Communications in Combinatorics and Optimization
Vol. xx, No. x (xxxx), pp. 1-21
https://doi.org/10.22049/cco.2024.28950.1791

CCO
Commun. Comb. Optim.

Research Article

A new measure for transmission irregularity extent of graphs

Mahdieh Azari

Department of Mathematics, Kazerun Branch, Islamic Azad University, P. O. Box: 73135-168, Kazerun, Iran
mahdieh.azari@iau.ac.ir

Received: 23 August 2023; Accepted: 29 March 2024
Published Online: 5 April 2024

Abstract: The transmission of a vertex ς in a connected graph $\mathcal{J}$ is the sum of distances between ς and all other vertices of $\mathcal{J}$. A graph $\mathcal{J}$ is called transmission regular if all vertices have the same transmission. In this paper, we propose a new graph invariant for measuring the transmission irregularity extent of transmission irregular graphs. This invariant which we call the total transmission irregularity number (TTI number for short) is defined as the sum of the absolute values of the difference of the vertex transmissions over all unordered vertex pairs of a graph. We investigate some lower and upper bounds on the TTI number which reveal its connection to a number of already established indices. In addition, we compute the TTI number for various families of composite graphs and for some chemical graphs and nanostructures derived from them.

Keywords: transmission of a vertex, transmission irregular graph, graph invariant, bound, composite graph.

AMS Subject classification: 05C12, 05C35, 05C76, 92E10.

1. Introduction

In this paper, all graphs are assumed to be simple, connected, and finite. Consider a graph $\mathcal{J}$ with vertex set $V(\mathcal{J})$ and edge set $E(\mathcal{J})$. For $\varsigma \in V(\mathcal{J})$, $N_{\mathcal{J}}(\varsigma)$ denotes the open neighborhood of the vertex ς in $\mathcal{J}$ and $d_{\mathcal{J}}(\varsigma)$ denotes the degree of ς which is the order of $N_{\mathcal{J}}(\varsigma)$. For $\iota, \varsigma \in V(\mathcal{J})$, the distance $d_{\mathcal{J}}(\iota, \varsigma)$ is the length of a shortest $\iota - \varsigma$ path in $\mathcal{J}$. The eccentricity $\varepsilon_{\mathcal{J}}(\varsigma)$ is the maximum amount of $d_{\mathcal{J}}(\iota, \varsigma)$ for any $\iota \in V(\mathcal{J})$. The maximum (minimum, resp.) amount of $\varepsilon_{\mathcal{J}}(\varsigma)$ for all $\varsigma \in V(\mathcal{J})$ is known as the diameter (radius, resp.) of $\mathcal{J}$ and shown by $d(\mathcal{J})$ ($r(\mathcal{J})$, resp.). For $\varsigma \in V(\mathcal{J})$, the transmission (also called distance sum or status) $tr_{\mathcal{J}}(\varsigma)$ is the sum of $d_{\mathcal{J}}(\iota, \varsigma)$ over all $\iota \in V(\mathcal{J})$. For each two distinct vertices $\iota, \varsigma \in V(\mathcal{J})$, $n_{\iota}(\mathcal{J})$ denotes the number of vertices of $\mathcal{J}$ lying closer to ι than to ς and $n_{\varsigma}(\mathcal{J})$ indicates the number of vertices of $\mathcal{J}$ lying closer to ς than to ι.

A *graph invariant* (also called *topological index*) is a real number associated to a graph that is invariant under isomorphism of graph.
A graph $\mathcal{J}$ in which $d_{\mathcal{J}}(\iota) = d_{\mathcal{J}}(\varsigma)$ for every $\iota, \varsigma \in V(\mathcal{J})$ is called *regular*, otherwise it is called *irregular*. For measuring irregularity of graphs, Abdo *et al.* [1] introduced the *total irregularity* of $\mathcal{J}$ as

$$irr_t(\mathcal{J}) = \sum_{\{\iota,\varsigma\}\subseteq V(\mathcal{J})} \left|d_{\mathcal{J}}(\iota) - d_{\mathcal{J}}(\varsigma)\right| = \frac{1}{2} \sum_{\iota,\varsigma\in V(\mathcal{J})} \left|d_{\mathcal{J}}(\iota) - d_{\mathcal{J}}(\varsigma)\right|,$$

where the first summation is over all unordered vertex pairs of $\mathcal{J}$. We refer the reader to [2–4, 6, 29, 30], for some recent researches on the total irregularity.
A graph $\mathcal{J}$ is called *self-centered* if $\varepsilon_{\mathcal{J}}(\iota) = \varepsilon_{\mathcal{J}}(\varsigma)$ for every $\iota, \varsigma \in V(\mathcal{J})$, otherwise it is called *non-self-centered*. In order to quantify the non-self-centrality extent of graphs, Xu *et al.* [28] proposed the *non-self-centrality number* of $\mathcal{J}$ as

$$N(\mathcal{J}) = \sum_{\{\iota,\varsigma\}\subseteq V(\mathcal{J})} \left|\varepsilon_{\mathcal{J}}(\iota) - \varepsilon_{\mathcal{J}}(\varsigma)\right| = \frac{1}{2} \sum_{\iota,\varsigma\in V(\mathcal{J})} \left|\varepsilon_{\mathcal{J}}(\iota) - \varepsilon_{\mathcal{J}}(\varsigma)\right|.$$

See, for example, [7, 15, 26, 29] for some recent results on the non-self-centrality number.
A graph $\mathcal{J}$ in which $n_\iota(\mathcal{J}) = n_\varsigma(\mathcal{J})$, for every edge $\iota\varsigma \in E(\mathcal{J})$ is said to be *distance-balanced*, otherwise it is called *distance-unbalanced*. In 2018, Došlić *et al.* [13] proposed a distance-based topological index namely *Mostar index* as a measure for the distance-unbalancedness extent of graphs. It is expressed by

$$Mo(\mathcal{J}) = \sum_{\iota\varsigma\in E(\mathcal{J})} \left|n_\iota(\mathcal{J}) - n_\varsigma(\mathcal{J})\right|,$$

in which the summation runs over all edges of $\mathcal{J}$.
A graph $\mathcal{J}$ is said to be *highly distance-balanced* if for every pair of vertices $\iota, \varsigma \in V(\mathcal{J})$, $n_\iota(\mathcal{J}) = n_\varsigma(\mathcal{J})$. In recent years, some measures for quantifying the highly distance-unbalancedness extent of graphs, i.e., the difference of graphs from being highly distance-balanced have been proposed. One of them is the *total Mostar index* (also called *distance unbalancedness index*) which was introduced by Miklaviča and Šparl [21] in 2021 as

$$Mo_t(\mathcal{J}) = \sum_{\{\iota,\varsigma\}\subseteq V(\mathcal{J})} \left|n_\iota(\mathcal{J}) - n_\varsigma(\mathcal{J})\right| = \frac{1}{2} \sum_{\iota,\varsigma\in V(\mathcal{J})} \left|n_\iota(\mathcal{J}) - n_\varsigma(\mathcal{J})\right|.$$

Another measure was proposed by Furtula [16] in 2022 as

$$NT(\mathcal{J}) = \sum_{\{\iota,\varsigma\}\subseteq V(\mathcal{J})} \left(n_\iota(\mathcal{J}) - n_\varsigma(\mathcal{J})\right)^2.$$

This index, due to the prominent rule of Nenad Trinajstić in chemical graph theory and particularly in the progress of distance-based topological indices, was named *Trinajstić index*. See, for example, [9, 12, 17–19] and specially the recent survey [5] for more details on the Mostar index and total Mostar index.
A graph $\mathcal{J}$ is *transmission regular* if $tr_{\mathcal{J}}(\iota) = tr_{\mathcal{J}}(\varsigma)$, for every vertices $\iota, \varsigma \in V(\mathcal{J})$. To provide a quantitative measure of transmission irregularity in transmission irregular graphs, i.e., the deviation of a graph from being transmission regular, a structural invariant named as *transmission irregularity* was proposed by Sharafdini and Réti [24] as

$$Irr_{Tr}(\mathcal{J}) = \sum_{\iota\varsigma \in E(\mathcal{J})} |tr_{\mathcal{J}}(\iota) - tr_{\mathcal{J}}(\varsigma)|.$$

Due to the fact that $tr_{\mathcal{J}}(\iota) - tr_{\mathcal{J}}(\varsigma) = n_{\varsigma}(\mathcal{J}) - n_{\iota}(\mathcal{J})$, for each edge $\iota\varsigma \in E(\mathcal{J})$ from [10], one can easily verified that this invariant is equal to the Mostar index. Clearly, $Irr_{Tr}(\mathcal{J}) = 0$ if and only if $\mathcal{J}$ is transmission regular.
Inspired by Abdo *et al.*'s definition for the total irregularity [1], Xu *et al.*'s definition for the non-self-centrality number [28], and Miklavič and Šparl's definition for the total Mostar index [21], we propose here a quantitative measure for the transmission irregularity of graphs. This measure which we call the *total transmission irregularity number* (TTI number for short) is defined for a graph $\mathcal{J}$ as

$$TTI(\mathcal{J}) = \sum_{\{\iota,\varsigma\} \subseteq V(\mathcal{J})} \big|tr_{\mathcal{J}}(\iota) - tr_{\mathcal{J}}(\varsigma)\big| = \frac{1}{2} \sum_{\iota,\varsigma \in V(\mathcal{J})} |tr_{\mathcal{J}}(\iota) - tr_{\mathcal{J}}(\varsigma)|.$$

Evidently, the value of $TTI(\mathcal{J})$ equals zero if and only if $\mathcal{J}$ is transmission regular and the higher values of the TTI number show the deviation of $\mathcal{J}$ from being transmission regular. Indeed, the transmission irregularity $Irr_{Tr}(\mathcal{J})$ is the contribution of pairs of adjacent vertices to the TTI number. Unlike the $Irr_{Tr}(\mathcal{J})$, $TTI(\mathcal{J})$ can be calculated immediately from the sequence of the vertex transmissions (transmission sequence) of $\mathcal{J}$. The transmission sequence $\mathcal{T}(\mathcal{J})$ of $\mathcal{J}$ is a monotonic non increasing sequence of the vertex transmissions. Assume that $\mathcal{T}(\mathcal{J})$ has exactly k distinct elements $t_1 > t_2 > \cdots > t_k$ with $l_1, l_2, \cdots, l_k$ as their respective multiplicities. Then we can express the TTI number of $\mathcal{J}$ as

$$TTI(\mathcal{J}) = \sum_{1 \le r < s \le k} l_r l_s (t_r - t_s).$$

Remark 1. It should be noted that the total transmission irregularity number and the total Mostar index are not equal in general. For example, for the path on 4 vertices, the value of the TTI number is 8, while the value of the Mostar index is 6.

The aim of this research is to investigate certain mathematical properties of the TTI number. We compute the values of the TTI number for some familiar graphs. Moreover, we give some bounds (upper and lower) on the TTI number which connects this invariant to some already established indices like the Wiener index [27], eccentric connectivity index [25], total eccentricity, total irregularity, and transmission irregularity. In addition, we study the TTI number for various families of composite graphs including join, disjunction, symmetric difference, Indu-Bala product, lexicographic product, generalized hierarchical product, Cartesian product, rooted product and corona product and apply our formulae for computing the TTI number for some chemical graphs and nanostructures.

2. Preliminaries

As usual, the notations P_ν, C_ν, K_ν, and $\bar{K}_\nu$ are respectively used for the path, cycle, complete graph, and empty graph on ν vertices. One can easily check that, $TTI(C_\nu) = TTI(K_\nu) = TTI(\bar{K}_\nu) = 0$.

Lemma 1. *The TTI number of P_ν is given by*

$$TTI(P_\nu) = \begin{cases} \frac{\nu^2(\nu^2-4)}{24} & \text{if } \nu \text{ is even,} \\ \frac{(\nu^2-1)(\nu^2-3)}{24} & \text{if } \nu \text{ is odd.} \end{cases} \tag{2.1}$$

Proof. Label the vertices of P_ν as $1, 2, \cdots, \nu$, consecutively. For the ςth vertex of P_ν, we have

$$\begin{aligned} tr_{P_\nu}(\varsigma) =& 1 + 2 + \ldots + (\varsigma - 1) + 1 + 2 + \ldots + (\nu - \varsigma) \\ =& \frac{\varsigma(\varsigma-1)}{2} + \frac{(\nu-\varsigma)(\nu-\varsigma+1)}{2} \\ =& \varsigma^2 - \varsigma(\nu+1) + \binom{\nu+1}{2}. \end{aligned}$$

Then,

$$\begin{aligned} TTI(P_\nu) =& \sum_{1\le\iota<\varsigma\le\nu} |tr_{P_\nu}(\iota) - tr_{P_\nu}(\varsigma)| \\ =& \sum_{1\le\iota<\varsigma\le\nu} \Big|(\iota^2 - \iota(\nu+1) + \binom{\nu+1}{2}) - (\varsigma^2 - \varsigma(\nu+1) + \binom{\nu+1}{2})\Big| \\ =& \sum_{1\le\iota<\varsigma\le\nu} |\iota^2 - \varsigma^2 - (\nu+1)(\iota-\varsigma)| \\ =& \sum_{\iota=1}^{\lfloor\frac{\nu}{2}\rfloor} \Big(\sum_{\varsigma=\iota+1}^{\nu+1-\iota} [\iota^2 - \varsigma^2 - (\nu+1)(\iota-\varsigma)] + \sum_{\varsigma=\nu+2-\iota}^{\nu} [\varsigma^2 - \iota^2 + (\nu+1)(\iota-\varsigma)] \Big) \\ &+ \sum_{\iota=\lfloor\frac{\nu}{2}\rfloor+1}^{\nu-1} \sum_{\varsigma=\iota+1}^{\nu} [\varsigma^2 - \iota^2 + (\nu+1)(\iota-\varsigma)]. \end{aligned}$$

Eq. (2.1) can be obtained after a direct calculation. □

Lemma 2. *Consider a graph $\mathcal{J}$ with ν vertices. For any $\varsigma \in V(\mathcal{J})$,*

$$2(\nu-1) - d_{\mathcal{J}}(\varsigma) \leq tr_{\mathcal{J}}(\varsigma) \leq d_{\mathcal{J}}(\varsigma) + (\nu - 1 - d_{\mathcal{J}}(\varsigma))\varepsilon_{\mathcal{J}}(\varsigma).$$

The equality holds in both sides if and only if $\varepsilon_{\mathcal{J}}(\varsigma) \leq 2$.

Proof. To prove the left hand side inequality for $\varsigma \in V(\mathcal{J})$, we obtain

$$\begin{aligned} tr_{\mathcal{J}}(\varsigma) = &\sum_{\iota\varsigma \in E(\mathcal{J})} 1 + \sum_{\iota\varsigma \notin E(\mathcal{J})} d_{\mathcal{J}}(\iota,\varsigma) \geq \sum_{\iota\varsigma \in E(\mathcal{J})} 1 + \sum_{\iota\varsigma \notin E(\mathcal{J})} 2 \\ =& d_{\mathcal{J}}(\varsigma) + 2(\nu - 1 - d_{\mathcal{J}}(\varsigma)) = 2(\nu-1) - d_{\mathcal{J}}(\varsigma). \end{aligned}$$

The left hand side equality holds if and only if for each $\iota\varsigma \notin E(\mathcal{J})$, $d_{\mathcal{J}}(\iota,\varsigma) = 2$, from which $\varepsilon_{\mathcal{J}}(\varsigma) \leq 2$. To prove the right hand side inequality, we have

$$\begin{aligned} tr_{\mathcal{J}}(\varsigma) = &\sum_{\iota\varsigma \in E(\mathcal{J})} 1 + \sum_{\iota\varsigma \notin E(\mathcal{J})} d_{\mathcal{J}}(\iota,\varsigma) \leq \sum_{\iota\varsigma \in E(\mathcal{J})} 1 + \sum_{\iota\varsigma \notin E(\mathcal{J})} \varepsilon_{\mathcal{J}}(\varsigma) \\ =& d_{\mathcal{J}}(\varsigma) + (\nu - 1 - d_{\mathcal{J}}(\varsigma))\varepsilon_{\mathcal{J}}(\varsigma). \end{aligned}$$

The right hand side equality holds if and only if for each $\iota\varsigma \notin E(\mathcal{J})$, $\varepsilon_{\mathcal{J}}(\varsigma) = 2$, from which $\varepsilon_{\mathcal{J}}(\varsigma) \leq 2$. □

Lemma 3. [14] *Let $\mathcal{J}$ be a graph of order ν and size μ. For any $\varsigma \in V(\mathcal{J})$,*

$$\nu - 1 \leq tr_{\mathcal{J}}(\varsigma) \leq \frac{1}{2}(\nu-1)(\nu+2) - \mu.$$

The equality holds in left side if and only if $\varepsilon_{\mathcal{J}}(\varsigma) = 1$ and the right bound is attained for each μ, $\nu - 1 \leq \mu \leq \binom{\nu}{2}$.

3. Relation with other parameters

In this section, we give some bounds (upper and lower) for the $TTI(\mathcal{J})$, where $\mathcal{J}$ is assumed to be a graph including ν vertices and μ edges. The given bounds are usually in terms of some structural parameter and/or certain already established indices of $\mathcal{J}$.

Theorem 1. *If $\mathcal{J}$ has diameter at most 2, then*

$$TTI(\mathcal{J}) = irr_t(\mathcal{J}).$$

Proof. Since $d(\mathcal{J}) \leq 2$, by Lemma 2, $tr_{\mathcal{J}}(\varsigma) = 2(\nu - 1) - d_{\mathcal{J}}(\varsigma)$. Now by definition of the TTI number, we get

$$\begin{aligned} TTI(\mathcal{J}) =& \frac{1}{2} \sum_{\iota,\varsigma \in V(\mathcal{J})} |tr_{\mathcal{J}}(\iota) - tr_{\mathcal{J}}(\varsigma)| \\ =& \frac{1}{2} \sum_{\iota,\varsigma \in V(\mathcal{J})} |(2(\nu-1) - d_{\mathcal{J}}(\iota)) - (2(\nu-1) - d_{\mathcal{J}}(\varsigma))| \\ =& \frac{1}{2} \sum_{\iota,\varsigma \in V(\mathcal{J})} |d_{\mathcal{J}}(\iota) - d_{\mathcal{J}}(\varsigma)| = irr_t(\mathcal{J}), \end{aligned}$$

from which the result follows. □

Theorem 2. *If $\mathcal{J}$ has diameter at most 2, then*

$$Mo_t(\mathcal{J}) = irr_t(\mathcal{J}).$$

Proof. Let ι, ς be a pair of distinct vertices in $V(\mathcal{J})$. If ι, ς are adjacent, then the vertex ι and all vertices of $\mathcal{J}$ other than ς which are adjacent to ι but not to ς are lying closer to ι than to ς. Hence, $n_\iota(\mathcal{J}) = d_{\mathcal{J}}(\iota) - |N_\iota(\mathcal{J}) \cap N_\varsigma(\mathcal{J})|$ and $n_\varsigma(\mathcal{J}) = d_{\mathcal{J}}(\varsigma) - |N_\iota(\mathcal{J}) \cap N_\varsigma(\mathcal{J})|$. If ι, ς are not adjacent, then the vertex ι and all vertices of $\mathcal{J}$ which are adjacent to ι but not to ς are lying closer to ι than to ς. Hence, $n_\iota(\mathcal{J}) = 1 + d_{\mathcal{J}}(\iota) - |N_\iota(\mathcal{J}) \cap N_\varsigma(\mathcal{J})|$ and $n_\varsigma(\mathcal{J}) = 1 + d_{\mathcal{J}}(\varsigma) - |N_\iota(\mathcal{J}) \cap N_\varsigma(\mathcal{J})|$. Now by definition of the total Mostar index, we have

$$\begin{aligned} Mo_t(\mathcal{J}) =& \sum_{\{\iota,\varsigma\} \subseteq V(\mathcal{J})} \big|n_\iota(\mathcal{J}) - n_\varsigma(\mathcal{J})\big| \\ =& \sum_{\iota\varsigma \in E(\mathcal{J})} \big|\big(d_{\mathcal{J}}(\iota) - |N_\iota(\mathcal{J}) \cap N_\varsigma(\mathcal{J})|\big) - \big(d_{\mathcal{J}}(\varsigma) - |N_\iota(\mathcal{J}) \cap N_\varsigma(\mathcal{J})|\big)\big| \\ &+ \sum_{\iota\varsigma \notin E(\mathcal{J})} \big|\big(1 + d_{\mathcal{J}}(\iota) - |N_\iota(\mathcal{J}) \cap N_\varsigma(\mathcal{J})|\big) \\ &- \big(1 + d_{\mathcal{J}}(\varsigma) - |N_\iota(\mathcal{J}) \cap N_\varsigma(\mathcal{J})|\big)\big| \\ =& \sum_{\iota\varsigma \in E(\mathcal{J})} \big|d_{\mathcal{J}}(\iota) - d_{\mathcal{J}}(\varsigma)\big| + \sum_{\iota\varsigma \notin E(\mathcal{J})} \big|d_{\mathcal{J}}(\iota) - d_{\mathcal{J}}(\varsigma)\big| = irr_t(G). \end{aligned}$$

This completes the proof. □

Combining Theorems 1 and 2 and considering the facts that for any graph $\mathcal{J}$, $Mo(\mathcal{J}) = Irr_{Tr}(\mathcal{J})$ and for any graph $\mathcal{J}$ with $d(\mathcal{J}) \leq 2$, $irr_t(\mathcal{J}) = Mo(\mathcal{J})$ from [12], we arrive at:

Corollary 1. *If $\mathcal{J}$ has diameter at most 2, then*

$$TTI(\mathcal{J}) = Mo_t(\mathcal{J}) = Irr_{Tr}(\mathcal{J}).$$

The *Wiener number* [27], the first and the most famous distance-based invariant, is formulated for a graph $\mathcal{J}$ as

$$W(\mathcal{J}) = \sum_{\{\iota,\varsigma\}\subseteq V(\mathcal{J})} d_{\mathcal{J}}(\iota,\varsigma) = \frac{1}{2}\sum_{\varsigma\in V(\mathcal{J})} tr_{\mathcal{J}}(\varsigma).$$

Here, a sharp upper bound on the TTI number in terms of the Wiener number is given.

Theorem 3.

$$TTI(\mathcal{J}) \leq \nu W(\mathcal{J}) - \nu^2(\nu-1) + \nu\mu, \tag{3.1}$$

with equality if and only if $d(\mathcal{J}) \leq 2$.

Proof. By Lemma 2,

$$\begin{aligned} TTI(\mathcal{J}) =& \frac{1}{2}\sum_{\iota,\varsigma\in V(\mathcal{J})} |tr_{\mathcal{J}}(\iota) - tr_{\mathcal{J}}(\varsigma)| \\ \leq& \frac{1}{2}\sum_{\iota,\varsigma\in V(\mathcal{J})} \big(tr_{\mathcal{J}}(\iota) - (2(\nu-1) - d_{\mathcal{J}}(\varsigma))\big) \\ =& \frac{1}{2}(2\nu W(\mathcal{J}) - 2\nu^2(\nu-1) + 2\nu\mu) \\ =& \nu W(\mathcal{J}) - \nu^2(\nu-1) + \nu\mu, \end{aligned}$$

from which Eq. (3.1) follows. By Lemma 2 the equality happens in Eq. (3.1) if and only if $\varepsilon_{\mathcal{J}}(\varsigma) \leq 2$ for all $\varsigma \in V(\mathcal{J})$, from which $d(\mathcal{J}) \leq 2$. □

The *eccentric connectivity index* $\xi^c(\mathcal{J})$ [25] and *total eccentricity* $\zeta(\mathcal{J})$ are among the most famous distance-based graph invariants of $\mathcal{J}$ which are respectively defined as

$$\xi^c(\mathcal{J}) = \sum_{\varsigma\in V(\mathcal{J})} d_{\mathcal{J}}(\varsigma)\varepsilon_{\mathcal{J}}(\varsigma), \quad \zeta(\mathcal{J}) = \sum_{\varsigma\in V(\mathcal{J})} \varepsilon_{\mathcal{J}}(\varsigma).$$

In what follows, a sharp upper bound on the TTI number based on the Wiener number, eccentric connectivity index, and total eccentricity is presented.

Theorem 4.

$$TTI(\mathcal{J}) \leq \binom{\nu}{2}\zeta(\mathcal{J}) - \frac{\nu}{2}\xi^c(\mathcal{J}) - \nu W(\mathcal{J}) + \nu\mu, \tag{3.2}$$

with equality if and only if $d(\mathcal{J}) \leq 2$.

Proof. By Lemma 2,

$$\begin{aligned}TTI(\mathcal{J}) =& \frac{1}{2}\sum_{\iota,\varsigma\in V(\mathcal{J})} |tr_{\mathcal{J}}(\iota) - tr_{\mathcal{J}}(\varsigma)| \\ \le& \frac{1}{2}\sum_{\iota,\varsigma\in V(\mathcal{J})} \Big[\big(d_{\mathcal{J}}(\iota) + (\nu - 1 - d_{\mathcal{J}}(\iota))\varepsilon_{\mathcal{J}}(\iota)\big) - tr_{\mathcal{J}}(\varsigma)\Big] \\ =& \frac{1}{2}\big(2\nu\mu + \nu(\nu-1)\zeta(\mathcal{J}) - \nu\xi^{c}(\mathcal{J}) - 2\nu W(\mathcal{J})\big),\end{aligned}$$

from which Eq. (3.2) follows and the equality holds if and only if $d(\mathcal{J}) \le 2$. □

Theorem 5.

$$TTI(\mathcal{J}) \le \big(\frac{1}{2}(\nu-1)(\nu+2) - \mu\big)\binom{\nu}{2} - \nu W(\mathcal{J}). \tag{3.3}$$

The bound is attained for each μ, $\nu - 1 \le \mu \le \binom{\nu}{2}$.

Proof. By Lemma 3,

$$\begin{aligned}TTI(\mathcal{J}) =& \sum_{\{\iota,\varsigma\}\subseteq V(\mathcal{J})} |tr_{\mathcal{J}}(\iota) - tr_{\mathcal{J}}(\varsigma)| \\ \le& \sum_{\{\iota,\varsigma\}\subseteq V(\mathcal{J})} \big(\frac{1}{2}(\nu-1)(\nu+2) - \mu - tr_{\mathcal{J}}(\varsigma)\big) \\ =& \big(\frac{1}{2}(\nu-1)(\nu+2) - \mu\big)\binom{\nu}{2} - \nu W(\mathcal{J}),\end{aligned}$$

from which Eq. (3.3) follows. By Lemma 3 the bound is attained for each μ, $\nu - 1 \le \mu \le \binom{\nu}{2}$. □

Theorem 6.

$$TTI(\mathcal{J}) \le \binom{\nu}{2}\big(\binom{\nu}{2} - \mu\big), \tag{3.4}$$

with equality if and only if $\mathcal{J} \cong K_{\nu}$.

Proof. By Lemma 3,

$$\begin{aligned}TTI(\mathcal{J}) =& \sum_{\{\iota,\varsigma\}\subseteq V(\mathcal{J})} |tr_{\mathcal{J}}(\iota) - tr_{\mathcal{J}}(\varsigma)| \\ \le& \sum_{\{\iota,\varsigma\}\subseteq V(\mathcal{J})} \big(\frac{1}{2}(\nu-1)(\nu+2) - \mu - (\nu-1)\big) \\ =& \sum_{\{\iota,\varsigma\}\subseteq V(\mathcal{J})} \big(\binom{\nu}{2} - \mu\big),\end{aligned}$$

from which Eq. (3.4) follows. The equality holds in Eq. (3.4), if and only if for each $\{\iota, \varsigma\} \subseteq V(\mathcal{J})$, $tr_{\mathcal{J}}(\iota) = \frac{1}{2}(\nu-1)(\nu+2) - \mu$ and $tr_{\mathcal{J}}(\varsigma) = \nu - 1$, from which $\mathcal{J} \cong K_\nu$. □

Applying Theorem 6, we can get a Nordhaus-Gaddum type result for the TTI number.

Corollary 2.

$$TTI(\mathcal{J}) + TTI(\bar{\mathcal{J}}) < \binom{\nu}{2}^2.$$

We introduce $W^{(2)}(\mathcal{J})$ as

$$W^{(2)}(\mathcal{J}) = \frac{1}{2} \sum_{\varsigma \in V(\mathcal{J})} tr_{\mathcal{J}}(\varsigma)^2.$$

In what follows, a sharp upper bound on the $TTI(\mathcal{J})$ in terms of $W(\mathcal{J})$ and $W^{(2)}(\mathcal{J})$ is obtained.

Theorem 7.

$$TTI(\mathcal{J}) \leq \sqrt{\binom{\nu}{2}\big(2\nu W^{(2)}(\mathcal{J}) - 4W(\mathcal{J})^2\big)}, \tag{3.5}$$

with equality if and only if $\mathcal{J}$ is a transmission regular graph.

Proof. Applying Cauchy–Schwarz inequality yields:

$$\begin{aligned} TTI(\mathcal{J})^2 &= \Big(\sum_{\{\iota,\varsigma\} \subseteq V(\mathcal{J})} |tr_{\mathcal{J}}(\iota) - tr_{\mathcal{J}}(\varsigma)| \Big)^2 \\ &\leq \binom{\nu}{2} \sum_{\{\iota,\varsigma\} \subseteq V(\mathcal{J})} |tr_{\mathcal{J}}(\iota) - tr_{\mathcal{J}}(\varsigma)|^2 \\ &= \frac{1}{2}\binom{\nu}{2} \sum_{\iota,\varsigma \in V(\mathcal{J})} \big(tr_{\mathcal{J}}(\iota)^2 + tr_{\mathcal{J}}(\varsigma)^2 - 2tr_{\mathcal{J}}(\iota)tr_{\mathcal{J}}(\varsigma) \big) \\ &= \binom{\nu}{2}\big(2\nu W^{(2)}(\mathcal{J}) - 4W(\mathcal{J})^2\big), \end{aligned}$$

from which Eq. (3.5) follows. The equality holds in (3.5), if and only if for each $\{\iota, \varsigma\} \subseteq V(\mathcal{J})$, $|tr_{\mathcal{J}}(\iota) - tr_{\mathcal{J}}(\varsigma)|$ is constant, from which we deduce that $\mathcal{J}$ is a transmission regular graph. □

Here, we obtain a lower bound on the TTI number of a non-complete graph $\mathcal{J}$ based on the order, size, and Wiener index of $\mathcal{J}$.

Theorem 8. *If $\mathcal{J}$ is non-complete, then*

$$TTI(\mathcal{J}) > \frac{2\nu W^{(2)}(\mathcal{J}) - 4W(\mathcal{J})^2}{\binom{\nu}{2} - \mu}. \tag{3.6}$$

Proof. By Lemma 3, for every $\{\iota, \varsigma\} \subseteq V(\mathcal{J})$,

$$|tr_{\mathcal{J}}(\iota) - tr_{\mathcal{J}}(\varsigma)| \leq \frac{1}{2}(\nu - 1)(\nu + 2) - m - (\nu - 1) = \binom{\nu}{2} - \mu.$$

Hence

$$\begin{aligned} 2\nu W^{(2)}(\mathcal{J}) - 4W(\mathcal{J})^2 &= \sum_{\{\iota,\varsigma\}\subseteq V(\mathcal{J})} |tr_{\mathcal{J}}(\iota) - tr_{\mathcal{J}}(\varsigma)|^2 \\ &< (\binom{\nu}{2} - \mu) \sum_{\{\iota,\varsigma\}\subseteq V(\mathcal{J})} |tr_{\mathcal{J}}(\iota) - tr_{\mathcal{J}}(\varsigma)| \\ &= (\binom{\nu}{2} - \mu) TTI(\mathcal{J}), \end{aligned}$$

from which Eq. (3.6) is concluded. The inequality in (3.6) is strict since $\mathcal{J}$ is a non-complete graph. □

4. Composite graphs

In this section, we study the TTI number for various families of composite graphs. We denote the components of each composite graphs by $\mathcal{J}_1$ and $\mathcal{J}_2$. The order and size of $\mathcal{J}_r$ are depicted by ν_r and μ_r, respectively, $r = 1, 2$. For composite graphs three or more components, the values of subscripts alter correspondingly. See, for example, [2, 7–9, 22, 23] for more information on graph invariants of composite graphs.
The *join* or *sum* $\mathcal{J}_1 \nabla \mathcal{J}_2$ is a graph with $V(\mathcal{J}_1 \nabla \mathcal{J}_2) = V(\mathcal{J}_1) \cup V(\mathcal{J}_2)$ and $E(\mathcal{J}_1 \nabla \mathcal{J}_2) = E(\mathcal{J}_1) \cup E(\mathcal{J}_2) \cup \{\varsigma_1\varsigma_2 : \varsigma_1 \in V(\mathcal{J}_1), \varsigma_2 \in V(\mathcal{J}_2)\}$.
We first consider the join of the empty graph $\bar{K}_1$ with an arbitrary graph $\mathcal{J}$ which is called the *suspension* of $\mathcal{J}$.

Theorem 9. *Let $\mathcal{J}$ be a graph of order ν and size μ. Then*

$$TTI(\bar{K}_1 \nabla \mathcal{J}) = irr_t(\mathcal{J}) + \nu(\nu - 1) - 2\mu. \tag{4.1}$$

Proof. As the diameter of $\bar{K}_1\nabla\mathcal{J}$ is at most 2, Theorem 1 implies

$$\begin{aligned} TTI(\bar{K}_1\nabla\mathcal{J}) =& irr_t(\bar{K}_1\nabla\mathcal{J}) = \sum_{\{\iota,\varsigma\}\subseteq V(\bar{K}_1\nabla\mathcal{J})} \left|d_{\bar{K}_1\nabla\mathcal{J}}(\iota) - d_{\bar{K}_1\nabla\mathcal{J}}(\varsigma)\right| \\ =& \sum_{\{\iota,\varsigma\}\subseteq V(\mathcal{J})} \left|(d_{\mathcal{J}}(\iota)+1) - (d_{\mathcal{J}}(\varsigma)+1)\right| + \sum_{\varsigma\in V(\mathcal{J})} \left|\nu - (d_{\mathcal{J}}(\varsigma)+1)\right| \\ =& \sum_{\{\iota,\varsigma\}\subseteq V(\mathcal{J})} \left|d_{\mathcal{J}}(\iota) - d_{\mathcal{J}}(\varsigma)\right| + \sum_{\varsigma\in V(\mathcal{J})} \left(\nu - 1 - d_{\mathcal{J}}(\varsigma)\right) \\ =& irr_t(\mathcal{J}) + \nu(\nu-1) - 2\mu, \end{aligned}$$

from which Eq. (4.1) follows. □

Star graph S_ν, *Fan graph* F_ν, *wheel graph* W_ν, and *Windmill graph* $D_\nu^{(\mu)}$ are suspension of $\bar{K}_{\nu-1}$, $P_{\nu-1}$, $C_{\nu-1}$, and $mK_{\nu-1}$, respectively, where $\mu K_{\nu-1}$ is the union of μ copies of $K_{\nu-1}$. Applying Theorem 9 yields:

Corollary 3. *For* $\nu \geq 4$,

(i) $TTI(S_\nu) = (\nu-1)(\nu-2)$;

(ii) $TTI(F_\nu) = \nu(\nu-3)$;

(iii) $TTI(W_\nu) = (\nu-1)(\nu-4)$;

(iv) $TTI(D_\nu^{(\mu)}) = \mu(\mu-1)(\nu-1)^2$.

By Theorem 1, for the ν*-gonal* μ*-cone graph* $C_\nu\nabla\bar{K}_\mu$, we have

Corollary 4.

$$TTI(C_\nu\nabla\bar{K}_\mu) = \nu\mu|\mu-\nu+2|.$$

Now, we consider the join of two arbitrary graphs. Note that $\mathcal{J}_1\nabla\mathcal{J}_2$ is of diameter at most 2. Now by Theorem 1 and the upper bound presented for $irr_t(\mathcal{J}_1\nabla\mathcal{J}_2)$ in Theorem 2 of [2], we get the following corollary.

Theorem 10. *If* $\nu_1 \geq \nu_2$, *then*

$$TTI(\mathcal{J}_1\nabla\mathcal{J}_2) \leq irr_t(\mathcal{J}_1) + irr_t(\mathcal{J}_2) + \nu_2(\nu_1-1)(\nu_1-2).$$

In addition, the bound is best possible.

In Theorem 10, if $\mathcal{J}_1$ is any tree of order ν_1 and $\mathcal{J}_2$ is complete graph of order ν_2, then the equality holds and hence the presented bound is best possible.
The *complete* p*-partite graph* $K_{\nu_1,\nu_2,\ldots,\nu_p}$ is a join of $\bar{K}_{\nu_1}, \bar{K}_{\nu_2}, \cdots, \bar{K}_{\nu_p}$ and according to Theorem 1, we get:

Corollary 5.

$$TTI(K_{\nu_1,\nu_2,\ldots,\nu_p}) = \sum_{r=1}^{p-1}\sum_{s=r+1}^{p} \nu_r\nu_s|\nu_s-\nu_r|.$$

The *disjunction* $\mathcal{J}_1 \vee \mathcal{J}_2$ is a graph with $V(\mathcal{J}_1 \vee \mathcal{J}_2) = V(\mathcal{J}_1) \times V(\mathcal{J}_2)$ and $E(\mathcal{J}_1 \vee \mathcal{J}_2) = \{(\iota_1,\iota_2)(\varsigma_1,\varsigma_2) : \iota_1\varsigma_1 \in E(\mathcal{J}_1) \text{ or } \iota_2\varsigma_2 \in E(\mathcal{J}_2)\}$. Due to the fact that, $\mathcal{J}_1 \vee \mathcal{J}_2$ has diameter at most 2, applying Theorem 1 and the upper bound for $irr_t(\mathcal{J}_1 \vee \mathcal{J}_2)$ obtained in Theorem 8 of [2] yield:

Theorem 11.

$$TTI(\mathcal{J}_1 \vee \mathcal{J}_2) \leq \nu_2\big({\nu_2}^2 + 2\mu_2\big)irr_t(\mathcal{J}_1) + \nu_1\big({\nu_1}^2 + 2\mu_1\big)irr_t(\mathcal{J}_2).$$

The *symmetric difference* $\mathcal{J}_1 \oplus \mathcal{J}_2$ is a graph with $V(\mathcal{J}_1 \oplus \mathcal{J}_2) = V(\mathcal{J}_1) \times V(\mathcal{J}_2)$ and $E(\mathcal{J}_1 \oplus \mathcal{J}_2) = \{(\iota_1,\iota_2)(\varsigma_1,\varsigma_2) : \iota_1\varsigma_1 \in E(\mathcal{J}_1) \text{ or } \iota_2\varsigma_2 \in E(\mathcal{J}_2), \text{but not both}\}$. As the diameter of $\mathcal{J}_1 \oplus \mathcal{J}_2$ equals 2, Theorem 1 and the upper bound for $irr(\mathcal{J}_1 \oplus \mathcal{J}_2)$ given in Theorem 9 of [2] imply:

Theorem 12.

$$TTI(\mathcal{J}_1 \oplus \mathcal{J}_2) \leq \nu_2\big({\nu_2}^2 + 4\mu_2\big)irr_t(\mathcal{J}_1) + \nu_1\big({\nu_1}^2 + 4\mu_1\big)irr_t(\mathcal{J}_2).$$

The *Indu-Bala product* $\mathcal{J}_1 \diamondsuit \mathcal{J}_2$ is a graph made from two disjoint copies of $\mathcal{J}_1 \nabla \mathcal{J}_2$ by joining each vertex in one copy of $\mathcal{J}_2$ to its corresponding vertex in another copy of $\mathcal{J}_2$ (see [20]).

Theorem 13. *If $\nu_1 \geq \nu_2$ and $\mathcal{J}_1$ is not trivial, then*

$$TTI(\mathcal{J}_1 \diamondsuit \mathcal{J}_2) \leq 8irr_t(\mathcal{J}_1) + 16irr_t(\mathcal{J}_2) + 8\nu_2(\nu_1^2 - \nu_1 + 1). \tag{4.2}$$

In addition, the bound is best possible.

Proof. If $\varsigma \in V(\mathcal{J}_1)$, then

$$\begin{aligned} tr_{\mathcal{J}_1\diamondsuit\mathcal{J}_2}(\varsigma) &= \sum_{\iota\varsigma\in E(\mathcal{J}_1)} 1 + \sum_{\iota\varsigma\notin E(\mathcal{J}_1)} 2 + \sum_{\varsigma\in V(\mathcal{J}_2)} (1+2) + \sum_{\varsigma\in V(\mathcal{J}_1)} 3 \\ &= d_{\mathcal{J}_1}(\varsigma) + 2\big(\nu_1 - 1 - d_{\mathcal{J}_1}(\varsigma)\big) + 3\nu_2 + 3\nu_1 = 5\nu_1 + 3\nu_2 - d_{\mathcal{J}_1}(\varsigma) - 2, \end{aligned}$$

and if $\varsigma \in V(\mathcal{J}_2)$, then

$$\begin{aligned} tr_{\mathcal{J}_1\diamondsuit\mathcal{J}_2}(\varsigma) &= \sum_{\iota\varsigma\in E(\mathcal{J}_2)} (1+2) + \sum_{\iota\varsigma\notin E(\mathcal{J}_2)} (2+3) + 1 + \sum_{\iota\in V(\mathcal{J}_1)} (1+2) \\ &= 3d_{\mathcal{J}_2}(\varsigma) + 5\big(\nu_2 - 1 - d_{\mathcal{J}_2}(\varsigma)\big) + 1 + 3\nu_1 = 3\nu_1 + 5\nu_2 - 2d_{\mathcal{J}_2}(\varsigma) - 4. \end{aligned}$$

Now from definition of the TTI number, we have

$$
\begin{aligned}
TTI(\mathcal{J}_1\diamondsuit\mathcal{J}_2) =& \sum_{\{\iota,\varsigma\}\subseteq V(\mathcal{J}_1\diamondsuit\mathcal{J}_2)} \left|tr_{\mathcal{J}_1\diamondsuit\mathcal{J}_2}(\iota) - tr_{\mathcal{J}_1\diamondsuit\mathcal{J}_2}(\varsigma)\right| \\
=&8\sum_{\{\iota,\varsigma\}\subseteq V(\mathcal{J}_1)} \left|tr_{\mathcal{J}_1\diamondsuit\mathcal{J}_2}(\iota) - tr_{\mathcal{J}_1\diamondsuit\mathcal{J}_2}(\varsigma)\right| \\
&+ 8\sum_{\{\iota,\varsigma\}\subseteq V(\mathcal{J}_2)} \left|tr_{\mathcal{J}_1\diamondsuit\mathcal{J}_2}(\iota) - tr_{\mathcal{J}_1\diamondsuit\mathcal{J}_2}(\varsigma)\right| \\
&+ 4\sum_{\iota\in V(\mathcal{J}_1)}\sum_{\varsigma\in V(\mathcal{J}_2)} \left|tr_{\mathcal{J}_1\diamondsuit\mathcal{J}_2}(\iota) - tr_{\mathcal{J}_1\diamondsuit\mathcal{J}_2}(\varsigma)\right| \\
=&8\sum_{\{\iota,\varsigma\}\subseteq V(\mathcal{J}_1)} \left|\big(5\nu_1+3\nu_2-d_{\mathcal{J}_1}(\iota)-2\big)-\big(5\nu_1+3\nu_2-d_{\mathcal{J}_1}(\varsigma)-2\big)\right| \\
&+ 8\sum_{\{\iota,\varsigma\}\subseteq V(\mathcal{J}_2)} \big|\big(3\nu_1+5\nu_2-2d_{\mathcal{J}_2}(\iota)-4\big) \\
&- \big(3\nu_1+5\nu_2-2d_{\mathcal{J}_2}(\varsigma)-4\big)\big| \\
&+ 4\sum_{\iota\in V(\mathcal{J}_1)}\sum_{\varsigma\in V(\mathcal{J}_2)} \big|\big(5\nu_1+3\nu_2-d_{\mathcal{J}_1}(\iota)-2\big) \\
&- \big(3\nu_1+5\nu_2-2d_{\mathcal{J}_2}(\varsigma)-4\big)\big| \\
=&8\sum_{\{\iota,\varsigma\}\subseteq V(\mathcal{J}_1)} \left|d_{\mathcal{J}_1}(\varsigma)-d_{\mathcal{J}_1}(\iota)\right| + 16\sum_{\{\iota,\varsigma\}\subseteq V(\mathcal{J}_2)} \left|d_{\mathcal{J}_2}(\varsigma)-d_{\mathcal{J}_2}(\iota)\right| \\
&+ 4\sum_{\iota\in V(\mathcal{J}_1)}\sum_{\varsigma\in V(\mathcal{J}_2)} \left|2\nu_1-2\nu_2-d_{\mathcal{J}_1}(\iota)+2d_{\mathcal{J}_2}(\varsigma)+2\right| \\
=&8irr_t(\mathcal{J}_1)+16irr_t(\mathcal{J}_2) \\
&+ 4\sum_{\iota\in V(\mathcal{J}_1)}\sum_{\varsigma\in V(\mathcal{J}_2)} \left|2\nu_1-2\nu_2-d_{\mathcal{J}_1}(\iota)+2d_{\mathcal{J}_2}(\varsigma)+2\right|.
\end{aligned}
$$

Evidently, $d_{\mathcal{J}_1}(\iota) < \nu_1$ and $d_{\mathcal{J}_2}(\varsigma) < \nu_2$ and under the constrain $\nu_1 \geq \nu_2$, the double sum $\sum_{\iota\in V(\mathcal{J}_1)}\sum_{\varsigma\in V(\mathcal{J}_2)} \left|2\nu_1-2\nu_2-d_{\mathcal{J}_1}(\iota)+2d_{\mathcal{J}_2}(\varsigma)+2\right|$ is maximum whenever $\mathcal{J}_1$ has minimum sum of vertex degrees, i.e., $\mathcal{J}_1 \cong T_{\nu_1}$, where T_{ν_1} is a tree on ν_1 vertices and $\mathcal{J}_2$ has maximum sum of vertex degrees, i.e., $\mathcal{J}_2 \cong K_{\nu_2}$. Hence

$$
\begin{aligned}
&\sum_{\iota\in V(\mathcal{J}_1)}\sum_{\varsigma\in V(\mathcal{J}_2)} \left|2\nu_1-2\nu_2-d_{\mathcal{J}_1}(\iota)+2d_{\mathcal{J}_2}(\varsigma)+2\right| \\
\leq& \sum_{\iota\in V(T_{\nu_1})}\sum_{\varsigma\in V(K_{\nu_2})} \left|2\nu_1-2\nu_2-d_{T_{\nu_1}}(\iota)+2d_{K_{\nu_2}}(\varsigma)+2\right| \\
=& \sum_{\iota\in V(T_{\nu_1})}\sum_{\varsigma\in V(K_{\nu_2})} \left|2\nu_1-2\nu_2-d_{T_{\nu_1}}(\iota)+2(\nu_2-1)+2\right|
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{\iota \in V(T_{\nu_1})} \sum_{\varsigma \in V(K_{\nu_2})} \left|2\nu_1 - d_{T_{\nu_1}}(\iota)\right| \\
&= \nu_2 \sum_{\iota \in V(T_{\nu_1})} \left(2\nu_1 - d_{T_{\nu_1}}(\iota)\right) \\
&= 2\nu_1^2\nu_2 - 2\nu_2(\nu_1 - 1) = 2\nu_2(\nu_1^2 - \nu_1 + 1),
\end{aligned}
$$

from which

$$
TTI(\mathcal{J}_1 \diamondsuit \mathcal{J}_2) \leq 8irr_t(\mathcal{J}_1) + 16irr_t(\mathcal{J}_2) + 8\nu_2(\nu_1^2 - \nu_1 + 1).
$$

If $\mathcal{J}_1$ is any tree of order ν_1 and $\mathcal{J}_2$ is complete graph of order ν_2, then the equality in (4.2) holds. Hence the bound in (4.2) is best possible. □

The *lexicographic product* (also called *composition*) of $\mathcal{J}_1$ and $\mathcal{J}_2$, denoted by $\mathcal{J}_1[\mathcal{J}_2]$, is a graph with $V(\mathcal{J}_1[\mathcal{J}_2]) = V(\mathcal{J}_1) \times V(\mathcal{J}_2)$ and $E(\mathcal{J}_1[\mathcal{J}_2]) = \{(\iota_1, \iota_2)(\varsigma_1, \varsigma_2) : \iota_1\varsigma_1 \in E(\mathcal{J}_1), \iota_2, \varsigma_2 \in V(\mathcal{J}_2)\} \cup \{(\iota_1, \iota_2)(\varsigma_1, \varsigma_2) : \iota_1 = \varsigma_1 \in V(\mathcal{J}_1), \iota_2\varsigma_2 \in E(\mathcal{J}_2)\}$.

Theorem 14.

$$
TTI(\mathcal{J}_1[\mathcal{J}_2]) \leq {\nu_2}^3 TTI(\mathcal{J}_1) + {\nu_1}^2 Irr_t(\mathcal{J}_2),
$$

with equality holds if and only if $\mathcal{J}_1$ is transmission regular or $\mathcal{J}_2$ is regular.

Proof. For $(\iota_1, \iota_2) \in V(\mathcal{J}_1[\mathcal{J}_2])$,

$$
\begin{aligned}
tr_{\mathcal{J}_1[\mathcal{J}_2]}((\iota_1, \iota_2)) &= \sum_{\iota_2\varsigma_2 \in E(\mathcal{J}_2)} 1 + \sum_{\iota_2\varsigma_2 \notin E(\mathcal{J}_2)} 2 + \sum_{\varsigma_1 \in V(\mathcal{J}_1)\setminus\{\iota_1\}} \sum_{\varsigma_2 \in V(\mathcal{J}_2)} d_{\mathcal{J}_1}(\iota_1, \varsigma_1) \\
&= d_{\mathcal{J}_2}(\iota_2) + 2(\nu_2 - 1 - d_{\mathcal{J}_2}(\iota_2)) + \nu_2 tr_{\mathcal{J}_1}(\iota_1) \\
&= \nu_2 tr_{\mathcal{J}_1}(\iota_1) + 2(\nu_2 - 1) - d_{\mathcal{J}_2}(\iota_2).
\end{aligned}
$$

Then from the definition of the TTI number, we get

$$
\begin{aligned}
TTI(\mathcal{J}_1[\mathcal{J}_2]) &= \frac{1}{2} \sum_{(\iota_1,\iota_2),(\varsigma_1,\varsigma_2) \in V(\mathcal{J}_1[\mathcal{J}_2])} \left|tr_{\mathcal{J}_1[\mathcal{J}_2]}((\iota_1, \iota_2)) - tr_{\mathcal{J}_1[\mathcal{J}_2]}((\varsigma_1, \varsigma_2))\right| \\
&= \frac{1}{2} \sum_{\iota_1,\varsigma_1 \in V(\mathcal{J}_1)} \sum_{\iota_2,\varsigma_2 \in V(\mathcal{J}_2)} \Big|\left(\nu_2 tr_{\mathcal{J}_1}(\iota_1) + 2(\nu_2 - 1) - d_{\mathcal{J}_2}(\iota_2)\right) \\
&\quad - \left(\nu_2 tr_{\mathcal{J}_1}(\varsigma_1) + 2(\nu_2 - 1) - d_{\mathcal{J}_2}(\varsigma_2)\right)\Big| \\
&= \frac{1}{2} \sum_{\iota_1,\varsigma_1 \in V(\mathcal{J}_1)} \sum_{\iota_2,\varsigma_2 \in V(\mathcal{J}_2)} \left|\nu_2\left(tr_{\mathcal{J}_1}(\iota_1) - tr_{\mathcal{J}_1}(\varsigma_1)\right) + \left(d_{\mathcal{J}_2}(\varsigma_2) - d_{\mathcal{J}_2}(\iota_2)\right)\right|.
\end{aligned}
$$

Using triangle inequality, we obtain

$$\begin{aligned} TTI(\mathcal{J}_1[\mathcal{J}_2]) \leq & \frac{1}{2} \sum_{\iota_1,\varsigma_1 \in V(\mathcal{J}_1)} \sum_{\iota_2,\varsigma_2 \in V(\mathcal{J}_2)} \left[\nu_2 \left| tr_{\mathcal{J}_1}(\iota_1) - tr_{\mathcal{J}_1}(\varsigma_1) \right| + \left| d_{\mathcal{J}_2}(\varsigma_2) - d_{\mathcal{J}_2}(\iota_2) \right| \right] \\ = & {\nu_2}^3 TTI(\mathcal{J}_1) + {\nu_1}^2 Irr_t(\mathcal{J}_2), \end{aligned}$$

in which the equality occurs if and only if $\mathcal{J}_1$ is transmission regular or $\mathcal{J}_2$ is regular.

□

According to Theorem 14, for the *open fence graph* $P_\nu[P_2]$ and *closed fence graph* $C_\nu[P_2]$, we reach to:

Corollary 6.

$$TTI(P_\nu[P_2]) = \begin{cases} \frac{\nu^2(\nu^2-4)}{3} & \text{if } \nu \text{ is even,} \\ \frac{(\nu^2-1)(\nu^2-3)}{3} & \text{if } \nu \text{ is odd,} \end{cases} \qquad TTI(C_\nu[P_2]) = 0.$$

The *generalized hierarchical product* $\mathcal{J}_1(\mathcal{S}) \sqcap \mathcal{J}_2$ is a graph with $V(\mathcal{J}_1(\mathcal{S}) \sqcap \mathcal{J}_2) = V(\mathcal{J}_1) \times V(\mathcal{J}_2)$ and $E(\mathcal{J}_1(\mathcal{S}) \sqcap \mathcal{J}_2) = \{(\iota_1,\iota_2)(\varsigma_1,\varsigma_2) : \iota_1 = \varsigma_1 \in \mathcal{S}, \iota_2\varsigma_2 \in E(\mathcal{J}_2)\} \cup \{(\iota_1,\iota_2)(\varsigma_1,\varsigma_2) : \iota_1\varsigma_1 \in E(\mathcal{J}_1), \iota_2 = \varsigma_2 \in V(\mathcal{J}_2)\}$, , where $\mathcal{S}$ is a nonempty subset of $V(\mathcal{J}_1)$ (see [11]).
For $\iota_1,\varsigma_1 \in V(\mathcal{J}_1)$, the distance $d_{\mathcal{J}_1(\mathcal{S})}(\iota_1,\varsigma_1)$ between ι_1 and ς_1 through $\mathcal{S}$ is the length of any shortest $\iota_1 - \varsigma_1$ path in $\mathcal{J}_1$ including some vertex $s \in \mathcal{S}$ (s can be ι_1 or ς_1). Obviously, if $\iota_1 \in \mathcal{S}$ or $\varsigma \in \mathcal{S}$, then $d_{\mathcal{J}_1(\mathcal{S})}(\iota_1,\varsigma_1) = d_{\mathcal{J}_1}(\iota_1,\varsigma_1)$. As an instanse, the *zig-zag polyhex nanotube* $TUC_6[2\nu, 2]$ is the generalized hierarchical product $C_{2\nu}(\mathcal{S}) \sqcap P_2$, where $S = \{\iota_1, \iota_3, ..., \iota_{2\nu-1}\}$ and $V(C_{2\nu}) = \{\iota_1, \iota_2, ..., \iota_{2\nu}\}$.
For the sake of simplicity, we define

$$TTI(\mathcal{J}_1(\mathcal{S})) = \sum_{\{\iota_1,\varsigma_1\} \subseteq V(\mathcal{J}_1)} \left| tr_{\mathcal{J}_1(\mathcal{S})}(\iota_1) - tr_{\mathcal{J}_1(\mathcal{S})}(\varsigma_1) \right|,$$

where $tr_{\mathcal{J}_1(\mathcal{S})}(\varsigma_1) = \sum_{\iota_1 \in V(\mathcal{J}_1)} d_{\mathcal{J}_1(\mathcal{S})}(\iota_1,\varsigma_1)$. We say $\mathcal{J}_1(\mathcal{S})$ is transmission regular if for every $\iota_1,\varsigma_1 \in V(\mathcal{J}_1)$, $tr_{\mathcal{J}_1(\mathcal{S})}(\iota_1) = tr_{\mathcal{J}_1(\mathcal{S})}(\varsigma_1)$.

Theorem 15. *Let $\mathcal{S}$ be a nonempty subset of $V(\mathcal{J}_1)$. Then*

$$TTI(\mathcal{J}_1(\mathcal{S}) \sqcap \mathcal{J}_2) \leq {\nu_2}^2 TTI(\mathcal{J}_1) + {\nu_2}^2(\nu_2 - 1) TTI(\mathcal{J}_1(\mathcal{S})) + {\nu_1}^3 TTI(\mathcal{J}_2),$$

with equality if and only if $\mathcal{J}_1$ and $\mathcal{J}_1(\mathcal{S})$ are transmission regular or $\mathcal{J}_2$ is transmission regular and for every $\iota_1,\varsigma_1 \in V(\mathcal{J}_1)$ with $tr_{\mathcal{J}_1(\mathcal{S})}(\iota_1) < tr_{\mathcal{J}_1(\mathcal{S})}(\varsigma_1)$, we have $tr_{\mathcal{J}_1}(\iota_1) \leq tr_{\mathcal{J}_1}(\varsigma_1)$.

Proof. For $(\iota_1, \iota_2) \in V(\mathcal{J}_1(\mathcal{S}) \sqcap \mathcal{J}_2)$,

$$\begin{aligned}
&tr_{\mathcal{J}_1(\mathcal{S})\sqcap\mathcal{J}_2}((\iota_1,\iota_2))\\
=&\sum_{\varsigma_1\in V(\mathcal{J}_1)} d_{\mathcal{J}_1}(\iota_1,\varsigma_1) + \sum_{\varsigma_1\in V(\mathcal{J}_1)}\sum_{\varsigma_2\in V(\mathcal{J}_2)\setminus\{\iota_2\}} \big(d_{\mathcal{J}_1(\mathcal{S})}(\iota_1,\varsigma_1) + d_{\mathcal{J}_2}(\iota_2,\varsigma_2)\big)\\
=&tr_{\mathcal{J}_1}(\iota_1) + (\nu_2-1)tr_{\mathcal{J}_1(\mathcal{S})}(\iota_1) + \nu_1 tr_{\mathcal{J}_2}(\iota_2).
\end{aligned}$$

Now by definition of the TTI number, we obtain

$$\begin{aligned}
&TTI(\mathcal{J}_1(\mathcal{S})\sqcap\mathcal{J}_2)\\
=&\frac{1}{2}\sum_{(\iota_1,\iota_2),(\varsigma_1,\varsigma_2)\in V(\mathcal{J}_1(\mathcal{S})\sqcap\mathcal{J}_2)} \big|tr_{\mathcal{J}_1(\mathcal{S})\sqcap\mathcal{J}_2}((\iota_1,\iota_2)) - tr_{\mathcal{J}_1(\mathcal{S})\sqcap\mathcal{J}_2}((\varsigma_1,\varsigma_2))\big|\\
=&\frac{1}{2}\sum_{\iota_1,\varsigma_1\in V(\mathcal{J}_1)}\sum_{\iota_2,\varsigma_2\in V(\mathcal{J}_2)} \big|\big(tr_{\mathcal{J}_1}(\iota_1) + (\nu_2-1)tr_{\mathcal{J}_1(\mathcal{S})}(\iota_1) + \nu_1 tr_{\mathcal{J}_2}(\iota_2)\big)\\
&-\big(tr_{\mathcal{J}_1}(\varsigma_1) + (\nu_2-1)tr_{\mathcal{J}_1(\mathcal{S})}(\varsigma_1) + \nu_1 tr_{\mathcal{J}_2}(\varsigma_2)\big)\big|\\
=&\frac{1}{2}\sum_{\iota_1,\varsigma_1\in V(\mathcal{J}_1)}\sum_{\iota_2,\varsigma_2\in V(\mathcal{J}_2)} \big|\big(tr_{\mathcal{J}_1}(\iota_1) - tr_{\mathcal{J}_1}(\varsigma_1)\big) + (\nu_2-1)\big(tr_{\mathcal{J}_1(\mathcal{S})}(\iota_1) - tr_{\mathcal{J}_1(\mathcal{S})}(\varsigma_1)\big)\\
&+\nu_1\big(tr_{\mathcal{J}_2}(\iota_2) - tr_{\mathcal{J}_2}(\varsigma_2)\big)\big|.
\end{aligned}$$

According to triangle inequality,

$$\begin{aligned}
&TTI(\mathcal{J}_1(\mathcal{S})\sqcap\mathcal{J}_2)\\
\leq&\frac{1}{2}\sum_{\iota_1,\varsigma_1\in V(\mathcal{J}_1)}\sum_{\iota_2,\varsigma_2\in V(\mathcal{J}_2)} \big[\big|tr_{\mathcal{J}_1}(\iota_1) - tr_{\mathcal{J}_1}(\varsigma_1)\big| + (\nu_2-1)\big|tr_{\mathcal{J}_1(\mathcal{S})}(\iota_1) - tr_{\mathcal{J}_1(\mathcal{S})}(\varsigma_1)\big|\\
&+\nu_1\big|tr_{\mathcal{J}_2}(\iota_2) - tr_{\mathcal{J}_2}(\varsigma_2)\big|\big]\\
=&{\nu_2}^2 TTI(\mathcal{J}_1) + {\nu_2}^2(\nu_2-1)TTI(\mathcal{J}_1(\mathcal{S})) + {\nu_1}^3 TTI(\mathcal{J}_2),
\end{aligned}$$

with equality if and only if $\mathcal{J}_1$ and $\mathcal{J}_1(\mathcal{S})$ are transmission regular or $\mathcal{J}_2$ is transmission regular and for every $\iota_1, \varsigma_1 \in V(\mathcal{J}_1)$ with $tr_{\mathcal{J}_1(\mathcal{S})}(\iota_1) < tr_{\mathcal{J}_1(\mathcal{S})}(\varsigma_1)$, we have $tr_{\mathcal{J}_1}(\iota_1) \leq tr_{\mathcal{J}_1}(\varsigma_1)$. □

The *Cartesian product* $\mathcal{J}_1 \times \mathcal{J}_2$, is a graph with $V(\mathcal{J}_1 \times \mathcal{J}_2) = V(\mathcal{J}_1) \times V(\mathcal{J}_2)$ and $E(\mathcal{J}_1 \times \mathcal{J}_2) = \{(\iota_1,\iota_2)(\varsigma_1,\varsigma_2) : \iota_1 = \varsigma_1 \in V(\mathcal{J}_1), \iota_2\varsigma_2 \in E(\mathcal{J}_2)\} \cup \{(\iota_1,\iota_2)(\varsigma_1,\varsigma_2) : \iota_1\varsigma_1 \in E(\mathcal{J}_1), \iota_2 = \varsigma_2 \in V(\mathcal{J}_2)\}$. One can extend the definition to more than two graphs, straightforwardly. Evidently, if $\mathcal{S} = V(\mathcal{J}_1)$, then $\mathcal{J}_1 \times \mathcal{J}_2 \cong \mathcal{J}_1(\mathcal{S}) \sqcap \mathcal{J}_2$ and by Theorem 15 we reach to:

Theorem 16.

$$TTI(\mathcal{J}_1 \times \mathcal{J}_2) \leq {\nu_2}^3 TTI(\mathcal{J}_1) + {\nu_1}^3 TTI(\mathcal{J}_2), \tag{4.3}$$

with equality occurs if and only if $\mathcal{J}_1$ or $\mathcal{J}_2$ is transmission regular.

The graphs $P_\nu \times C_\mu$ and $C_\nu \times C_\mu$ are called C_4-*nanotube* and C_4-*nanotorus* and denoted by $TUC_4(\mu,\nu)$ and $TC_4(\mu,\nu)$, respectively and the graph $K_\nu \times K_\mu$ is known as *Rook's graph*. Using Theorem 16, we arrive at:

Corollary 7.

$$\begin{aligned} TTI(TUC_4(\mu,\nu)) &= \begin{cases} \frac{\mu^3\nu^2(\nu^2-4)}{24} & \text{if } \nu \text{ is even},\\ \frac{\mu^3(\nu^2-1)(\nu^2-3)}{24} & \text{if } \nu \text{ is odd},\end{cases}\\ TTI(TC_4(\mu,\nu)) &= TTI(K_\nu \times K_\mu) = 0. \end{aligned}$$

By an inductive argument, we can extend Eq. (4.3) to a Cartesian product of any desired number of graphs.

Corollary 8.

$$TTI(\mathcal{J}_1 \times \mathcal{J}_2 \times ... \times \mathcal{J}_k) \le {\nu_1}^3{\nu_2}^3\cdots{\nu_k}^3\sum_{r=1}^{k}\frac{TTI(\mathcal{J}_\iota)}{{\nu_r}^3},$$

with equality happens if and only if at most one of the components $\mathcal{J}_1, \mathcal{J}_2, \cdots, \mathcal{J}_k$ is transmission irregular.

The *Hamming graph* $H_{\nu_1,\nu_2,\ldots,\nu_r}$ can be considered as the Cartesian product of K_{ν_1}, $K_{\nu_2}, \cdots, K_{\nu_r}$ and by Corollary 8, we arrive at:

Corollary 9.

$$TTI(H_{\nu_1,\nu_2,\ldots,\nu_r}) = 0.$$

By considering $\mathcal{J}_2$ as a graph rooted at $\rho \in V(\mathcal{J}_2)$, the *rooted product* or *cluster* $\mathcal{J}_1\{\mathcal{J}_2\}$ is a graph constructed from a copy of $\mathcal{J}_1$ and ν_1 copies of $\mathcal{J}_2$ by identifying the rth vertex of $\mathcal{J}_1$ with the root vertex ρ in the rth copy of $\mathcal{J}_2$, for $r = 1, 2, \ldots, \nu_1$. If $\mathcal{S} = \{\rho\} \subseteq V(\mathcal{J}_2)$, then $\mathcal{J}_1\{\mathcal{J}_2\} \cong \mathcal{J}_2(\mathcal{S}) \sqcap \mathcal{J}_1$ and Theorem 15 yields:

Theorem 17. *Let $\mathcal{J}_1$ be non-trivial and $\mathcal{J}_2$ be rooted at $\rho \in V(\mathcal{J}_2)$. Then*

$$TTI(\mathcal{J}_1\{\mathcal{J}_2\}) \le {\nu_2}^3TTI(\mathcal{J}_1) + {\nu_1}^2TTI(\mathcal{J}_2) + {\nu_1}^2\nu_2(\nu_1-1)TTI(\rho|\mathcal{J}_2), \tag{4.4}$$

where $TTI(\rho|\mathcal{J}_2) = \sum_{\{\iota,\varsigma\}\subseteq V(\mathcal{J}_2)} \left|d_{\mathcal{J}_2}(\iota,\rho) - d_{\mathcal{J}_2}(\varsigma,\rho)\right|$. The equality happens if and only if $\mathcal{J}_1$ is transmission regular and for every $\iota,\varsigma \in V(\mathcal{J}_2)$ with $d_{\mathcal{J}_2}(\iota,\rho) < d_{\mathcal{J}_2}(\varsigma,\rho)$, we have $tr_{\mathcal{J}_2}(\iota) \le tr_{\mathcal{J}_2}(\varsigma)$.

The *corona product* $\mathcal{J}_1 \circ \mathcal{J}_2$ is a graph made from $\mathcal{J}_1$ and ν_1 copies of $\mathcal{J}_2$ with $V(\mathcal{J}_1 \circ \mathcal{J}_2) = V(\mathcal{J}_1) \cup \{V(\mathcal{J}_{2\iota}) : \iota \in V(\mathcal{J}_1)\}$ and $E(\mathcal{J}_1 \circ \mathcal{J}_2) = E(\mathcal{J}_1) \cup \{E(\mathcal{J}_{2\iota}) : \iota \in V(\mathcal{J}_1)\} \cup \{\iota\varsigma : \iota \in V(\mathcal{J}_1), \varsigma \in V(\mathcal{J}_{2\iota})\}$, where $\mathcal{J}_{2\iota}$ is a copy of $\mathcal{J}_2$ correspond to $\iota \in V(\mathcal{J}_1)$. One can easily see that $\mathcal{J}_1 \circ \mathcal{J}_2 \cong \mathcal{J}_1\{\bar{K}_1 \nabla \mathcal{J}_2\}$, where $\bar{K}_1 \nabla \mathcal{J}_2$ is supposed to be rooted at the vertex of $\bar{K}_1$. Now Theorem 17 yields:

Theorem 18. *Let $\mathcal{J}_1$ be a non-trivial graph. Then*

$$TTI(\mathcal{J}_1 \circ \mathcal{J}_2) \leq (\nu_2 + 1)^3 TTI(\mathcal{J}_1) + {\nu_1}^2 irr_t(\mathcal{J}_2) + {\nu_1}^2(\nu_1 {\nu_2}^2 + \nu_1\nu_2 - 2\nu_2 - 2\mu_2), \quad (4.5)$$

with equality happens if and only if $\mathcal{J}_1$ is transmission regular.

The corona product of K_2 and a given graph $\mathcal{J}$ is named the *bottleneck graph* of $\mathcal{J}$. According to Theorem 18, we get:

Corollary 10. *For a given graph $\mathcal{J}$ with ν vertices and μ edges,*

$$TTI(K_2 \circ \mathcal{J}) = 4irr_t(\mathcal{J}) + 8(\nu^2 - \mu).$$

In particular, for the bottleneck graph of path and cycle, we obtain:

Corollary 11. *For $\nu \geq 3$, $TTI(K_2 \circ P_\nu) = 8(\nu^2 - 1)$, $TTI(K_2 \circ C_\nu) = 8\nu(\nu - 1)$.*

The corona product of a graph $\mathcal{J}$ and $\overline{K}_t$ is called the *t-thorny graph* of $\mathcal{J}$ and denoted by $\mathcal{J}^t$. Using Theorem 18, we arrive at:

Corollary 12. *For a non-trivial graph $\mathcal{J}$ with ν vertices,*

$$TTI(\mathcal{J}^t) \leq (t + 1)^3 TTI(\mathcal{J}) + \nu^2 t(\nu t + \nu - 2),$$

with equality occurs if and only if $\mathcal{J}$ is transmission regular.

In particular, the TTI number of the *$t-$thorny cycle* is given by:

Corollary 13. $TTI(C_\nu^t) = \nu^2 t(\nu t + \nu - 2)$.

Moreover, Theorem 18 implies:

Corollary 14. *For a given graph $\mathcal{J}$ with n vertices,*

$$TTI(\mathcal{J} \circ tK_2) \leq (2t + 1)^3 TTI(\mathcal{J}_1) + \nu^2 t(4\nu t + 2\nu - 6),$$

in which t is a positive integer and the equality happens if and only if $\mathcal{J}$ is transmission regular.

In particular, for the *flower graph* with ν petals, we obtain:

Corollary 15. $TTI(C_\nu \circ K_2) = 6\nu^2(\nu - 1)$.

5. Conclusion

This paper is concerned with the introduction of a new graph invariant namely the total transmission irregularity (TTI) number as an indicator for measuring the transmission irregularity extent in transmission irregular graphs. We investigated some mathematical properties of the new invariant such as computing its value for some families of graphs, giving lower and upper bounds on the invariant and its relation with some already established indices, and studying it for various families of composite graphs and for some structures of mathematico-chemical interest. Investigating other properties of the TTI number such as its extremal values over various graph classes such as trees and c-cyclic graphs with given parameter, its potential applications in chemistry and other scientific fields and computing its values for various families of chemical graphs and nano-structures are suggested for further studies. In particular, it would be interesting to find a general relation between the TTI number and total Mostar index of connected graphs.

Acknowledgements. We would like to extend our sincerest gratitude to the editor and the referees who dedicated their time and expertise to review and provide valuable feedback on this manuscript. Their insightful comments, constructive criticism, and thorough review have significantly contributed to improving the quality and clarity of this paper.

Conflict of interest. The authors declare that they have no conflict of interest.

Data Availability. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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