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Research Article

The crossing numbers of join products of $K_4 \cup K_1$ with cycles

Michal Staš* and Mária Timková†

Department of Mathematics and Theoretical Informatics,
Faculty of Electrical Engineering and Informatics,
Technical University, 042 00 Košice, Slovak Republic.
*michal.stas@tuke.sk
†maria.timkova@tuke.sk

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Abstract: The crossing number $\mathrm{cr}(G)$ of a graph G is the minimum number of edge crossings over all drawings of G in the plane. In the paper, we extend known results concerning crossing numbers of join products of two small graphs with cycles. The crossing number of the join product $G^* + C_n$ for the disconnected graph G^* consisting of the complete graph K_4 and one isolated vertex is given, where C_n is the cycle on n vertices. The proof of the main result is done with the help of lemma whose proof is based on a special redrawing technique. Up to now, the crossing numbers of $G + C_n$ are done only for a few disconnected graphs G. Finally, by adding new edge to the graph G^*, we are able to obtain the crossing number of $G_1 + C_n$ for one other graph G_1 of order five.

Keywords: graph, crossing number, join product, separating cycle, cycle.

AMS Subject classification: 05C10, 05C38.

1. Introduction

The problem of reducing the number of crossings is interesting in many areas. In network visualizations, it can help to understand the network's underlying structure and identify important nodes and connections [1]. In electronic circuit design, minimizing the number of edge crossings is important for reducing signal interference and improving circuit performance. Graph drawings with fewer crossings can lead to more efficient and reliable circuit designs [20]. Crossing numbers were also studied

* *Corresponding Author*

to improve the readability of hierarchical structures and automated graphs. The visualized graph should be easy to read and understand. For the sake of the clarity of the graphical drawings, the reduction in crossings is likely the most important. Therefore, the investigation on the crossing number of simple graphs is a classical but very difficult problem. Garey and Johnson [6] proved that determining $\mathrm{cr}(G)$ is an NP-complete problem. Nevertheless, many researchers are trying to solve this problem. Note that the exact values of the crossing numbers are known for some families of graphs, see Clancy *et al.* [4].

Let G be a simple graph (without loops or multiple edges). We use $V(G)$ and $E(G)$ to denote the vertex set and the edge set of G, respectively. The used graph terminology is taken from the book [31]. A drawing of G is a representation of G in the plane such that its vertices are represented by distinct points and its edges by simple continuous arcs connecting the corresponding point pairs. The *crossing number* $\mathrm{cr}(G)$ is the smallest number of edge crossings over all drawings of G in the plane. It is easy to see that a drawing with a minimum number of crossings (an optimal drawing) is always a *good* drawing, meaning that no edge crosses itself, no two edges cross more than once, and no two edges are incident with the same vertex cross. Let D be a good drawing of the graph G. We denote the number of crossings in D by $\mathrm{cr}_D(G)$. Let G_i and G_j be edge-disjoint subgraphs of G. We denote the number of crossings between edges of G_i and edges of G_j by $\mathrm{cr}_D(G_i, G_j)$, and the number of crossings among edges of G_i in D by $\mathrm{cr}_D(G_i)$. It is easy to see that for any three mutually edge-disjoint subgraphs G_i, G_j, and G_k of G, the following equations hold:

$$\mathrm{cr}_D(G_i \cup G_j) = \mathrm{cr}_D(G_i) + \mathrm{cr}_D(G_j) + \mathrm{cr}_D(G_i, G_j)\,, \tag{1.1}$$

$$\mathrm{cr}_D(G_i \cup G_j, G_k) = \mathrm{cr}_D(G_i, G_k) + \mathrm{cr}_D(G_j, G_k)\,. \tag{1.2}$$

It was Turán [30] who introduced the concept of crossing numbers. In his Brick Factory Problem, he investigated the minimal number of crossings among edges of the complete bipartite graphs $K_{m,n}$. Kleitman in [9] showed that

$$\mathrm{cr}(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor, \quad \text{if} \quad \min\{m,n\} \leq 6. \tag{1.3}$$

The join product of two different graphs G_i and G_j, denoted $G_i + G_j$, is obtained from vertex-disjoint copies of G_i and G_j by adding all edges between $V(G_i)$ and $V(G_j)$. For $|V(G_i)| = m$ and $|V(G_j)| = n$, the edge set of $G_i + G_j$ is the union of the disjoint edge sets of the graphs G_i, G_j, and the complete bipartite graph $K_{m,n}$. Let P_n and C_n be the *path* and the *cycle* on n vertices, respectively, and let D_n denote the *discrete graph* (sometimes called *empty graph*) on n vertices. The crossings numbers of join products of paths and cycles with all graphs of order at most four have been well-known for a long time by Klešč [10, 12], and Klešč and Schrötter [15]. It is understandable that our immediate aim is to establish exact values for crossing numbers of $G + P_n$ and $G + C_n$ also for all graphs G of order five and six. Of course, the crossing numbers

of $G + P_n$ and $G + C_n$ are already known for a lot of graphs G of order five and six [2, 11, 13, 16, 18, 21, 27–29]. In all these cases, the graph G is connected and usually contains at least one cycle. Note that $\mathrm{cr}(G + P_n)$ and $\mathrm{cr}(G + C_n)$ are known only for some disconnected graphs G on five or six vertices [5, 17, 23–26]. To date, the crossing number of $K_3 \cup 2K_1 + P_n$, $K_3 \cup 2K_1 + C_n$ and $K_4 \cup K_1 + C_n$ can only be given as a conjecture. The last open problem will be solved in our paper. For this purpose, we present a new technique regarding the use of knowledge from the subgraphs whose values of crossing numbers are already known. It is appropriate to combine this idea with possibility of an existence of a separating cycle in some particular drawing of investigated graph. In a good drawing D of some graph G, we say that a cycle C *separates* some two different vertices of the subgraph $G \setminus C$ (obtained by removing all vertices of C with their corresponding incident edges from G) if they are contained in different components of $\mathbb{R}^2 \setminus C$, where $\mathbb{R}^2$ means a two-dimensional space. This considered cycle C is said to be a *separating* cycle of the graph G in D.
Let G^* be the disconnected graph consisting of the complete graph K_4 and one isolated vertex. The crossing numbers of the join products of G^* with the discrete graphs D_n were well-known by Staš [22] using a lot of properties of cyclic permutations. This established result has been extended to the crossing number of $G^* + P_n$ thanks to Staš and Švecová [24]. The main aim of the paper is to establish $\mathrm{cr}(G^* + C_n)$ for all n at least three. The crossing number of $G^* + C_n$ equal to $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 3$ is determined in Theorem 3 with the proof that is strongly based on Lemma 7. This lemma with a very special innovative proof could also be used to establish crossing numbers of other graphs. The paper concludes by giving the crossing numbers of $G_1 + C_n$ in Corollary 4 for the graph G_1 by adding one new edge to G^*, the result of which has already been claimed by Li [19]. Since this paper does not seem to be available in English, we have not been able to verify this result but we can certainly say that the author's result is incorrect for the graph $G_1 + P_2$ thanks to Theorem 2.
In the proofs of the paper, we will often use the term "region" also in nonplanar subdrawings. In this case, crossings are considered to be vertices of the "map".

2. The crossing numbers of $G^* + D_n$ and $G^* + P_n$

Let $G^* = (V(G^*), E(G^*))$ be the disconnected graph on five vertices consisting of the complete graph K_4 and one isolated vertex, and let also $V(G^*) = \{v_1, v_2, \dots, v_5\}$. In the rest of the paper, let v_5 be the vertex notation of the isolated vertex of G^* in all considered good subdrawings of the graph G^*. In [22], three possible non isomorphic drawings of G^* were described. They are presented in Figure 1 with the corresponding vertex notation.
We consider the join product of the graph G^* with the discrete graph D_n, which yields that $G^* + D_n$ (sometimes used notation $G^* + nK_1$) consists of just one copy of G^* and n vertices $t_1, t_2, \dots, t_n$. Here, each vertex t_i, $i = 1, 2, \dots, n$, is adjacent to every vertex of the graph G^*. Let T^i, $1 \le i \le n$, denote the subgraph induced by five edges incident with the fixed vertex t_i. This means that the graph $T^1 \cup \dots \cup T^n$ is

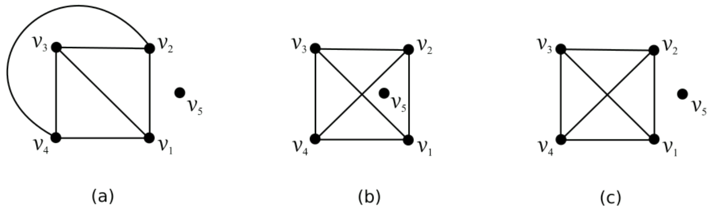

Figure 1. **Three possible non isomorphic drawings of the graph G^*.**

isomorphic to the complete bipartite graph $K_{5,n}$ and

$$G^* + D_n = G^* \cup \left(\bigcup_{i=1}^{n} T^i \right). \tag{2.1}$$

The obtained equality (2.1) together with the crossing property (1.1) produce

$$\mathrm{cr}_D(G^* + D_n) = \mathrm{cr}_D(G^*) + \mathrm{cr}_D\left(\bigcup_{i=1}^{n} T^i \right) + \mathrm{cr}_D\left(G^*, \bigcup_{i=1}^{n} T^i \right) \tag{2.2}$$

for all good drawings D of $G^* + D_n$. The graph $G^* + P_n$ contains $G^* + D_n$ as a subgraph, and therefore let P_n^* denote the path induced on n vertices of $G^* + P_n$ not belonging to the subgraph G^*. The path P_n^* consists of the vertices $t_1, t_2, \dots, t_n$ and $n-1$ edges $t_i t_{i+1}$ for $i = 1, 2, \dots, n-1$, and thus

$$G^* + P_n = G^* \cup \left(\bigcup_{i=1}^{n} T^i \right) \cup P_n^*. \tag{2.3}$$

Similarly, the graph $G^* + C_n$ contains both $G^* + D_n$ and $G^* + P_n$ as subgraphs. Let C_n^* denote the subgraph of $G^* + C_n$ induced on the vertices $t_1, t_2, \dots, t_n$. Therefore,

$$G^* + C_n = G^* \cup \left(\bigcup_{i=1}^{n} T^i \right) \cup C_n^*. \tag{2.4}$$

We consider a good drawing D of $G^* + D_n$. The *rotation* $\mathrm{rot}_D(t_i)$ of a vertex t_i in the drawing D as the cyclic permutation that records the (cyclic) counter-clockwise order in which the edges leave t_i have been defined by Hernández-Vélez et al. [8] or Woodall [32]. We use the notation (12345) if the counter-clockwise order the edges incident with the vertex t_i is t_iv_1, t_iv_2, t_iv_3, t_iv_4, and t_iv_5. We recall that rotation is a cyclic permutation. In the given drawing D, it is highly desirable to separate n subgraphs T^i into three mutually disjoint subsets of subgraphs depending on the number of crossings between T^i and G^* in D. Let us denote by R_D and S_D the set of subgraphs for which $\mathrm{cr}_D(G^*, T^i) = 0$ and $\mathrm{cr}_D(G^*, T^i) = 1$, respectively. Every other subgraph T^i crosses G^* at least twice in D. For $T^i \in R_D \cup S_D$, let F^i denote

the subgraph $G^* \cup T^i$, $i \in \{1, 2, \dots, n\}$, of $G^* + D_n$ and let $D(F^i)$ be its subdrawing induced by D. Clearly, this idea of dividing all subgraphs T^i into three mentioned subsets of subgraphs will be also retained in all drawings of the join products $G^* + P_n$ and $G^* + C_n$.
The crossing numbers of $G^* + D_n$ equal to $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor$ were established by Staš [22]. Using the results of Staš and Švecová [24], the crossing numbers of the graphs $G^* + P_n$ have already been well-known for any $n \geq 2$.

Theorem 1 ([22], Theorem 3.1). $\mathrm{cr}(G^* + D_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor$ *for* $n \geq 1$.

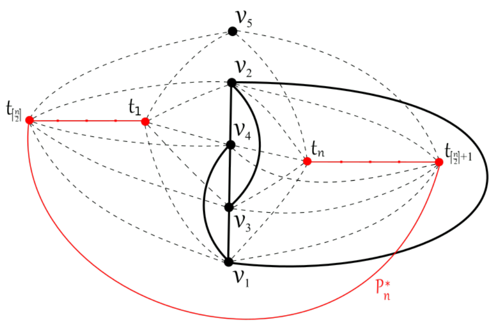

Figure 2. **The drawing of $G^* + P_n$ with $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 1$ crossings for $n \geq 3$.**

Theorem 2 ([24], Lemma 2.2, Theorem 2.4). $\mathrm{cr}(G^* + P_2) = 3$ *and* $\mathrm{cr}(G^* + P_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 1$ *for* $n \geq 3$.

Due to Theorem 2, the good drawing of $G^* + P_n$ in Figure 2 is optimal. To date, the crossing number of $G^* + C_n$ can only be given as a conjecture. This open problem will be solved in the next section.

3. The crossing number of $G^* + C_n$

For the vertices $v_1, v_2, \dots, v_5$ of the graph G^*, let T^{v_i} denote the subgraph induced by n edges joining the vertex v_i with n vertices of C_n^*. The edges joining the vertices of G^* with the vertices of C_n^* form the complete bipartite graph $K_{5,n}$, and so

$$G^* + C_n = G^* \cup \left(\bigcup_{i=1}^{5} T^{v_i} \right) \cup C_n^*. \tag{3.1}$$

In the proof of main theorem of this section, the following three statements related to some restricted subdrawings of $G + C_n$ will be also required.

Lemma 1 ([10], Lemma 2.2). *For $m \geq 2$ and $n \geq 3$, let D be a good drawing of $D_m + C_n$ in which no edge of C_n^* is crossed, and C_n^* does not separate the other vertices of the graph. Then, for all $i, j = 1, 2, \dots, m$, two different subgraphs T^{v_i} and T^{v_j} cross each other in D at least $\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ times.*

Corollary 1 ([14], Corollary 4). *For $m \geq 2$ and $n \geq 3$, let D be a good drawing of the join product $D_m + C_n$ in which the edges of C_n^* do not cross each other and C_n^* does not separate p vertices $v_1, v_2, \dots, v_p$, $2 \leq p \leq m$. Let $T^{v_1}, T^{v_2}, \dots, T^{v_q}$, $q < p$, be the subgraphs induced on the edges incident with the vertices $v_1, v_2, \dots, v_q$ that do not cross C_n^*. If k edges of some subgraph T^{v_j} induced on the edges incident with the vertex v_j, $j \in \{q+1, q+2, \dots, p\}$, cross the cycle C_n^*, then the subgraph T^{v_j} crosses each of the subgraphs $T^{v_1}, T^{v_2}, \dots, T^{v_q}$ at least $\lfloor \frac{n-k}{2} \rfloor \lfloor \frac{(n-k)-1}{2} \rfloor$ times in D.*

Lemma 2 ([14], Lemma 1). *For $m \geq 1$, let G be a graph of order m. In an optimal drawing of the join product $G + C_n$, $n \geq 3$, the edges of C_n^* do not cross each other.*

We can always redraw a crossing of two edges of C_n^* in an effort to get a new drawing of C_n^* (with vertices in a different order) with less number of edge crossings. Based on the arguments above, we will assume that edges of C_n^* do not cross each other in all considered subdrawings $D(C_n^*)$ induced by a good drawing D of $G^* + C_n$.
In the following, we are able to compute the exact values of crossing numbers of the join products of the graph G^* with both cycles C_3 and C_4 using the algorithm located on the website `http://crossings.uos.de/`. This algorithm can find the crossing numbers of small undirected graphs. It uses an ILP formulation, based on Kuratowski subgraphs, and solves it via branch-and-cut-and-price. The system also generates verifiable formal proofs, as described by Chimani and Wiedera [3]. Unfortunately, the capacity of this system is restricted.

Lemma 3. $\mathrm{cr}(G^* + C_3) = 11$ *and* $\mathrm{cr}(G^* + C_4) = 17$.

Lemma 4. *For $n \geq 5$, let D be a good drawing of $G^* + C_n$ in which all vertices of the cycle C_n^* are not placed in one region of $D(G^*)$, then there are at least $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 3$ crossings in D.*

Proof. By Theorem 1, the required number of crossings can be obtained if the edges of the cycle C_n^* are crossed at least three times in D. Now, let us consider $\mathrm{cr}_D(G^*, C_n^*) = 2$, and therefore each of five subgraphs T^{v_i} does not cross any edge of C_n^*. At least four vertices of the graph G^* must be placed in the same region of $D(C_n^*)$ because G^* contains K_4 as a subgraph. We have at least four distinct $i, j \in \{1, 2, 3, 4\}$ such that any two different considered subgraphs T^{v_i} and T^{v_j} cross each other at least $\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ times by Lemma 1, which yields that there are at least $\binom{4}{2} \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2$ crossings in D. As $\binom{4}{2} \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2 \geq 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 3$ for n at least five, the proof of Lemma 4 is done. $\square$

The crossing numbers of $G^* + P_n$ equal to $4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + n + \lfloor\frac{n}{2}\rfloor + 1$ were established by Staš and Švecová [24]. The proof of Theorem 2.4 for the planar drawing of G^* in Figure 1(a) (with the same notation of vertices preserved) was strongly based on twelve possible configurations $\mathcal{A}_p$ and $\mathcal{B}_q$, $p, q \in \{1, \dots, 6\}$ for possible subgraphs by which edges of G^* are crossed exactly once. As the same fixation in all four subcases of Case 1 in Theorem 2.4 can be also applied, the proof of Corollary 2 can be omitted.

Corollary 2. *For $n \geq 5$, let $D^\star$ be a good drawing of $G^* + D_n$ with the planar subdrawing of G^* induced by $D^\star$. If $|S_{D^\star}| = n$, then there are at least $4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + n + \lfloor\frac{n}{2}\rfloor + 1$ crossings in $D^\star$.*

Lemma 5. *For $n \geq 5$, let D be a good drawing of $G^* + C_n$ with the planar subdrawing of G^* induced by D. If edges of C_n^* are crossed twice, then there are at least $4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + n + \lfloor\frac{n}{2}\rfloor + 3$ crossings in D.*

Proof. Let D be a good drawing of $G^* + C_n$ with at most $4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + n + \lfloor\frac{n}{2}\rfloor + 2$ crossings in which the edges of C_n^* are crossed twice. By Theorem 1, let $D^{'}$ be the optimal subdrawing of $G^* + D_n$ induced by D without n edges of C_n^* with exactly $4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + n + \lfloor\frac{n}{2}\rfloor$ crossings. Let us consider a separating cycle $C_3 = v_1v_2v_4v_1$ of G^* in the planar subdrawing $D^{'}(G^*)$ shown in Figure 1(a). Since two remaining vertices v_3 and v_5 of G^* lie in different regions of $D^{'}(C_3)$, there is no subgraph T^i by which the edges of C_3 are not crossed. Hence, each subgraph T^i crosses edges of C_3 at least once, which yields that $\mathrm{cr}_{D'}(C_3, \bigcup_{i=1}^{n} T^i) \geq n$. Let H be the graph difference of graphs G^* and C_3, i.e., H is isomorphic to the graph $K_{1,3} \cup K_1$. The exact value for the crossing number of $H + D_n$ is given by Klešč and Staš [17], that is, $\mathrm{cr}(H + D_n) = 4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + \lfloor\frac{n}{2}\rfloor$. Based on these arguments and using Lemma 4, the edges of the separating cycle C_3 must be crossed by each subgraph T^i just once, i.e., $|S_{D'}| = n$. In this case, Corollary 2 contradicts the considered number of crossings in the optimal drawing $D^{'}$ of $G^* + C_n$. □

Lemma 6. *For $n \geq 5$, let D be a good drawing of $G^* + C_n$ with the nonplanar subdrawing of G^* induced by D and one region with all five vertices of G^* located on its boundary. If $2|R_D| \geq n+1$ and edges of C_n^* are crossed at least once, then there are at least $4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + n + \lfloor\frac{n}{2}\rfloor + 3$ crossings in D.*

Proof. Let D be a good drawing of $G^* + C_n$ with at least one crossing on edges of C_n^*. The nonplanar subdrawing of the graph G^* can be obtained from the unique drawing (with respect to isomorphisms) in Figure 1(c). Note that the set S_D is empty for a such drawing of G^*. For easier reading, let $r = |R_D|$. Assuming $2r \geq n+1$, there are at least two different subgraphs by which edges of G^* are not crossed. For some $T^i \in R_D$, there is only one subdrawing of $(G^* \cup T^i) \setminus v_5$ represented by the rotation (1432) and therefore we have four possibilities how to obtain the subdrawing of $G^* \cup T^i$ depending on which region the vertex v_5 is placed in. Let $\mathcal{N}_D$ be the set of all configurations for the drawing D belonging to $\mathcal{N} = \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4\}$, where a subdrawing of any subgraph $G^* \cup T^i$ has the configuration $\mathcal{E}_p$ represented by some cyclic permutation with $\mathrm{rot}_D(t_i) = \mathcal{E}_p$ for $p \in \{1, 2, 3, 4\}$, see also Figure 3. Namely, $\mathcal{E}_1 = (14325)$, $\mathcal{E}_2 = (14532)$, $\mathcal{E}_3 = (14352)$ and $\mathcal{E}_4 = (15432)$.

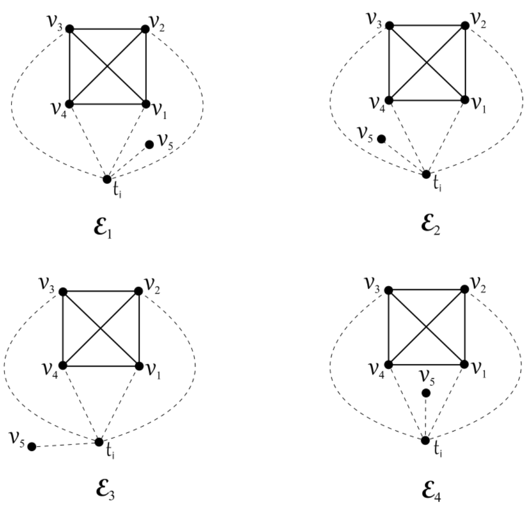

Figure 3. **Four drawings of possible configurations from $\mathcal{N}$ of subgraph $G^* \cup T^i$.**

In Table 1, there are all necessary numbers of crossings between two subgraphs T^i and T^j with configurations $\mathcal{E}_p$ and $\mathcal{E}_q$ of the subgraphs $F^i = G^* \cup T^i$ and $F^j = G^* \cup T^j$, respectively. The obtained values can be obtained using their drawings, but they were also established in Table 2 of [22] using properties of cycles permutations.

$-$	$\mathcal{E}_1$	$\mathcal{E}_2$	$\mathcal{E}_3$	$\mathcal{E}_4$
$\mathcal{E}_1$	4	2	3	3
$\mathcal{E}_2$	2	4	3	3
$\mathcal{E}_3$	3	3	4	2
$\mathcal{E}_4$	3	3	2	4

Table 1. The minimum number of crossings between T^i and T^j with $\mathrm{conf}(F^i) = \mathcal{E}_p$ and $\mathrm{conf}(F^j) = \mathcal{E}_q$.

For these $T^i, T^j \in R_D$, we will discuss the existence of possible configurations of subgraphs $F^i = G^* \cup T^i$ and $F^j = G^* \cup T^j$, and we will show that in all cases it is possible to verify at least $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 3$ crossings in the drawing D. We remind that all vertices t_i of the cycle C_n^* are placed in the same outer region of $D(G^*)$ in all three following subcases due to Lemma 4.

a) $\{\mathcal{E}_p, \mathcal{E}_{p+1}\} \subseteq \mathcal{N}_D$ for some $p \in \{1, 3\}$. Without lost of generality, let us consider two different subgraphs $T^i, T^j \in R_D$ such that F^i and F^j have configurations $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively. Then, $\mathrm{cr}_D(T^i \cup T^j, T^k) \geq 6$ holds for each other $T^k \in R_D$ by summing of two corresponding values of Table 1. For any $T^k \notin R_D$ and

$l = 1, \dots, 5$, it is not difficult to verify that the edge $t_k v_l$ with at least one crossing on edges of G^* can be redrawn to an edge (preserving the incidence of the given two vertices t_k and v_l) with no crossing on edges of G^* without increasing the number of crossings in $D(G^* \cup T^i \cup T^j)$. This means that each of $n-r$ subgraphs $T^k \notin R_D$ of $K_{5,n-2}$ crosses $G^* \cup T^i \cup T^j$ at least six times. Thus, by fixing the subgraph $G^* \cup T^i \cup T^j$, we have

$$\begin{aligned}\mathrm{cr}_D(G^* + C_n) &= \mathrm{cr}_D(K_{5,n-2}) + \mathrm{cr}_D(K_{5,n-2}, G^* \cup T^i \cup T^j) + \mathrm{cr}_D(G^* \cup T^i \cup T^j)\\ &\quad + \mathrm{cr}_D(K_{5,n} \cup G^*, C_n^*)\\ &\geq 4\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 6(r-2) + 6(n-r) + 3 + 1\\ &= 4\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 6n - 8 \geq 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n + \left\lfloor \frac{n}{2} \right\rfloor + 3.\end{aligned}$$

In the following, suppose that $\{\mathcal{E}_1, \mathcal{E}_2\} \not\subseteq \mathcal{N}_D$ and $\{\mathcal{E}_3, \mathcal{E}_4\} \not\subseteq \mathcal{N}_D$.

b) $\mathcal{N}_D = \{\mathcal{E}_p, \mathcal{E}_q\}$ for two different $p, q = 1, 2, 3, 4$. Without lost of generality, let us assume the configurations $\mathcal{E}_1$ of F^i with $\mathrm{rot}_D(t_i) = (14325)$ and $\mathcal{E}_3$ of F^j with $\mathrm{rot}_D(t_j) = (14352)$. Again, by summing of two corresponding values of Table 1, $\mathrm{cr}_D(T^i \cup T^j, T^k) \geq 4 + 3 = 7$ is fulfilling for each other $T^k \in R_D$. Since the minimum number of interchanges of adjacent elements of (14325) required to produce (14352) is one, each other subgraph T^k crosses edges of $T^i \cup T^j$ at least once, that is, $\mathrm{cr}_D(T^i \cup T^j, T^k) \geq 3$ due to the Woodall's result [32] for our graph G^* of odd order five. As the set S_D is empty, by fixing the subgraph $G^* \cup T^i \cup T^j$, we have

$$\begin{aligned}\mathrm{cr}_D(G^* + C_n) &\geq 4\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 7(r-2) + 5(n-r) + 4 + 1\\ &= 4\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 5n + 2r - 9 \geq 4\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor\\ &\quad + 5n + n + 1 - 9 \geq 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n + \left\lfloor \frac{n}{2} \right\rfloor + 3.\end{aligned}$$

c) $\mathcal{N}_D = \{\mathcal{E}_p\}$ for only one $p \in \{1, 2, 3, 4\}$. In the rest of the proof, we can assume that $T^i, T^j \in R_D$ with the same configuration $\mathcal{E}_1$ of the subgraphs F^i, F^j. In the same way as in the previous case, by fixing the subgraph $G^* \cup T^i \cup T^j$, we obtain

$$\begin{aligned}\mathrm{cr}_D(G^* + C_n) &\geq 4\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 8(r-2) + 4(n-r) + 5 + 1\\ &= 4\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 4n + 4r - 10 \geq 4\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor\\ &\quad + 4n + 2(n+1) - 10 \geq 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n + \left\lfloor \frac{n}{2} \right\rfloor + 3.\end{aligned}$$

□

By Lemma 6 for the subdrawing G^* in Figure 1(c), there are at most $\lfloor \frac{n}{2} \rfloor$ subgraphs $T^i \in R_D$ for any good drawing D of $G^* + C_n$ with at most $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 2$ crossings if edges of C_n^* are crossed at least once. Thus, we obtain $\mathrm{cr}_D(G^*, \bigcup_{i=1}^n T^i) \geq n$ in D because the set S_D is empty.

Corollary 3. *For $n \geq 6$, let D be a good drawing of $G^* + C_n$ with the nonplanar subdrawing of G^* induced by D and one region with all five vertices of G^* located on its boundary. If $2|R_D| \geq n$ and edges of C_n^* are crossed twice, then there are at least $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 3$ crossings in D.*

Proof. For $n \geq 7$, the same fixations as in the proof of Lemma 6 can be applied. The same idea applies to $n = 6$ in the first two subcases. It is sufficient to fix the subgraph $G^* \cup T^i$ for some $T^i \in R_D$ in the last subcase of $\mathcal{N}_D = \{\mathcal{E}_p\}$, and we also achieve the desired result provided by

$$\mathrm{cr}_D(G^* + C_6) \geq 4\left\lfloor \frac{6-1}{2} \right\rfloor \left\lfloor \frac{6-2}{2} \right\rfloor + 4(|R_D| - 1) + 3(6 - |R_D|) + 1 + 2 = 33 + |R_D| \geq 36.$$

$\square$

By Corollary 3 for the subdrawing G^* in Figure 1(c), similarly as by Lemma 6, we obtain $\mathrm{cr}_D(G^*, \bigcup_{i=1}^n T^i) \geq n + 1$ in any drawing D of $G^* + C_n$ with at most $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 2$ crossings two of which are on edges of C_n^*.

Lemma 7. *Let D be a good drawing of $G^* + C_5$ with $\mathrm{cr}_D(T^{v_5}, C_5^*) = 0$. If the edges of T^{v_5} are crossed at most five times, then there are at least 26 crossings in D.*

Proof. Due to Lemma 4 together with the assumption $\mathrm{cr}_D(T^{v_5}, C_5^*) = 0$, consider a good drawing D of $G^* + C_5$ in which the edges of the cycle C_5^* can only be crossed by some subgraph T^{v_j} for $j \in \{1, 2, 3, 4\}$. Let α and β denote the numbers of crossings on edges of subgraphs T^{v_5} and $(G^* + C_5) - T^{v_5}$ in D, respectively. In the following, by eliminating the vertex v_5 of G^*, we will transform the drawing D into a new drawing $D^\star$ of a graph isomorphic to the complete graph K_9.
The plane is a normal space. Hence, in the plane there is an open set A_{v_5} such that A_{v_5} contains v_5 together with the corresponding segments of uncrossed parts of five edges. All remaining edges of the drawing D are disjoint with A_{v_5}, see Figure 4(a). Figure 4(b) shows that uncrossed parts of five edges can be removed in A_{v_5} and the subsequent duplication of remaining parts of five edges outside A_{v_5} enforces exactly α new crossings. Finally, let $G^\star$ be the graph obtained by removing the vertex v_5 and adding five new continuous arcs connecting the corresponding pair of points from the boundary of A_{v_5} as shown in Figure 4(c). The obtained graph $G^\star$ contains K_5 induced by the vertices of C_5^*, and therefore, a new drawing $D^\star$ (not necessarily a good drawing) of a graph isomorphic to K_9 with just $2\alpha + \beta + 5$ crossings is achieved (five new crossings have been created in A_{v_5}).

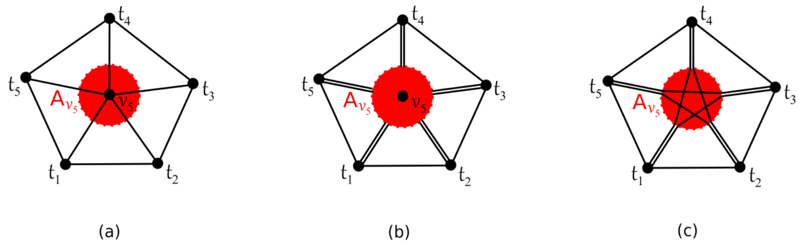

Figure 4. **Elimination of the isolated vertex v_5 of G^*.**

It was proved by Guy [7] that $\mathrm{cr}(K_9) = 36$. Taking into account the assumption that $\alpha \leq 5$, we obtain $\alpha + \beta \geq 26$ in D according to at least 36 crossings on edges of K_9 in the drawing $D^\star$.

□

Let D be a good drawing of the graph G^*+C_n, we distinguish two types of this drawing D. A drawing D of $G^* + C_n$ is of type A_D or B_D if C_n^* is not a separating cycle of the graph G^* in D or C_n^* is a separating cycle of the graph G^* in D, respectively.

Theorem 3. $\mathrm{cr}(G^* + C_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 3$ *for* $n \geq 3$.

Proof. By Lemma 3, the result holds for $n = 3$ and $n = 4$. Into the drawing in Figure 2, it is possible to add the edge t_1t_n which forms the cycle C_n^* on vertices of P_n^* with just two additional crossings, i.e., C_n^* is crossed by two edges v_2v_3 and v_2v_4 of the graph G^*. Thus $\mathrm{cr}(G^* + C_n) \leq 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 3$ for $n \geq 3$, and let us suppose that there is an optimal drawing D of $G^* + C_n$ such that

$$\mathrm{cr}_D(G^* + C_n) \leq 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n + \left\lfloor \frac{n}{2} \right\rfloor + 2 \quad \text{for some } n \geq 5. \tag{3.2}$$

By Theorem 1, there are at most two crossings on the edges of C_n^* in D, and we can also suppose that edges of C_n^* do not cross each other using Lemma 2. If the drawing D is of type B_D, then the isolated vertex v_5 lies in different region of $D(C_n^*)$ as the four vertices v_1, v_2, v_3, v_4 of G^*. Now, three possible cases may occur:

Case 1: There is no crossing on edges of C_n^*. Any two different considered subgraphs T^{v_i} and T^{v_j} for $i, j \in \{1, 2, 3, 4\}$, cross each other at least $\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ times by Lemma 1. There are at least $\binom{4}{2} \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ crossings in D which confirms a contradiction with the assumption (3.2) for every $n \geq 6$. If $n = 5$, by Lemma 7 we have at least six others crossings on the edges of T^{v_5} in D which contradicts the assumption (3.2).

Case 2: There is exactly one crossing on edges of C_n^*. If the drawing D is of type A_D, then by Lemma 1, Corollary 1 for $p = 5$, $q = 4$, $k = 1$, we have at least $\binom{4}{2} \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 4\lfloor \frac{n-1}{2} \rfloor \lfloor \frac{n-2}{2} \rfloor > 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 2$ crossings in D which

confirms a contradiction with the assumption (3.2). For the drawing D of type B_D, we discuss two subcases:
(a) Let $\mathrm{cr}_D(T^{v_5}, C_n^*) = 1$. Using Lemma 1, one crossing on C_n^* and at least one crossing between T^{v_5} and T^{v_i}, $i \in \{1,2,3,4\}$, we have at least $\binom{4}{2}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 1 + 1 > 4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + n + \lfloor\frac{n}{2}\rfloor + 2$ crossings in D which also confirms a contradiction with the assumption (3.2).
(b) Let $\mathrm{cr}_D(T^{v_i}, C_n^*) = 1$ for only one $i \in \{1,2,3,4\}$. Note that $R_D = \emptyset$ for both subdrawings of G^* induced by D given in Figure 1(a) and 1(b). This fact together with Lemma 6 for the subdrawing of G^* given in Figure 1(c) imply $\mathrm{cr}_D(G^*, \bigcup_{m=1}^n T^m) \geq n$. In the following, by Lemma 1 and Corollary 1 for $p = 4$, $q = 3$, $k = 1$, we have at least $\binom{3}{2}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 3\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor + n$ crossings in D. This again confirms a contradiction with the assumption (3.2) for every $n \geq 6$. Lemma 7 contradicts (3.2) for $n = 5$ because T^{v_i} cannot cross T^{v_5} at least six times.
Case 3: There are exactly two crossings on edges of C_n^*. In this case we consider also one crossing among the edges of G^* in D thanks to Lemma 5. If the subdrawing of G^* in D is given in Figure 1(b), then $R_D = \emptyset$. In addition if the drawing D is of type B_D then $S_D = \emptyset$ due to Lemma 4. For the subdrawing of G^* given in Figure 1(c) we use Corollary 3, so there are at least $n+1$ crossings between the edges of the graph G^* and $\bigcup_{m=1}^n T^m$ for $n \geq 6$. For both drawings we consider two cases. If the drawing D is of type A_D, then $\mathrm{cr}_D(G^*, \bigcup_{m=1}^n T^m) \geq n$ for $n \geq 5$. If the drawing D is of type B_D then $\mathrm{cr}_D(G^*, \bigcup_{m=1}^n T^m) \geq n+1$ for $n \geq 6$. We discuss four subcases:
(a) Let $\mathrm{cr}_D(G^*, C_n^*) = 2$ or $\mathrm{cr}_D(T^{v_5}, C_n^*) = 2$. By Lemma 1, we have at least $\binom{4}{2}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + n + 3 > 4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + n + \lfloor\frac{n}{2}\rfloor + 2$ crossings in D which also confirms a contradiction with the assumption (3.2).
(b) Let $\mathrm{cr}_D(T^{v_i}, C_n^*) = 2$ for only one $i \in \{1,2,3,4\}$. If the drawing D is of type A_D, then the same idea as in Case 3(a) contradicts the assumption (3.2). If the drawing D is of type B_D, then by Lemma 1 and Corollary 1 for $p = 4$, $q = 3$, $k = 2$, we have at least $\binom{3}{2}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 3\lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor + n + 1 + 3 + 2$ crossings in D also due to two crossings between T^{v_5} and T^{v_i}. This confirms a contradiction with (3.2) for every $n \geq 6$. If $n = 5$, then using Lemma 1, Corollary 1, Lemma 6 or Lemma 4 and Lemma 7, we have $12 + 3 + 5 + 6 > 25$ crossings in D which again contradicts (3.2).
(c) Let $\mathrm{cr}_D(T^{v_i}, C_n^*) = 1$ and $\mathrm{cr}_D(T^{v_j}, C_n^*) = 1$ for two distinct $i, j \in \{1,2,3,4\}$. For such a index pair i, j, the subgraph $T^{v_i} \cup T^{v_j} \cup C_n^*$ is isomorphic to the graph $D_2 + C_n$. Consider $n-2$ vertices of the cycle C_n^* incident with edges of T^{v_i} and T^{v_j} which do not cross C_n^*. Let us delete all edges of T^{v_i} and T^{v_j} which are not incident with these $n-2$ vertices. The resulting subgraph is homeomorphic to the graph $D_2 + C_{n-2}$ and, in its subdrawing D' induced by D, we obtain $\mathrm{cr}_{D'}(T^{v_i}, T^{v_j}) \geq \lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor$ thanks to Lemma 1. Now, if the drawing D is of type A_D then by Lemma 1 and Corollary 1 for $p = 5$, $q = 3$, $k = 1$ we have at least $\binom{3}{2}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + (3+3)\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor + \lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor + n + 3 > 4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + n + \lfloor\frac{n}{2}\rfloor + 2$ crossings in D which confirms a contradiction with the assumption (3.2). For the drawing D of type B_D by Lemma 1, Corollary 1 for $p = 4$, $q = 2$, $k = 1$ with at least two crossings on the edges of T^{v_5}, we have at least $\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + (2+2)\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor + \lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor + (n+1) + 3 + 2$ crossings in D which confirms a contradiction with (3.2) for every $n \geq 6$. For $n = 5$ using Lemma 7, we

have at least six crossings on the edges of T^{v_5} in D which also contradicts (3.2).
(d) Let $\mathrm{cr}_D(T^{v_5}, C_n^*) = 1$ and $\mathrm{cr}_D(T^{v_i}, C_n^*) = 1$ for only one $i \in \{1,2,3,4\}$. Now, the same idea as in first part of Case 3(c) or in Case 2(b) for the drawing D of type A_D or of type B_D contradicts the assumption (3.2), respectively.
We have shown that there is no good drawing D of the graph $G^* + C_n$ with less than $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 3$ crossings, and this completes the proof of Theorem 3. □

4. Some consequence of the main result

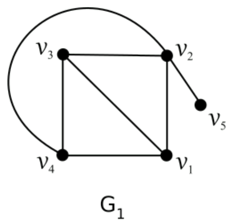

Figure 5. **The graph G_1 by adding one edge to the graph G^*.**

In Figure 5, let G_1 be the graph obtained from G^* by adding the edge v_2v_5 into the drawing in Figure 1(a). Since it is possible to add this edge to the graph G^* without additional crossings in Figure 2, the drawing of $G_1 + C_n$ with just $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 3$ crossings can be obtained. Thus, the next result is obvious.

Corollary 4. $\mathrm{cr}(G_1 + C_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 3$ *for* $n \geq 3$.

5. Conclusions

We suppose that similar forms of discussions can be helpful to determine unknown values of the crossing numbers of other symmetric graphs on five vertices with a much larger number of edges in the join products with cycles on n vertices. Especially for the complete graph K_5 and the graph $K_5 \setminus e$ obtained by removing one edge from K_5. The result of $G^* + C_n$ could also be useful to establish their crossings numbers with cycles C_n provided by the crossings numbers of both graph differences $K_5 - G^*$ and $(K_5 \setminus e) - G^*$ in the join products with C_n have already been well-known.

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Data Availability. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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