x) = \frac{x^2 + nm}{x^2} P_s(G_1^{\sigma_1}, x) P_s(G_2^{\sigma_2}, x).$$

Proof. We have

$$S(G_1^{\sigma_1} \to G_2^{\sigma_2}) = \begin{bmatrix} S(G_1^{\sigma_1}) & J \\ -J^T & S(G_2^{\sigma_2}) \end{bmatrix},$$

where J is the $n \times m$ matrix with all its entries are 1.
Since $S(G_1^{\sigma_1})$ and $S(G_1^{\sigma_1})$ are normal matrices, they are unitarily diagonalizable. Now, as $S(G_1^{\sigma_1})\mathbf{1} = 0$ and $S(G_2^{\sigma_2})\mathbf{1} = 0$, we have $S(G_1^{\sigma_1}) =$

$U_1D_1U_1^*$ and $S(G_2^{\sigma_2}) = U_2D_2U_2^*$, where U_1 and U_2 are unitary matrix having its first column vector as $\frac{1}{\sqrt{n}}(1,1,\ldots 1)$ and $\frac{1}{\sqrt{m}}(1,1,\ldots 1)$, respectively. Also $D_1 = diag(\lambda_1(G_1^{\sigma_1}), \lambda_2(G_1^{\sigma_1}), \ldots, \lambda_n(G_1^{\sigma_1}))$ and $D_2 = diag(\lambda_1(G_2^{\sigma_2}), \lambda_2(G_2^{\sigma_2}), \ldots, \lambda_m(G_2^{\sigma_2}))$. Hence the above equation can be rewritten as follows.

$$\begin{aligned} S(G_1^{\sigma_1} \to G_2^{\sigma_2}) &= \begin{bmatrix} U_1D_1U_1^* & J \\ -J^T & U_2D_2U_2^* \end{bmatrix} \\ &= \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix} \begin{bmatrix} D_1 & U_1^*JU_2 \\ -U_2^*J^TU_1 & D_2 \end{bmatrix} \begin{bmatrix} U_1^* & 0 \\ 0 & U_2^* \end{bmatrix} \\ &= \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix} \begin{bmatrix} D_1 & \sqrt{nm}J' \\ -\sqrt{nm}{J'}^T & D_2 \end{bmatrix} \begin{bmatrix} U_1^* & 0 \\ 0 & U_2^* \end{bmatrix}, \end{aligned}$$

where J' is the matrix obtained from J by replacing every entry by 0, except the first diagonal entry. So, by above equation we see that

$$|xI - S(G_1^{\sigma_1} \to G_2^{\sigma_2})| = \begin{vmatrix} xI_n - D_1 & -\sqrt{nm}J' \\ \sqrt{nm}{J'}^T & xI_m - D_2 \end{vmatrix}.$$

Now by applying Lemma 2 to the above equation, we obtain the following.

$$\begin{aligned} |xI - S(G_1^{\sigma_1} \to G_2^{\sigma_2})| &= |xI_n - D_1||xI_m - D_2 + nmJ'{J'}^T(xI_n - D_1)^{-1}| \\ &= \frac{(x^2+nm)}{x^2}|xI_n - D_1||xI_m - D_2|. \end{aligned}$$

Thus

$$P_s(G_1^{\sigma_1} \to G_2^{\sigma_2}, x) = \frac{x^2+nm}{x^2} P_s(G_1^{\sigma_1}, x) P_s(G_2^{\sigma_2}, x).$$

□

As an immediate consequence of the above theorem, we have the following corollary.

Corollary 2. *[20] Let $G_1^{\sigma_1}$ and $G_2^{\sigma_2}$ be two digraphs of order n and m, respectively. Suppose that the in-vertex degree of each vertex v of $G_1^{\sigma_1}$ (respectively, $G_2^{\sigma_2}$) is same as the out-vertex degree of v, then*

$$\varepsilon_s(G_1^{\sigma_1} \to G_2^{\sigma_2}, x) = \varepsilon_s(G_1^{\sigma_1}, x) + \varepsilon_s(G_2^{\sigma_2}, x) + 2\sqrt{nm}.$$

In the following remark we give a method to construct a digraph G^{γ} such that $in{-}deg_{{G_1}^{\gamma}}(v) = out{-}deg_{{G_1}^{\gamma}}(v)$, for all vertex v in ${G_1}^{\gamma}$, starting with a digraph G^{σ}.

Remark 1. Let G^{σ} be a digraph with vertex set $V(G^{\sigma}) = \{v_1, v_2, \ldots, v_n\}$. Let $D^*(G^{\sigma})$ be the digraph with vertex set $V(D^*(G^{\sigma})) = \{v_1, v_2, \ldots, v_n, v_1', v_2', \ldots, v_n'\}$ and arc set defined as follows.

a. (v_i, v_j) is an arc in $D^*(G^{\sigma})$ if (v_i, v_j) is an arc in G^{σ}.

b. (v_i, v_j') is an arc in $D^*(G^{\sigma})$ if (v_j, v_i) is an arc in G^{σ}.

c. (v_i', v_j') is an arc in $D^*(G^{\sigma})$ if (v_i, v_j) is an arc in G^{σ}.

Then $in{-}deg_{D^*(G^{\sigma})}(v) = out{-}deg_{D^*(G^{\sigma})}(v)$, for all vertex v in $D^*(G^{\sigma})$.

As an application of Theorem 4, we construct a digraph G^{σ} of order n $(n \geq 4)$ such that $\varepsilon(G) < \varepsilon_s(G^{\sigma})$.

Corollary 3. *Let G_i $(i = 1, 2)$ be r_i-regular graphs of order n_i together with an orientation σ_i such that $\varepsilon(G_i) = \varepsilon_s(G_i^{\sigma_i})$, $r_1 + r_2 \neq 0$ and $in{-}deg_{G_i^{\sigma_i}}(v) = out{-}deg_{G_i^{\sigma_i}}(v)$ for all vertices v in $G_i^{\sigma_i}$. Then*

$$\varepsilon(G_1 \vee G_2) < \varepsilon_s(G_1^{\sigma_1} \to G_2^{\sigma_2}).$$

Proof. By our hypothesis and Theorems 1 and 4, we obtain

$$\varepsilon_s(G_1^{\sigma_1} \to G_2^{\sigma_2}) - \varepsilon(G_1 \vee G_2) = 2\sqrt{n_1 n_2} + r_1 + r_2 - \sqrt{(r_1 - r_2)^2 + 4 n_1 n_2} > 0.$$

This completes the proof. □

Example 1. Let G^{σ} be a digraph as shown in Figure 1. Then

$$\varepsilon(G \vee \overline{K}_m) < \varepsilon_s(G^{\sigma} \to \overline{K}_m),$$

for all $m \geq 1$.

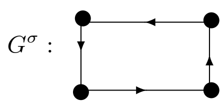

Fig. 1. 4-cycle together with an orientation σ.

The following theorems extends the class of digraphs $G_1^{\sigma_1} \to G_2^{\sigma_2}$, whose spectrum is completely known.
Let $B_i^{\sigma_i} := B_i^{\sigma_i}(U_i, W_i)$ $(i = 1, 2)$ be bipartite digraphs such that $|U_i| = |W_i| = n_i$ and $S(B_i^{\sigma_i}(U_i, W_i)) = \begin{bmatrix} 0 & X_i \\ -X_i & 0 \end{bmatrix}$, where X_i is a (0,1)-$n_i \times n_i$ matrix and $X_i\mathbf{1} = r_i\mathbf{1}$.

Theorem 5. *The characteristic polynomial of* $B_1^{\sigma_1} \to B_2^{\sigma_2}$ *is*

$$P_s(B_1^{\sigma_1} \to B_2^{\sigma_2}, x) = \frac{x^4 + (r_1^2 + r_2^2 + 4n_1n_2)x^2 + r_1^2r_2^2}{(x^2 + r_1^2)(x^2 + r_2^2)} P_s(B_1^{\sigma_1}, x)P_s(B_2^{\sigma_2}, x).$$

Proof. We have

$$S(B_1^{\sigma_1} \to B_2^{\sigma_2}) = \begin{bmatrix} \begin{matrix} 0 & X_1 \\ -X_1 & 0 \end{matrix} & J \\ -J^T & \begin{matrix} 0 & X_2 \\ -X_2 & 0 \end{matrix} \end{bmatrix},$$

where J is the $2n_1 \times 2n_2$ matrix whose entries are all 1. Denote the eigenvalues of X_i by λ_{ij}, $1 \le j \le n_i$. Using the fact that X_i $(i = 1, 2)$ are orthogonally diagonalizable and $X_i\mathbf{1} = r_i\mathbf{1}$, one can easily see that the above matrix is similar to

$$\begin{bmatrix} \begin{matrix} 0 & D_1 \\ -D_1 & 0 \end{matrix} & \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \otimes \sqrt{n_1n_2}J' \\ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \otimes -\sqrt{n_1n_2}J'^T & \begin{matrix} 0 & D_2 \\ -D_2 & 0 \end{matrix} \end{bmatrix},$$

where J' is the $n_1 \times n_2$ matrix having its first diagonal entry as 1 and remaining entries as 0 and $D_1 = diag(r_1, \lambda_{12}, \ldots, \lambda_{1n_1})$ and $D_2 = diag(r_2, \lambda_{22}, \ldots, \lambda_{2n_2})$.

So,

$$|xI - S(B_1^{\sigma_1} \to B_2^{\sigma_2})| = \begin{vmatrix} \begin{matrix} xI_{n_1} & -D_1 \\ \\ D_1 & xI_{n_1} \end{matrix} & \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \otimes -\sqrt{n_1 n_2} J' \\ \\ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \otimes \sqrt{n_1 n_2} {J'}^T & \begin{matrix} xI_{n_2} & -D_2 \\ \\ D_2 & xI_{n_2} \end{matrix} \end{vmatrix}.$$

Applying Lemma 2 to the above determinant we obtain

$$|xI - S(B_1^{\sigma_1} \to B_2^{\sigma_2})| = \begin{vmatrix} xI_{n_2} & -D_2 \\ D_2 & xI_{n_2} \end{vmatrix} \times$$
$$\left| \begin{bmatrix} xI_{n_1} & -D_1 \\ \\ D_1 & xI_{n_1} \end{bmatrix} + n_1 n_2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \otimes J' \begin{bmatrix} xI_{n_2} & -D_2 \\ \\ D_2 & xI_{n_2} \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \otimes {J'}^T \right|. \quad (1)$$

Now, since

$$\begin{bmatrix} x^2 I_{n_2} + D_2^2 & 0 \\ 0 & x^2 I_{n_2} + D_2^2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \otimes {J'}^T = \begin{bmatrix} x^2 + r_2^2 & x^2 + r_2^2 \\ x^2 + r_2^2 & x^2 + r_2^2 \end{bmatrix} \otimes {J'}^T,$$

we have

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \otimes J' \begin{bmatrix} xI_{n_2} & -D_2 \\ \\ D_2 & xI_{n_2} \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \otimes {J'}^T = \frac{2x}{(x^2 + r_2^2)} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \otimes J' {J'}^T.$$

Hence the equation (1) can be rewritten as

$$|xI - S(B_1^{\sigma_1} \to B_2^{\sigma_2})| = \begin{vmatrix} xI_{n_2} & -D_2 \\ \\ D_2 & xI_{n_2} \end{vmatrix}$$
$$\times \left| \begin{bmatrix} xI_{n_1} & -D_1 \\ \\ D_1 & xI_{n_1} \end{bmatrix} + \frac{2 n_1 n_2 x}{(x^2 + r_2^2)} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \otimes J' {J'}^T \right|$$

$$= \begin{vmatrix} xI_{n_2} & -D_2 \\ D_2 & xI_{n_2} \end{vmatrix} \times \begin{vmatrix} x + \frac{2n_1n_2x}{(x^2+r_2^2)} & 0 & -r_1 + \frac{2n_1n_2x}{(x^2+r_2^2)} & 0 \\ 0 & xI_{n_1-1} & 0 & -D_1' \\ r_1 + \frac{2n_1n_2x}{(x^2+r_2^2)} & 0 & x + \frac{2n_1n_2x}{(x^2+r_2^2)} & 0 \\ 0 & D_1' & 0 & xI_{n_1-1} \end{vmatrix},$$

where $D_1' = diag(\lambda_{12}, \lambda_{13}, \ldots, \lambda_{1n_1})$. Again applying Lemma 2 to the above equation, we see that

$$|xI - S(B_1^{\sigma_1} \to B_2^{\sigma_2})| = \frac{x^4 + (r_1^2 + r_2^2 + 4n_1n_2)x^2 + r_1^2r_2^2}{(x^2+r_1^2)(x^2+r_2^2)} P_s(B_1^{\sigma_1}, x)P_s(B_2^{\sigma_2}, x).$$

□

The following theorem follows immediately from the above theorem.

Theorem 6. *The skew-energy of the digraph $B_1^{\sigma_1} \to B_2^{\sigma_2}$ is given by*

$$\sqrt{2}\left(\sqrt{{r_1}^2 + {r_2}^2 + 4\,n_1n_2 - A} + \sqrt{{r_1}^2 + {r_2}^2 + 4\,n_1n_2 + A}\right) + \varepsilon_s(B_1^{\sigma_1}) + \varepsilon_s(B_2^{\sigma_2}) - 2(r_1 + r_2),$$

where $A = \sqrt{{r_1}^4 - 2\,{r_1}^2{r_2}^2 + 8\,{r_1}^2n_1n_2 + {r_2}^4 + 8\,{r_2}^2n_1n_2 + 16\,{n_2}^2{n_1}^2}$.

As a consequence of the above theorem we have the following corollary, which gives us a method to construct skew-equienergetic digraphs.

Corollary 4. *Let $B_i^{\sigma_i} := B_i^{\sigma_i}(U_i, W_i)$ $(i = 1, 2)$ be skew-equienergetic bipartite digraphs such that $|U_i| = |W_i| = n$ and $S(B_1^{\sigma_i}(U_i, W_i)) = \begin{bmatrix} 0 & X_i \\ -X_i & 0 \end{bmatrix}$, where X_i is a (0,1)-matrix of order n and $X_i\mathbf{1} = r_1\mathbf{1}$. Let $B_i^{\sigma_i} := B_1^{\sigma_i}(U_i, W_i)$ $(i = 3, 4)$ be skew-equienergetic bipartite digraphs such that $|U_i| = |W_i| = m$ and $S(B_1^{\sigma_i}(U_i, W_i)) = \begin{bmatrix} 0 & X_i \\ -X_i & 0 \end{bmatrix}$, where X_i is a (0,1)-matrix of order m and $X_i\mathbf{1} = r_2\mathbf{1}$. Then*

$$\varepsilon_s(B_1^{\sigma_1} \to B_3^{\sigma_3}) = \varepsilon_s((B_2^{\sigma_2} \to B_4^{\sigma_4}).$$

Remark 2. The above corollary enables us to construct skew-equienergetic digraphs via equienergetic graphs. In particular, if G_1 and G_2 are equienergetic regular graphs of same degree, also if G_3 and G_4 are equienergetic regular graphs of same degree, then

$$\varepsilon_s((D^{\sigma_1}(G_1) \to D^{\sigma_3}(G_3)) = \varepsilon_s((D^{\sigma_2}(G_2) \to D^{\sigma_4}(G_4)),$$

where $D^{\sigma_i}(G_i), (i = 1, 2, 3, 4)$ is the duplication graph together with partition sets U_i, W_i and an orientation σ_i such that all arcs are from U_i to W_i.

As the proof of the following theorem is analogous to that of Theorem 5 we omit the details.

Theorem 7. *Let G^σ be a digraph of order m such that in$-$deg$_{G^\sigma}(v)$ = out$-$deg$_{G^\sigma}(v)$, for all vertices v in G^σ. Then*

$$P_s(B_1^{\sigma_1} \to G^\sigma, x) = \frac{x^2 + r_1^2 + 2mn_1}{(x^2 + r_1^2)} P_s(B_1^{\sigma_1}, x) P_s(G^\sigma, x),$$

and

$$\varepsilon_s(B_1^{\sigma_1} \to G^\sigma) = 2\sqrt{r_1^2 + 2n_1 m} + \varepsilon_s(B_1^{\sigma_1}) + \varepsilon_s(G^\sigma) - 2r_1.$$

The following result follows immediately from the above theorem.

Corollary 5. *Let $B_i^{\sigma_i} := B_i^{\sigma_i}(U_i, W_i)$, $(i = 1, 2)$ be skew-equienergetic bipartite digraphs such that $|U_i| = |W_i| = n$ and $S(B_1^{\sigma_i}(U_i, W_i)) = \begin{bmatrix} 0 & X_i \\ -X_i & 0 \end{bmatrix}$, where X_i is a (0,1)-matrix of order n and $X_i\mathbf{1} = r_1\mathbf{1}$. Also let $G_1^{\gamma_1}$ and $G_2^{\gamma_2}$ be skew -equienergetic digraphs such that in-vertex degree of each vertex in $G_1^{\gamma_1}$ (respectively, $G_2^{\gamma_2}$) is same as the out-vertex degree. Then*

$$\varepsilon_s(B_1^{\sigma_1} \to G_1^{\gamma_1}) = \varepsilon_s(B_2^{\sigma_2} \to G_2^{\gamma_2}).$$

Corollary 6. *There exists a pair of skew-equienergetic digraphs of order $2m$, for all $m \geq 3$.*

Proof. Let C^σ be a 3-cycle as depicted in Figure 2.

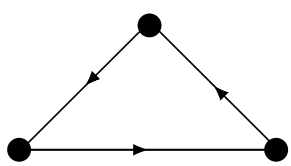

Fig. 2. 3-cycle together with an orientation σ.

Consider the digraphs $D(C^\sigma)$ and $D^*(C^\sigma)$. Clearly $S(D(C^\sigma)) = \begin{bmatrix} 0 & S(C^\sigma) \\ S(C^\sigma) & 0 \end{bmatrix}$ and $S(D^*(C^\sigma)) = \begin{bmatrix} S(C^\sigma) & -S(C^\sigma) \\ -S(C^\sigma) & S(C^\sigma) \end{bmatrix}$. Hence the skew-adjacency spectrum of $D(C^\sigma)$ and $D^*(C^\sigma)$ are $(\pm\sqrt{3}i, \pm\sqrt{3}i, 0, 0)$ and $(\pm 2\sqrt{3}i, 0, 0, 0, 0)$, respectively. And so

$$\varepsilon_s(D(C^\sigma)) = \varepsilon_s(D^*(C^\sigma)) = 2\varepsilon_s(C^\sigma) = 4\sqrt{3}.$$

Thus the digraphs $D(C^\sigma)$ and $D^*(C^\sigma)$ are skew-equienergetic of order 6. Moreover in-vertex degree of each vertex in $D(C^\sigma)$ (respectively, $D^*(C^\sigma)$) is same as the out-vertex degree. Therefore by above corollary we see that the digraphs $mK_2^\gamma \to D(C^\sigma)$ and $mK_2^\gamma \to D^*(C^\sigma)$ are skew-equienergetic for all $m \geq 1$. This completes the proof. □

3. Some new joins of digraphs

Let $G_1^{\sigma_1}$ and $G_2^{\sigma_2}$ be two bipartite digraphs with partition sets U_1, V_1 and U_2, V_2, respectively. We now define new join operations as follows

Definition 1. The join-1 of digraphs $G_1^{\sigma_1}$ and $G_2^{\sigma_2}$, denoted by $G_1^{\sigma_1} j_1 G_2^{\sigma_2}$ is a digraph obtained from $G_1^{\sigma_1}$ and $G_2^{\sigma_2}$, by adding arcs from each vertex in U_1 to every vertex in U_2 and V_2.

Definition 2. The join-2 of digraphs $G_1^{\sigma_1}$ and $G_2^{\sigma_2}$, denoted by $G_1^{\sigma_1} j_2 G_2^{\sigma_2}$ is a digraph obtained from $G_1^{\sigma_1}$ and $G_2^{\sigma_2}$, by adding arcs from each vertex in U_1 (respectively, V_1) to every vertex in U_2, (respectively, V_2).

Definition 3. The join-3 of digraphs $G_1^{\sigma_1}$ and $G_2^{\sigma_2}$, denoted by $G_1^{\sigma_1} j_3 G_2^{\sigma_2}$ is a digraph obtained from $G_1^{\sigma_1}$ and $G_2^{\sigma_2}$, by adding arcs from each vertex in U_1 to every vertex in U_2.

Definition 4. The join-4 of digraphs $G_1^{\sigma_1}$ and $G_2^{\sigma_2}$, denoted by $G_1^{\sigma_1} j_4 G_2^{\sigma_2}$ is a digraph obtained from $G_1^{\sigma_1}$ and $G_2^{\sigma_2}$, by adding arcs from each vertex in U_1 to every vertex in U_2 and V_2, also adding arcs from each vertex in V_1 to every vertex in V_2.

Definition 5. Let H^σ be a digraph. The join-5 of digraphs $G_1^{\sigma_1}$ and H^σ, denoted by $G_1^{\sigma_1} j_5 G_2^{\sigma_2}$ is a digraph obtained by $G_1^{\sigma_1}$ and H^σ, by adding arcs from each vertex in U_1 to every vertex in H^σ.

As the proof of the following theorem is similar to that of Theorem 5, we omit the details.

Theorem 8. *Let $B_i^{\sigma_i} := B_i^{\sigma_i}(U_i, W_i)$ $(i = 1, 2)$ be bipartite digraphs such that $|U_i| = |W_i| = n_i$ and $S(B_i^{\sigma_i}(U_i, W_i)) = \begin{bmatrix} 0 & X_i \\ -X_i^T & 0 \end{bmatrix}$, where X_i is either a (0,1)-symmetric matrix of order n_i or a (0,1,-1)-skew-symmetric matrix of order n_i and $X_i\mathbf{1} = r_i\mathbf{1}$. Let H^σ be a digraph of order m such that the in-vertex degree of each vertex in H^σ is same as the out-vertex degree. Then*

1. $P_s(B_1^{\sigma_1} j_1 B_2^{\sigma_2}, x) = \dfrac{x^4 + (r_1^2 + r_2^2 + 2n_1n_2)x^2 + r_1^2r_2^2}{(x^2 + r_1^2)(x^2 + r_2^2)} P_s(B_1^{\sigma_1}, x) P_s(B_2^{\sigma_2}, x).$

2. $$P_s(B_1^{\sigma_1} j_2 B_2^{\sigma_2}, x) = \frac{x^4 + (r_1^2 + r_2^2 + 2n_1n_2)x^2 + r_1^2r_2^2 - 2r_1r_2n_1n_2 + n_1^2n_2^2}{(x^2 + r_1^2)(x^2 + r_2^2)} \times P_s(B_1^{\sigma_1}, x) P_s(B_2^{\sigma_2}, x).$$

3. $P_s(B_1^{\sigma_1} j_3 B_2^{\sigma_2}, x) = \dfrac{x^4 + (r_1^2 + r_2^2 + n_1n_2)x^2 + r_1^2r_2^2}{(x^2 + r_1^2)(x^2 + r_2^2)} P_s(B_1^{\sigma_1}, x) P_s(B_2^{\sigma_2}, x).$

4. $$P_s(B_1^{\sigma_1} j_4 B_2^{\sigma_2}, x) = \frac{x^4 + (r_1^2 + r_2^2 + 3n_1n_2)x^2 + r_1^2r_2^2 - 2r_1r_2n_1n_2 + n_1^2n_2^2}{(x^2 + r_1^2)(x^2 + r_2^2)} \times P_s(B_1^{\sigma_1}, x) P_s(B_2^{\sigma_2}, x).$$

5. $P_s(B_1^{\sigma_1} j_5 H^\sigma, x) = \dfrac{x^2 + r_1^2 + n_1m}{(x^2 + r_1^2)} P_s(B_1^{\sigma_1}, x) P_s(H^\sigma, x).$

As a consequence of the above theorem, we obtain the following result.

Theorem 9. *Let $B_i^{\sigma_i} := B_i^{\sigma_i}(U_i, W_i)$ $(i = 1, 2)$ be bipartite digraphs such that $|U_i| = |W_i| = n_i$ and $S(B_i^{\sigma_i}(U_i, W_i)) = \begin{bmatrix} 0 & X_i \\ -X_i & 0 \end{bmatrix}$, where X_i is either a (0,1)-symmetric matrix of order n_i or a (0,1,-1)-skew-symmetric matrix of order n_i and $X_i\mathbf{1} = r_i\mathbf{1}$. Let H^σ be a digraph of order m such that the in-vertex degree of each vertex in H^σ is same as the out-vertex degree. Then $\varepsilon_s(B_1^{\sigma_1} j_1 B_2^{\sigma_2}), \varepsilon_s(B_1^{\sigma_1} j_2 B_2^{\sigma_2})$, $\varepsilon_s(B_1^{\sigma_1} j_3 B_2^{\sigma_2})$, $\varepsilon_s(B_1^{\sigma_1} j_4 B_2^{\sigma_2})$ and $\varepsilon_s(B_1^{\sigma_1} j_5 H^\sigma)$ are respectively*

1. $$\sqrt{2}\left(\sqrt{{r_1}^2 + {r_2}^2 + 2\,n_1n_2 - A_1} + \sqrt{{r_1}^2 + {r_2}^2 + 2\,n_1n_2 + A_1}\right) + \varepsilon_s(B_1^{\sigma_1}) + \varepsilon_s(B_2^{\sigma_2}) - 2(r_1 + r_2),$$
   where
   $$A_1 = \sqrt{{r_1}^4 - 2\,{r_1}^2{r_2}^2 + 4\,{r_1}^2n_1n_2 + {r_2}^4 + 4\,{r_2}^2n_1n_2 + 4\,{n_2}^2{n_1}^2}.$$

2. $$\sqrt{2}\left(\sqrt{{r_1}^2 + {r_2}^2 + 2n_1n_2 - A_2} + \sqrt{{r_1}^2 + {r_2}^2 + 2n_1n_2 + A_2}\right) + \varepsilon_s(B_1^{\sigma_1}) + \varepsilon_s(B_2^{\sigma_2}) - 2(r_1 + r_2),$$
   where
   $$A_2 = \sqrt{{r_1}^4 - 2\,{r_1}^2{r_2}^2 + 4\,{r_1}^2n_1n_2 + {r_2}^4 + 4\,{r_2}^2n_1n_2 + 8\,r_1r_2n_1n_2}.$$

3. $\sqrt{2}\left(\sqrt{{r_1}^2+{r_2}^2+n_1n_2-A_3}+\sqrt{{r_1}^2+{r_2}^2+n_1n_2+A_3}\right)$
$+\varepsilon_s(B_1^{\sigma_1})+\varepsilon_s(B_2^{\sigma_2})-2(r_1+r_2),$
where

$$A_3=\sqrt{{r_1}^4-2\,{r_1}^2{r_2}^2+2\,{r_1}^2n_1n_2+{r_2}^4+2\,{r_2}^2n_1n_2+{n_2}^2{n_1}^2}.$$

4. $\sqrt{2}\left(\sqrt{{r_1}^2+{r_2}^2+3n_1n_2-A_4}+\sqrt{{r_1}^2+{r_2}^2+3n_1n_2+A_4}\right)$
$+\varepsilon_s(B_1^{\sigma_1})+\varepsilon_s(B_2^{\sigma_2})-2(r_1+r_2),$
where

$$A_4=\sqrt{{r_1}^4-2\,{r_1}^2{r_2}^2+6\,{r_1}^2n_1n_2+{r_2}^4+6\,{r_2}^2n_1n_2+8\,r_1r_2n_1n_2+5n_1^2n_2^2}.$$

5. $2\sqrt{r_1^2+n_1m}+\varepsilon_s(B_1^{\sigma_1})+\varepsilon_s(H^{\sigma})-2r_1.$

The following corollary follows immediately by the above theorem.

Corollary 7. *Let $B_i^{\sigma_i}:=B_i^{\sigma_i}(U_i,W_i)$ $(i=1,2)$ be skew-equienergetic bipartite digraphs such that $|U_i|=|W_i|=n$ and $S(B_1^{\sigma_i}(U_i,W_i))=\begin{bmatrix}0 & X_i\\ -X_i & 0\end{bmatrix}$, where X_i is either a (0,1)-symmetric matrix of order n_i or a (0,1,-1)-skew-symmetric matrix of order n and $X_i\mathbf{1}=r_1\mathbf{1}$. Let $B_i^{\sigma_i}:=B_1^{\sigma_i}(U_i,W_i)$ $(i=3,4)$ be skew-equienergetic bipartite digraphs such that $|U_i|=|W_i|=m$, $S(B_1^{\sigma_i}(U_i,W_i))=\begin{bmatrix}0 & X_i\\ -X_i & 0\end{bmatrix}$, where X_i is either a (0,1)-symmetric matrix of order n_i or a (0,1,-1)-skew-symmetric matrix of order m and $X_i\mathbf{1}=r_2\mathbf{1}$. Also let $H_1^{\gamma_1}$ and $H_2^{\gamma_2}$ be skew-equienergetic digraphs such that in-vertex degree of each vertex in $H_1^{\gamma_1}$ (respectively, $H_2^{\gamma_2}$) is same as the out-vertex degree. Then*

1. $\varepsilon_s(B_1^{\sigma_1}j_1B_3^{\sigma_3})=\varepsilon_s(B_2^{\sigma_2}j_1B_4^{\sigma_4}).$
2. $\varepsilon_s(B_1^{\sigma_1}j_2B_3^{\sigma_3})=\varepsilon_s(B_2^{\sigma_2}j_2B_4^{\sigma_4}).$
3. $\varepsilon_s(B_1^{\sigma_1}j_3B_3^{\sigma_3})=\varepsilon_s(B_2^{\sigma_2}j_3B_4^{\sigma_4}).$
4. $\varepsilon_s(B_1^{\sigma_1}j_4B_3^{\sigma_3})=\varepsilon_s(B_2^{\sigma_2}j_4B_4^{\sigma_4}).$
5. $\varepsilon_s(B_1^{\sigma_1}j_5H_1^{\gamma_1})=\varepsilon_s(B_2^{\sigma_2}j_5H_2^{\gamma_2}).$

Corollary 8. *There exists a pair of skew-equienergetic digraphs of order $2m$, for all $m\geq 3$.*

Proof. The digraphs $D(C^{\sigma})$ and $D^*(C^{\sigma})$ are skew-equienergetic of order 6. Also, the in-vertex degree of each vertex in $D(C^{\sigma})$ (respectively, $D^*(C^{\sigma})$) is same as the out-vertex degree. Therefore by above corollary we see that the digraphs $mK_2^{\gamma}j_5D(C^{\sigma})$ and $mK_2^{\gamma}j_5D^*(C^{\sigma})$ are skew-equienergetic for all $m\geq 1$. This completes the proof. □

Let G^σ be digraph with vertex set $V := V(G^\sigma)$ and arc set $E := E(G^\sigma)$. We now define Mycielskian digraph of a digraph as follows.

Definition 6. The Mycielskian digraph $\mu(G^\sigma)$ is the digraph with the vertex set $V(\mu(G)) = V \cup V' \cup \{u\}$, where $V' = \{x' : x \in V\}$ and the arc set $E(\mu(G)) = E \cup \{(x, y') : (x, y) \in E\} \cup \{(x', u) : x' \in V'\}$.

Theorem 10. *Let G^σ be a digraph on n vertices such that the in- vertex degree of each vertex is same as the out-vertex degree. Then the energy of $\mu(G^\sigma)$ is given by*

$$\varepsilon_s(\mu(G^\sigma)) = 2\sqrt{n} + \sqrt{5}\varepsilon_s(G^\sigma).$$

Proof. With suitable labelling of the graph $\mu(G^\sigma)$, the skew adjacency matrix of $\mu(G^\sigma)$ can be formulated as follows.

$$S(\mu(G^\sigma)) = \begin{bmatrix} 0 & S(G^\sigma) & e \\ S(G^\sigma) & S(G^\sigma) & 0 \\ -e^T & 0 & 0 \end{bmatrix},$$

where e is the column vector of size n with all its entries are 1. So,

$$|xI - S(\mu(G^\sigma))| = \begin{vmatrix} xI_n & -S(G^\sigma) & -e \\ -S(G^\sigma) & xI_n - S(G^\sigma) & 0 \\ e^T & 0 & x \end{vmatrix}.$$

Now using Lemma 2, we see that

$$|xI - S(\mu(G^\sigma))| = x \begin{vmatrix} xI_n + (J/x) & -S(G^\sigma) \\ -S(G^\sigma) & xI_n - S(G^\sigma) \end{vmatrix},$$

where J is the square matrix of order n with all its entries are 1. Using the fact that $S(G^\sigma)$ is unitarily diagonalizable, one can rewrite the above equation as follows.

$$|xI - S(\mu(G^\sigma))| = x \begin{vmatrix} xI_n + (nJ'/x) & -D(G^\sigma) \\ -D(G^\sigma) & xI_n - D(G^\sigma) \end{vmatrix},$$

where $D = (\lambda_1(G^\sigma) = 0, \lambda_2(G^\sigma), \ldots, \lambda_n(G^\sigma))$ and J' is the $n \times n$ matrix obtained by replacing all the entries of J by 0, expect the first diagonal entry. Again applying Lemma 2 to the above equation, we have

$$\begin{aligned} |xI - S(\mu(G^\sigma))| &= x|xI_n - D(G^\sigma)||xI_n + (nJ'/x) + (xI_n - D(G^\sigma))^{-1}D^2(G^\sigma)| \\ &= x(x^2 + n)\prod_{i=2}^{n}(x^2 - \lambda_i(G^\sigma)x - \lambda_i^2(G^\sigma)). \end{aligned}$$

Thus the spectrum of $\mu(G^\sigma)$ is

$$\left\{0, \pm i\sqrt{n}, \lambda_2(G^\sigma)\left(\frac{1\pm\sqrt{5}}{2}\right), \ldots, \lambda_n(G^\sigma)\left(\frac{1\pm\sqrt{5}}{2}\right)\right\}.$$

Hence

$$\begin{aligned}\varepsilon_s(\mu(G^\sigma)) &= 2\sqrt{n} + \left(\left|\frac{1+\sqrt{5}}{2}\right| + \left|\frac{1-\sqrt{5}}{2}\right|\right)(|\lambda_2(G^\sigma)| + \ldots + |\lambda_n(G^\sigma)|) \\ &= 2\sqrt{n} + \sqrt{5}\varepsilon_s(G^\sigma).\end{aligned}$$

□

Corollary 9. *Let G^σ and H^γ be skew-equienergetic digraphs on n vertices such that the in-vertex degree of each vertex in G^σ (respectively H^γ) is same as the out-vertex degree. Then*

$$\varepsilon_s(\mu(G^\sigma)) = \varepsilon_s(\mu(H^\gamma)).$$

Theorem 11. *Let $B_1^{\sigma_1} := B_1^{\sigma_1}(U_1, W_1)$ be a bipartite digraph such that $|U_1| = |W_1| = n$ and $S(B_1^{\sigma_1}(U_1, W_1)) = \begin{bmatrix} 0 & X_1 \\ -X_1 & 0 \end{bmatrix}$, where X_1 is a (0,1) matrix of order n and $X_1\mathbf{1} = r_1\mathbf{1}$. Then*

$$\varepsilon_s(\mu(B_1^{\sigma_1})) = \sqrt{2}\left[\sqrt{3r_1^2 + 2n + A} + \sqrt{3r_1^2 + 2n - A}\right] + \sqrt{5}\varepsilon_s(B_1^{\sigma_1}),$$

where $A = \sqrt{5r_1^4 - 4r_1^2 n + 4n^2}$.

Corollary 10. *Let $B_i^{\sigma_i} := B_i^{\sigma_i}(U_i, W_i)$ $(i = 1, 2)$ be skew-equienergetic bipartite digraph such that $|U_i| = |W_i| = n$ and $S(B_i^{\sigma_i}(U_i, W_i)) = \begin{bmatrix} 0 & X_i \\ -X_i & 0 \end{bmatrix}$, where X_i is a (0,1) matrix of order n and $X_i\mathbf{1} = r_1\mathbf{1}$. Then*

$$\varepsilon_s(\mu(B_1^{\sigma_1})) = \varepsilon_s(\mu(B_2^{\sigma_2})).$$

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