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Research Article

Complete solutions on local antimagic chromatic number of three families of disconnected graphs

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Abstract: An edge labeling of a graph $G = (V, E)$ is said to be local antimagic if it is a bijection $f : E \to \{1, \dots, |E|\}$ such that for any pair of adjacent vertices x and y, $f^+(x) \neq f^+(y)$, where the induced vertex label $f^+(x) = \sum f(e)$, with e ranging over all the edges incident to x. The local antimagic chromatic number of G, denoted by $\chi_{la}(G)$, is the minimum number of distinct induced vertex labels over all local antimagic labelings of G. In this paper, we study local antimagic labeling of disjoint unions of stars, paths and cycles whose components need not be identical. Consequently, we completely determined the local antimagic chromatic numbers of disjoint union of two stars, paths, and 2-regular graphs with at most one odd order component respectively.

Keywords: local antimagic labeling, local antimagic chromatic number, disconnected graphs.

AMS Subject classification: 05C78, 05C15.

1. Introduction

A graph $G = (V, E)$ is said to be *local antimagic* if it admits a *local antimagic edge labeling*, i.e., a bijection $f : E \to \{1, \dots, |E|\}$ such that the induced vertex labeling $f^+ : V \to \mathbb{Z}$ given by $f^+(u) = \sum f(e)$, with e ranging over all the edges incident to u, has the property that any two adjacent vertices have distinct induced vertex labels (see [1, 3]). Thus, f^+ is a coloring of G. Clearly, the order of G must be at least 3. The vertex label $f^+(u)$ is called the *induced color* of u under f (the *color* of

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u, for short, if no ambiguity occurs). The number of distinct induced colors under f is denoted by $c(f)$, and is called the *color number* of f. The labeling f is called a *local antimagic* $c(f)$*-coloring of* G. The *local antimagic chromatic number* of G, denoted by $\chi_{la}(G)$, is $\min\{c(f) \mid f$ is a local antimagic labeling of $G\}$. Haslegrave [7] proved that every connected graph other than K_2 is local antimagic. Hence $\chi_{la}(G)$ is well-defined for every connected or disconnected graph G not containing any isolated edge.

For graphs G and H, let $G + H$ be the disjoint union of G and H with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. For convenient, mG is the disjoint union of $m \geq 2$ isomorphic copies of G. We shall use the notation $[a, b] = \{c \in \mathbb{Z} \mid a \leq c \leq b\}$, for integers $a \leq b$. Unless stated otherwise, all graphs considered in this paper are simple, undirected and of order at least 3. A *pendant* vertex is a vertex of degree 1.

In [10], the authors provided some results on local antimagic labeling of disjoint unions of identical paths, identical stars, and identical cycles. However, certain parts of their proofs contain mistakes. The authors in [2] obtained many bounds on local antimagic chromatic number of disconnected graphs. In this paper, the local antimagic chromatic number of the disjoint union of two stars, paths and 2-regular graphs with at most one odd order cycle are completely determined.

We restate the following lemma in [8, Lemma 1] or [9, Lemma 2.1] below. It will be used later.

Lemma 1. [8, 9] *Let G be a graph of size q. Suppose there is a local antimagic labeling of G inducing a 2-coloring of G with colors x and y, where $x < y$. Let X and Y be the sets of vertices colored x and y, respectively. Then G is a bipartite graph with bipartition (X, Y) and $|X| > |Y|$. Moreover,*

$$x|X| = y|Y| = \frac{q(q+1)}{2}.$$

The contrapositive of Lemma 1 gives a sufficient condition for a bipartite graph G to have $\chi_{la}(G) \geq 3$.

2. Disjoint union of stars

In this section, we study the local antimagic chromatic number of a disjoint union of stars. Note that any bijective edge labeling of a disjoint union of stars, all of order at least 3, must be a local antimagic labeling. We need the following lemma, which is easily extended from the theorem in [8, Theorem 6] or [2, Lemma 2].

Lemma 2. *Let G be a graph with l pendant vertices. If G does not contain a K_2 as a component, then $\chi_{la}(G) \geq l + 1$.*

Proposition 1. *Let $G = K_{1,p_1} + K_{1,p_2} + \cdots + K_{1,p_n}$ where $2 \leq p_1 \leq p_2 \leq \cdots \leq p_n$, and q be the size of G, then $q + 1 \leq \chi_{la}(G) \leq q + n$.*

Proof. Note that $p_1+p_2+\cdots+p_n=q$. Since the order of G is $q+n$, $\chi_{la}(G)\le q+n$. By Lemma 2, $\chi_{la}(G)\ge q+1$. $\square$

Suppose all stars are identical. There are only two possible values for the local antimagic chromatic number as was proved in [2]. Here we give a shorter proof using magic rectangles. We will use the fact in [4–6] that an $a\times b$ magic rectangle exists if and only if $a,b\ge 2$, $a\equiv b \pmod 2$, and $(a,b)\ne(2,2)$.

Theorem 1. *Let $G=mK_{1,n}$ where $m\ge 3$ and $n\ge 2$, then*

$$\chi_{la}(G)=\begin{cases} mn+1 & \text{if } n \text{ is even,}\\ mn+1 & \text{if } m,n \text{ are both odd,}\\ mn+1 & \text{if } m \text{ is even, } n \text{ is odd, } n\le 2m-1,\\ mn+2 & \text{if } m \text{ is even, } n \text{ is odd, } n\ge 2m.\end{cases}$$

Proof. Obviously, $\chi_{la}(K_{1,n})=n+1$. By Proposition 1, $\chi_{la}(G)\ge mn+1$. Denote the m stars by $S_1,S_2,\dots,S_m$.
Suppose n is even. We label an edge of $S_1,S_2,\dots,S_m$ by $1,2,\dots,m$ respectively. Then label an edge of $S_m,S_{m-1},\dots,S_1$ by $m+1,m+2,\dots,2m$ respectively. Repeat this labeling process by $1,2,\dots,mn$ through alternating between the ascending and descending order of the stars. All centers will have identical label. So $\chi_{la}(G)=mn+1$.

Suppose m,n are both odd. Consider a magic rectangle of size $n\times m$ with entries $\{1,2,\dots,mn\}$. Use the i-th column as the edge labels of S_i for $1\le i\le m$. All centers have identical label and $\chi_{la}(G)=mn+1$.

Suppose m is even and n is odd. Note that $m\ge 4$ and each center label is at least $1+2+\cdots+n=\frac{n(n+1)}{2}$. Moreover, $\frac{n(n+1)}{2}\le mn$ if and only if $n\le 2m-1$.

For $n\le 2m-1$, since $m\ge 3$, consider a magic rectangle of size $n\times(m-1)$ with entries $\{n+1,n+2,\dots,mn\}$. Use the i-th column as the edge labels of S_i for $1\le i\le m-1$. Label the edges of S_m by $1,2,\dots,n$. The centers of $S_1,S_2,\dots,S_{m-1}$ have identical label while the center of S_m has label $\frac{n(n+1)}{2}$. Since $n+1<\frac{n(n+1)}{2}\le mn$, this center label equals a pendant vertex label of S_i, $1\le i\le m-1$. Hence, $\chi_{la}(G)=mn+1$.

For $n\ge 2m$, each center label is greater than mn. Since $\frac{1}{m}(1+2+\cdots+mn)=\frac{n(mn+1)}{2}$ is not an integer, the center labels cannot be all identical. So $\chi_{la}(G)\ge mn+2$. Since $m\ge 3$, consider a magic rectangle of size $n\times(m-1)$ with entries $\{1,2,\dots,(m-1)n\}$. Use the i-th column as the edge labels of S_i for $1\le i\le m-1$. Label the edges of S_m by $(m-1)n+1,(m-1)n+2,\dots,mn$. The centers of $S_1,S_2,\dots,S_{m-1}$ have identical label. Hence, $\chi_{la}(G)=mn+2$. $\square$

For $m=1$, it is trivial that $\chi_{la}(K_{1,n})=n+1$. The remaining part focuses on the disjoint union of two stars, $G=K_{1,p}+K_{1,q}$ where $2\le p\le q$. From Proposition 1, $p+q+1\le\chi_{la}(G)\le p+q+2$. We will derive the necessary and sufficient conditions for $\chi_{la}(G)=p+q+1$.

Lemma 3. *Consider $K_{1,p}$ where $p \geq 2$. Let q be any postive integer. Label the edges of $K_{1,p}$ using $\{1, 2, \ldots, p+q\}$ injectively. For any $x \in [\frac{p(p+1)}{2}, \frac{p(p+2q+1)}{2}]$, there exists a labeling such that the center of $K_{1,p}$ has label x.*

Proof. Start with the edge labels $1, 2, \ldots, p$. Now, keep adding 1 to the last edge label starting from p until $p+q$. Next, keep adding 1 to the second last edge label starting from $p-1$ until $p+q-1$. Repeat this process. Finally, keep adding 1 to the first label starting from 1 until $p+q-(p-1)$. The final edge labels will be $p+q-(p-1), p+q-(p-2), \ldots, p+q-1, p+q$. Each time the center label increases by 1. Therefore, it can attain every possible value from $\frac{p(p+1)}{2}$ to $\frac{p(p+2q+1)}{2}$. □

Theorem 2. *Let $G = K_{1,p} + K_{1,q}$ where $2 \leq p \leq q$, then $\chi_{la}(G) = p + q + 1$ if and only if $q \geq \frac{p(p-1)}{2}$; or $p + q \equiv 0, -1 \pmod 4$ and $q \leq \left\lfloor \frac{2p-1+\sqrt{8p^2+1}}{2} \right\rfloor$.*

Proof. Denote the label of the centers of $K_{1,p}$ and $K_{1,q}$ by x and y, respectively. Then

(1) $\frac{p(p+1)}{2} = 1+2+\cdots+p \leq x \leq (p+q)+(p+q-1)+\cdots+(p+q-(p-1)) = \frac{p(p+2q+1)}{2}$,

(2) $\frac{q(q+1)}{2} = 1+2+\cdots+q \leq y \leq (p+q)+(p+q-1)+\cdots+(p+q-(q-1)) = \frac{q(q+2p+1)}{2}$,

(3) $x + y = 1 + 2 + \cdots + (p + q) = \frac{(p+q)(p+q+1)}{2}$.

($\Rightarrow$) Suppose $\chi_{la}(G) = p+q+1$. Then at least one of x, y is greater than $p+q$. Both x and y are greater than $p+q$ if and only if $x = y$. There are three cases to consider.

(a) $x \leq p + q < y$. By (1), $\frac{p(p+1)}{2} \leq x \leq p + q$. This implies $q \geq \frac{p(p-1)}{2}$.

(b) $y \leq p + q < x$. By (2), $\frac{q(q+1)}{2} \leq y \leq p + q$. Hence, $q \geq p \geq \frac{q(q-1)}{2} \geq \frac{p(p-1)}{2}$.

(c) $p+q < x = y$. By (3), $(p+q)(p+q+1) = 4x$. Therefore, $p+q \equiv 0, -1 \pmod 4$. Since $x \leq \frac{p(p+2q+1)}{2}$ by (1), $x = \frac{(p+q)(p+q+1)}{4} \leq \frac{p(p+2q+1)}{2}$. Upon simplifications, $q^2 - (2p-1)q - p(p+1) \leq 0$ which is equivalent to $q \leq \left\lfloor \frac{2p-1+\sqrt{8p^2+1}}{2} \right\rfloor$.

($\Leftarrow$) Suppose $q \geq \frac{p(p-1)}{2}$. By labeling the edges of $K_{1,p}$ by $1, 2, \ldots, p$, $x = \frac{p(p+1)}{2}$. Since $p + q \geq p + \frac{p(p-1)}{2} = \frac{p(p+1)}{2} = x$, $\chi_{la}(G) = p + q + 1$ by Proposition 1.
Suppose $p + q \equiv 0, -1 \pmod 4$ and $q \leq \left\lfloor \frac{2p-1+\sqrt{8p^2+1}}{2} \right\rfloor$. Then $\frac{(p+q)(p+q+1)}{4}$ is an integer and from (c), $\frac{p(p+1)}{2} \leq \frac{(p+q)(p+q+1)}{4} \leq \frac{p(p+2q+1)}{2}$. By Lemma 3, there is an edge labeling of $K_{1,p}$ using $\{1, 2, \ldots, p+q\}$ such that $x = \frac{(p+q)(p+q+1)}{4} = y$. Hence, $\chi_{la}(G) = p + q + 1$ by Proposition 1. □

Corollary 1. *For $n \geq 2$,* $\chi_{la}(2K_{1,n}) = \begin{cases} 2n+1 & \text{if } n \text{ is even or } n = 3, \\ 2n+2 & \text{otherwise.} \end{cases}$

3. Disjoint union of paths and cycles

In [2], Bača et al. proved that $\chi_{la}(mC_{2n}) = 3$, $\chi_{la}(mC_{2n+1}) \le m+2$ and $\chi_{la}(mP_n) = 2m+1$. Here we generalize these results to disjoint union of paths and cycles which need not be identical. The proofs are direct applications of the labelings introduced in [1] showing that $\chi_{la}(P_n) = 3$ and $\chi_{la}(C_n) = 3$.

Let $P_n = v_1v_2\cdots v_n$ with $n \ge 3$. Let $e_i = v_iv_{i+1}$, $1 \le i \le n-1$. Define $f : E \to \{1, 2, \ldots, n-1\}$ by

$$f(e_i) = \begin{cases} n - \frac{i+1}{2} & \text{if } i \text{ is odd,} \\ \frac{i}{2} & \text{if } i \text{ is even,} \end{cases} \tag{3.1}$$

then

$$f^+(v_i) = \begin{cases} n-1 & \text{if } i \text{ is odd, } i \ne n, \\ n & \text{if } i \text{ is even, } i \ne n, \\ \lfloor \frac{n}{2} \rfloor & \text{if } i = n. \end{cases}$$

Let $C_n = v_1v_2\cdots v_nv_1$ with $n \ge 3$. Let $e_i = v_iv_{i+1}$, $1 \le i \le n-1$ and $e_n = v_nv_1$. Define $f : E \to \{1, 2, \ldots, n\}$ by

$$f(e_i) = \begin{cases} n - \frac{i-1}{2} & \text{if } i \text{ is odd,} \\ \frac{i}{2} & \text{if } i \text{ is even,} \end{cases} \tag{3.2}$$

then

$$f^+(v_i) = \begin{cases} n & \text{if } i \text{ is odd, } i \ne 1, \\ n+1 & \text{if } i \text{ is even,} \\ 2n - \lfloor \frac{n}{2} \rfloor & \text{if } i = 1. \end{cases}$$

Let P be a path. Suppose $a, b \in V(P)$. Let aPb denote the subpath of P starting from a to b. Let $C = v_1v_2\cdots v_nv_1$ be an n-cycle and be drawn as a plane graph. Suppose $a, b \in V(C)$. Let aCb be the subpath of C from a to b clockwise.

Theorem 3. *Let $G = P_{n_1} + P_{n_2} + \cdots + P_{n_m}$, where $n_i \ge 3$, $1 \le i \le m$, then $\chi_{la}(G) = 2m+1$.*

Proof. By Lemma 2, $\chi_{la}(G) \ge 2m+1$. Let $n = n_1+n_2+\cdots+n_m-(m-1)$. Consider the labeling of P_n using (3.1). Clearly, every two adjacent vertices are with distinct labels. Decompose P_n into m paths $P_{n_1} \cup P_{n_2} \cup \cdots \cup P_{n_m}$ defined by $P_{n_1} = v_1P_nv_{n_1}$ and $P_{n_k} = v_{n_1+n_2+\cdots+n_{k-1}-(k-2)}P_nv_{n_1+n_2+\cdots+n_k-(k-1)}$ for $1 < k \le m$. Note that the label of the first vertex of P_{n_1} is $n-1$, the labels of the internal vertices of P_{n_k} are either $n-1$ or n, and the labels of the two end vertices of P_{n_k} are all distinct and at most $n-1$. Thus, G is local antimagic. Hence, $\chi_{la}(G) = 2m+1$. □

Theorem 4. *Let $G = C_{n_1} + C_{n_2} + \cdots + C_{n_m}$, where $n_i \geq 3$, $1 \leq i \leq m$, then $3 \leq \chi_{la}(G) \leq m+2$.*

Proof. Suppose one of n_i's is odd, then $3 = \chi(G) \leq \chi_{la}(G)$. Now assume all n_i's are even, then G is a bipartite graph with the same size of partitions. By Lemma 1, $\chi_{la}(G) \geq 3$.

For the second inequality, consider the labeling f of C_n using (3.2), where $n = n_1 + n_2 + \cdots + n_m$. Decompose C_n into m paths $R_{n_1}, R_{n_2}, \ldots, R_{n_m}$ defined by $R_{n_1} = v_1 C_n v_{n_1+1}$, $R_{n_k} = v_{n_1+n_2+\cdots+n_{k-1}+1} C_n v_{n_1+n_2+\cdots+n_k+1}$ for $1 < k < m$, and $R_{n_m} = v_{n_1+n_2+\cdots+n_{m-1}+1} C_n v_1$. Note that the labels of the internal vertices of R_{n_k} are either n or $n+1$ alternately. Now identify the two end vertices of R_{n_k} to form C_{n_k}. The resulting 2-regular graph G admits a labeling such that every vertex, except at most one, must have label n or $n+1$. Note that no two adjacent vertices have the same label because they are incident to a common edge and the other two edges incident to them have different labels. Therefore, there are at most $m+2$ distinct vertex labels and $\chi_{la}(G) \leq m+2$. □

Remark 1. Suppose $n_1 = n_2 = \cdots = n_m = 2n$. The labels of the internal vertices of R_{n_k} are either $2mn$ or $2mn+1$. The labels of the first edge and the last edge of R_{n_k} are $(2m-k+1)n$ and kn respectively which add up to $(2m+1)n$. Therefore, $\chi_{la}(mC_{2n}) = 3$.

Let $G = C_{n_1} + C_{n_2} + \cdots + C_{n_m}$. A natural question to ask is: under what conditions, is $\chi_{la}(G) = 3$? As we shall see, this holds if all but at most one of the cycles are even. The following lemma is easy to obtain.

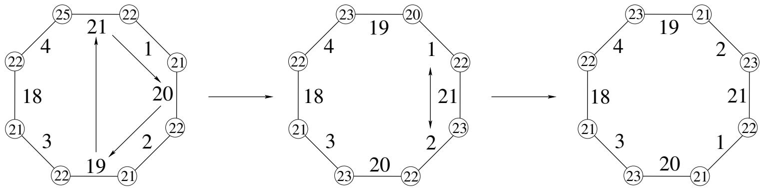
Lemma 4. *Let $P = v_1 v_2 \cdots v_{2k+2}$ be a subpath of a graph G, where $k \geq 1$. Suppose there is an edge labeling f of G such that $f(v_{2i+1}v_{2i+2}) = c \pm i$ for $0 \leq i \leq k$ and a fixed integer c, so that either all are plus signs or all are minus signs. Define a new labeling g of G such that*

$$g(e) = \begin{cases} f(e) & \text{if } e \notin E(P); \\ f(v_{2k+1}v_{2k+2}) & \text{if } e = v_1 v_2; \\ f(v_{2i-1}v_{2i}) & \text{if } e = v_{2i+1}v_{2i+2},\ 1 \leq i \leq k, \end{cases}$$
$$= \begin{cases} f(e) & \text{if } e \notin E(P); \\ c \pm k & \text{if } e = v_1 v_2; \\ c \pm (i-1) & \text{if } e = v_{2i+1}v_{2i+2},\ 1 \leq i \leq k, \end{cases}$$

then $g^+(v_1) = f^+(v_1) \pm k$, $g^+(v_2) = f^+(v_2) \pm k$ and $g^+(v_i) = f^+(v_i) \mp 1$ for $3 \leq i \leq 2k+2$. Moreover, $g^+(v) = f^+(v)$ for all $v \in V(G) \setminus V(P)$.

Such labeling g is called a *cyclic permutation of f* on a $(2k+2)$-path.

Example 1. Consider a labeling of C_8 shown in the leftmost graph of the following figure. By using Lemma 4 repeatedly, we can make the vertex labels of a cycle as close as possible. Firstly, we consider the subpath starting from the edge labeled by 21 to the edge labeled by 19 clockwise. Secondly, we consider the subpath starting from the edge labeled by 1 to the edge labeled by 2 clockwise.



$c = 21, k = 2$ $\qquad$ $c = 1, k = 1$

■

Lemma 5. *Let $C = v_1v_2\cdots v_nv_1$, $e_i = v_iv_{i+1}$ for $1 \le i \le n-1$ and $e_n = v_nv_1$. Let f be an edge labeling of C defined by*

$$f(e_i) = \begin{cases} a - \frac{i-1}{2} & \text{if } i \text{ is odd;} \\ b - 1 + \frac{i}{2} & \text{if } i \text{ is even,} \end{cases}$$

for some fixed integers a, b. If n is odd, additionally assume that $a - b = n - 1$. Then there exists an edge labeling of C using the same set of labels such that the induced vertex labels are $a+b-1$, $a+b$, $a+b+1$.

Proof. It is easy to see that

$$\begin{aligned} f^+(v_i) &= a + b + \frac{(-1)^i - 1}{2}, \quad \text{for } i \neq 1 \text{ and} \\ f^+(v_1) &= \begin{cases} a + b + \lfloor \frac{n-1}{2} \rfloor & \text{if } n \text{ is even;} \\ 2a - \lfloor \frac{n-1}{2} \rfloor & \text{if } n \text{ is odd.} \end{cases} \end{aligned} \tag{3.3}$$

When n is odd, under the additional assumption, we have $2a - \lfloor \frac{n-1}{2} \rfloor = a + (b + n - 1) - \frac{n-1}{2} = a+b+\frac{n-1}{2} = a+b+\lfloor \frac{n-1}{2} \rfloor$. So we may always write $f^+(v_1) = a+b+\lfloor \frac{n-1}{2} \rfloor$ for each n.

Let $m = \lfloor \frac{n-1}{2} \rfloor - 1$. Note that

$$n = \begin{cases} 2m + 3 & \text{if } n \text{ is odd;} \\ 2m + 4 & \text{if } n \text{ is even.} \end{cases}$$

Let f_1 be the cyclic permutation of f on the path $v_1v_2\cdots v_{2m+1}v_{2m+2}$ and f_k be the cyclic permutation of f_{k-1} on the path $v_kv_{k+1}\cdots v_{2m-k+2}v_{2m-k+3}$ for $2 \le k \le m$.

By (3.3) and Lemma 4, we have

$$
\begin{aligned}
f_m^+(v_1) &= f_1^+(v_1) = f^+(v_1) - m \\
&= a + b + m + 1 - m = a + b + 1. \\
f_m^+(v_{2m+2}) &= f_1^+(v_{2m+2}) = f^+(v_{2m+2}) + 1 \\
&= a + b + 1. \\
f_m^+(v_2) &= f_2^+(v_2) = f_1^+(v_2) + (m-1) \\
&= f^+(v_2) - m + (m-1) \\
&= a + b - m + (m-1) = a + b - 1. \\
f_m^+(v_{2m+1}) &= f_2^+(v_{2m+1}) = f_1^+(v_{2m+1}) - 1 \\
&= f^+(v_{2m+1}) + 1 - 1 \\
&= a + b - 1 + 1 - 1 = a + b - 1.
\end{aligned}
$$

For $3 \leq k \leq m$, by (3.3) and Lemma 4 ,

$$
\begin{aligned}
f_m^+(v_k) &= f_k^+(v_k) \\
&= f_{k-1}^+(v_k) + (-1)^k[m - (k-1)] \\
&= f_{k-2}^+(v_k) + (-1)^{k-1}[m - (k-2)] + (-1)^k[m - (k-1)] \\
&= [f_{k-3}^+(v_k) + (-1)^{k-1}] + (-1)^{k-1} \\
&\quad \vdots \\
&= [f^+(v_k) + (-1)^2 + (-1)^3 + \cdots + (-1)^{k-1}] + (-1)^{k-1} \\
&= [a + b + \frac{(-1)^k - 1}{2} + (-1)^2 + (-1)^3 + \cdots + (-1)^{k-1}] + (-1)^{k-1} \\
&= \begin{cases} a + b + 1 & \text{if } k \text{ is odd;} \\ a + b - 1 & \text{if } k \text{ is even.} \end{cases}
\end{aligned}
$$

For $3 \leq k \leq m$, by (3.3) and Lemma 4,

$$
\begin{aligned}
f_m^+(v_{2m+3-k}) &= f_k^+(v_{2m+3-k}) \\
&= f_{k-1}^+(v_{2m+3-k}) + (-1)^{k-1} \\
&= f_{k-2}^+(v_{2m+3-k}) + (-1)^{k-2} + (-1)^{k-1} \\
&\quad \vdots \\
&= f^+(v_{2m+3-k}) + (-1)^0 + (-1)^1 + \cdots + (-1)^{k-2} + (-1)^{k-1} \\
&= a + b + \frac{(-1)^{2m+3-k} - 1}{2} + (-1)^0 + (-1)^1 + \cdots + (-1)^{k-2} + (-1)^{k-1} \\
&= \begin{cases} a + b + 1 & \text{if } k \text{ is odd;} \\ a + b - 1 & \text{if } k \text{ is even.} \end{cases}
\end{aligned}
$$

$$\begin{aligned}
f^+_m(v_{m+1}) &= f^+_{m-1}(v_{m+1}) + (-1)^m \\
&= f^+_{m-2}(v_{m+1}) + (-1)^{m-2} + (-1)^m \\
&\quad \vdots \\
&= f^+(v_{m+1}) + (-1)^0 + (-1)^1 + \cdots + (-1)^{m-2} + (-1)^m \\
&= a + b + \frac{(-1)^{m+1} - 1}{2} + (-1)^0 + (-1)^1 + \cdots + (-1)^{m-2} + (-1)^m \\
&= \begin{cases} a+b+1 & \text{if } m+1 \text{ is odd;} \\ a+b-1 & \text{if } m+1 \text{ is even.} \end{cases}
\end{aligned}$$

$$\begin{aligned}
f^+_m(v_{m+2}) &= f^+_{m-1}(v_{m+2}) + (-1)^{m-1} \\
&= f^+_{m-2}(v_{m+2}) + (-1)^{m-2} + (-1)^{m-1} \\
&\quad \vdots \\
&= f^+(v_{m+2}) + (-1)^0 + (-1)^1 + \cdots + (-1)^{m-2} + (-1)^{m-1} \\
&= a + b + \frac{(-1)^{m+2} - 1}{2} + (-1)^0 + (-1)^1 + \cdots + (-1)^{m-2} + (-1)^{m-1} \\
&= a + b.
\end{aligned}$$

Finally, if n is odd, $f^+_m(v_{2m+3}) = f^+_m(v_n) = f^+(v_n) = a+b-1$ by (3.3). If n is even, $f^+_m(v_{2m+3}) = f^+_m(v_{n-1}) = f^+(v_{n-1}) = a+b-1$ and $f^+_m(v_{2m+4}) = f^+_m(v_n) = f^+(v_n) = a+b$ by (3.3). Summing up, f_m is a required labeling. $\square$

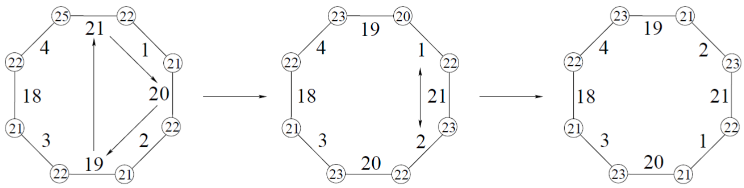
Theorem 5. *Let $G = C_{n_1} + C_{n_2} + \cdots + C_{n_m}$, where $n_i \geq 3$ for $1 \leq i \leq m$, and n_i is even for $1 \leq i \leq m-1$, then $\chi_{la}(G) = 3$.*

Proof. Let $n = n_1 + n_2 + \cdots + n_m$. For $1 \leq j \leq m$, denote $C_{n_j} = v^j_1 v^j_2 \cdots v^j_{n_j} v^j_1$, $e^j_i = v^j_i v^j_{i+1}$ for $1 \leq i \leq n_j - 1$ and $e^j_{n_j} = v^j_{n_j} v^j_1$. Let $a_1 = n$ and $b_1 = 1$. For $2 \leq j \leq m$, define $a_j = n - \frac{1}{2}(n_1 + n_2 + \cdots + n_{j-1})$ and $b_j = \frac{1}{2}(n_1 + n_2 + \cdots + n_{j-1}) + 1$. Note that $a_j + b_j = n+1$ for $1 \leq j \leq m$, and $a_m - b_m = n_m - 1$. Define an edge labeling f of G as follows.
For $1 \leq j \leq m$ and $1 \leq i \leq n_j$,

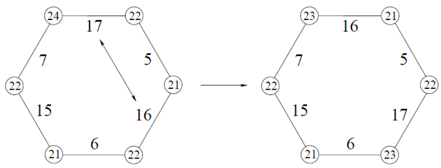
$$f(e^j_i) = \begin{cases} a_j - \frac{i-1}{2} & \text{if } i \text{ is odd} \\ b_j - 1 + \frac{i}{2} & \text{if } i \text{ is even} \end{cases}$$

By Lemma 5, there is an edge labeling g_j for C_{n_j} with induced vertex labels $a_j + b_j - 1 = n$, $a_j + b_j = n+1$ and $a_j + b_j + 1 = n+2$ for $1 \leq j \leq m$. Since the domain of each g_j is the same as the restriction of f on C_{n_j}, we have a local antimagic 3-coloring of C with induced vertex labels n, $n+1$ and $n+2$. By Theorem 4, $\chi_{la}(G) = 3$. $\square$

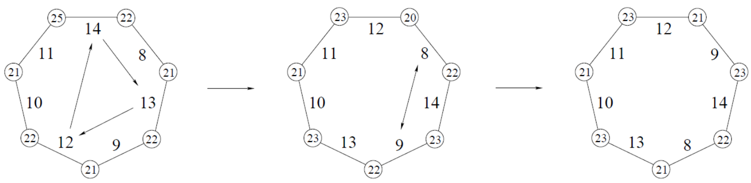
Example 2. Consider $C_8 + C_6 + C_7$. Here $n_1 = 8$, $n_2 = 6$, $n_3 = 7$ and $n = 21$. First we use the labeling of C_8 obtained in Example 1.



For C_6, we have



For C_7, we have



Thus, we have a local antimagic 3-coloring for $C_8 + C_6 + C_7$ with vertex labels 21, 22, 23. ■

4. Disjoint union of two odd cycles

In this section, we study disjoint union of two cycles, $C_p + C_q$. By Theorem 4, $3 \leq \chi_{la}(C_p + C_q) \leq 4$. By Theorem 5, $\chi_{la}(C_p + C_q) = 3$ if p or q is even. Therefore, we will focus on the disjoint union of two odd cycles. First, we give two families that have local antimagic chromatic number 3.

Theorem 6. *For $k \geq 2$, $\chi_{la}(C_{2k-1} + C_{2k+1}) = 3$.*

Proof. It suffices to construct a local antimagic 3-colorings g for $C_{2k-1} + C_{2k+1}$.

For C_{2k+1}, let e_i be the i-th edge in the clockwise order for $1 \leq i \leq 2k+1$. Define $g : E(C_{2k+1}) \to \{2j-1 \mid 1 \leq j \leq k+1\} \cup \{2j \mid k+1 \leq j \leq 2k\}$ by

$$g(e_i) = \begin{cases} i & \text{if } i \text{ is odd;} \\ 4k+2-i & \text{if } i \text{ is even.} \end{cases}$$

For C_{2k-1}, let e_i be the i-th edge in the clockwise order for $1 \le i \le 2k-1$. Define $g : E(C_{2k-1}) \to \{2j \mid 1 \le i \le k\} \cup \{2j-1 \mid k+2 \le j \le 2k\}$ by

$$g(e_i) = \begin{cases} i+1 & \text{if } i \text{ is odd;} \\ 4k+1-i & \text{if } i \text{ is even.} \end{cases}$$

It is easy to check that g is a local antimagic labeling of $C_{2k+1} + C_{2k-1}$ with induced vertex labels $2k+2$, $4k+1$, $4k+3$. □

Theorem 7. *For $k \ge 1$, $\chi_{la}(C_3 + C_{4k+1}) = 3$.*

Proof. We construct a local antimagic 3-coloring of C_3+C_{4k+1}. For $C_3 = u_1u_2u_3u_1$, define $g(u_1u_2) = k+1$, $g(u_2u_3) = 2k+2$, and $g(u_3u_1) = 3k+4$. Note that $g^+(u_1) = 4k+5$, $g^+(u_2) = 3k+3$, $g^+(u_3) = 5k+6$.

For C_{4k+1}, let $e_i = v_iv_{i+1}$ be the i-th edge in the clockwise order for $1 \le i \le 4k+1$, where $v_{4k+2} = v_1$. Define $g : E(C_{4k+1}) \to [1, 4k+4] \setminus \{k+1, 2k+2, 3k+4\}$ by

$$\begin{aligned}
g(e_{4i-3}) &= i, \quad 1 \le i \le k. \\
g(e_{4i-2}) &= 3k+3-i, \quad 1 \le i \le k. \\
g(e_{4i-1}) &= \begin{cases} k+2+i & \text{if } 1 \le i \le k-1; \\ 3k+3 & \text{if } i = k. \end{cases} \\
g(e_{4i}) &= \begin{cases} 4k+4-i & \text{if } 1 \le i \le k-1; \\ k+2 & \text{if } i = k. \end{cases} \\
g(e_{4k+1}) &= 4k+4.
\end{aligned}$$

We have

$$\begin{aligned}
g^+(v_1) &= g(e_{4k+1}) + g(e_1) = 4k+4+1 = 4k+5. \\
g^+(v_{4i-3}) &= g(e_{4i-4}) + g(e_{4i-3}) = 4k+4-(i-1)+i = 4k+5, \quad 2 \le i \le k. \\
g^+(v_{4i-2}) &= g(e_{4i-3}) + g(e_{4i-2}) = i+3k+3-i = 3k+3, \quad 1 \le i \le k. \\
g^+(v_{4i-1}) &= g(e_{4i-2}) + g(e_{4i-1}) = 3k+3-i+k+2+i = 4k+5, \quad 1 \le i \le k-1. \\
g^+(v_{4k-1}) &= g(e_{4k-2}) + g(e_{4k-1}) = 3k+3-k+3k+3 = 5k+6. \\
g^+(v_{4i}) &= g(e_{4i-1}) + g(e_{4i}) = k+2+i+4k+4-i = 5k+6, \quad 1 \le i \le k-1. \\
g^+(v_{4k}) &= g(e_{4k-1}) + g(e_{4k}) = 3k+3+k+2 = 4k+5. \\
g^+(v_{4k+1}) &= g(e_{4k}) + g(e_{4k+1}) = k+2+4k+4 = 5k+6.
\end{aligned}$$

□

Next, we will prove that some disjoint unions of two odd cycles have local antimagic chromatic number 4. Below are some useful lemmas.

Lemma 6. *Let G be a disjoint union of cycles with n vertices. Suppose $\chi_{la}(G) = 3$, then the edge labeled 1 is adjacent to the edge labeled n. Moreover, one vertex label is less than $n+1$, one equals to $n+1$, and one is greater than $n+1$.*

Proof. Consider a local antimagic 3-coloring f of G. Let st be the edge with $f(st) = 1$ and uv be the edge with $f(uv) = n$. Then $f^+(s), f^+(t) \leq n+1$ and $f^+(u), f^+(v) \geq n+1$. Since $\chi_{la}(G) = 3$, $\{f^+(s), f^+(t)\} \cap \{f^+(u), f^+(v)\} \neq \varnothing$. Without loss of generality, assume $f^+(s) = f^+(u)$. Then $f^+(s) = f^+(u) = n+1$ and $s = u$. This implies st is adjacent to uv. Also, $f^+(t) < n+1$ and $f^+(v) > n+1$. The results follow. □

Lemma 7. *Let G be a disjoint union of cycles with n vertices and $\chi_{la}(G) = 3$. Let f be a local antimagic 3-coloring of G with colors $a < b = n+1 < c$. Suppose uv is an edge with $f(uv) = z$.*

(1) If $f^+(u) = a$ and $f^+(v) = b$, then $c-z < a$ or $c-z > n$, i.e., $z \in [1, c-n-1] \cup [c-a+1, n]$;

(2) If $f^+(u) = a$ and $f^+(v) = c$, then $c-n \leq b-z \leq a-1$, i.e., $z \in [n+2-a, 2n+1-c]$;

(3) If $f^+(u) = b$ and $f^+(v) = c$, then $a-z \leq 0$ or $a-z \geq c-n$, i.e., $z \in [1, n+a-c] \cup [a, n]$.

Proof.

(1) Suppose $a \leq c-z \leq n$. Let st be the edge with $f(st) = c-z$. Since $a \leq c-z$, $f^+(s), f^+(t) > a$. Assume $f^+(s) = b$ and $f^+(t) = c$. This implies st is adjcent to the edge labeled z at t. Hence, c is an induced vertex label of u or v which is impossible. Therefore, $c-z < a$ or $c-z > n$.

(2) Let st be the edge with $f(st) = b-z$. Suppose $b-z \geq a$. Then $f^+(s), f^+(t) > a$. Assume $f^+(s) = b$ and $f^+(t) = c$. This implies st is adjacent to the edge labeled z at s. Hence, b is an induced vertex label of u or v which is impossible. Therefore, $b-z \leq a-1$.

    Suppose $b-z < c-n$. Since $b-z+n < c$, $f^+(s), f^+(t) \neq c$. Assume $f^+(s) = a$ and $f^+(t) = b$. This implies st is adjcent to the edge labeled z at t. Hence, b is an induced vertex label of u or v which is impossible. Therefore, $c-n \leq b-z$.

(3) Suppose $0 < a-z < c-n$. Let st be the edge with $f(st) = a-z$. Since $a-z+n < c$, $f^+(s), f^+(t) \neq c$. Assume $f^+(s) = a$ and $f^+(t) = b$. This implies st is adjcent to the edge labeled z at s. Hence, a is an induced vertex label of u or v which is impossible. Therefore, $a-z \leq 0$ or $a-z \geq c-n$.

□

Theorem 8. *For $k \geq 1$, suppose $G = C_3 + C_{2k+1}$ and $\chi_{la}(G) = 3$. Then for any local antimagic 3-coloring of G, no edge in C_3 has label 1.*

Proof. Let f be a local antimagic 3-coloring of G. Suppose the edge with label 1 lies in $C_3 = uvwu$. By Lemma 6, we can assume $f(uv) = 1$, $f(vw) = 2k+4$ and $f(uw) = x$. Then $f^+(u) = x+1$, $f^+(v) = 2k+5$, $f^+(w) = 2k+4+x$. By applying Lemma 7(2) to uw, $2k+6-(x+1) \le x \le 4k+9-(2k+4+x)$. This implies $2x = 2k+5$ which is impossible. Therefore, no edge in C_3 has label 1. $\square$

Lemma 8. *For $k \ge 2$, suppose $G = C_3 + C_{4k-1}$ and $\chi_{la}(G) = 3$. Then there exists a local antimagic 3-coloring f of $C_3 + C_{4k-1}$ containing 4 consecutive edges in C_{4k-1} with labels $x, 1, 4k+2, y$ in clockwise order such that x and y satisfy all the conditions below.*

(1) $x \equiv y \equiv 1 \pmod 2$,

(2) $2y-1 \le x \le 4k+1$,

(3) $3 \le y \le 2k+1$,

(4) $3x-3y \ge 8k+2$ *or* $3y-x \ge 4$,

(5) $x+3y \le 8k+6 \le 3x+y$,

(6) $3x-3y \ge 8k+2$ *or* $3x-y \le 8k+2$.

In addition, the edge labels in C_3 are $\frac{x-y}{2}+1, \frac{x+y}{2}, 4k+2-\frac{x-y}{2}$, and the induced vertex labels are $x+1, 4k+3, 4k+2+y$.

Proof. By Lemma 6 and Theorem 8, assume the edge labels $x, 1, 4k+2, y$ are in clockwise order in C_{4k-1}, then the induced vertex labels are $x+1$, $4k+3$, $4k+2+y$. It is easy to solve that the edge labels in $C_3 = uvwu$ which are $\frac{x-y}{2}+1 = f(uv)$, $\frac{x+y}{2} = f(uw)$, $4k+2-\frac{x-y}{2} = f(vw)$, say. This implies that $x > y > 1$, and x, y have the same parity. By considering the local antimagic 3-coloring $4k+3-f$ of G, the edge labels $x, 1, 4k+2, y$ will be transformed to $4k+3-x, 4k+2, 1, 4k+3-y$ respectively. Therefore, without loss of generality, we can assume both x and y are odd integers with $3 \le y \le x-2 \le 4k-1$.

Consider the edge st with $f(st) = \frac{x+1}{2}$. Obviously, $\frac{x-y}{2}+1 < \frac{x+1}{2} < \frac{x+y}{2}$. If $\frac{x+1}{2} = 4k+2-\frac{x-y}{2}$, then $x = 4k+\frac{y+3}{2} > 4k+2$ which is impossible. Hence, st lies in C_{4k-1} and $\{f^+(s), f^+(t)\} = \{4k+3, 4k+2+y\}$. Therefore, $4k+2+y-\frac{x+1}{2}$ is an edge label implying $y \le \frac{x+1}{2}$, i.e., $2y-1 \le x$. Hence, $2y-1 \le x \le 4k+1$ and $3 \le y \le 2k+1$.

By applying Lemma 7(1) to the edge uv in C_3, we have

$$\frac{x-y}{2}+1 \le 4k+2+y-(4k+2)-1 \quad \text{or} \quad \frac{x-y}{2}+1 \ge 4k+2+y-(x+1)+1$$
$$\Leftrightarrow \qquad 3y-x \ge 4 \quad \text{or} \quad 3x-3y \ge 8k+2.$$

By applying Lemma 7(2) to the edge uw in C_3, we have

$$4k+4-(x+1) \le \frac{x+y}{2} \le 8k+5-(4k+2+y)$$
$$\Leftrightarrow \qquad x+3y \le 8k+6 \le 3x+y.$$

By applying Lemma 7(3) to the edge vw in C_3, we have

$$\begin{aligned} 4k+2-\frac{x-y}{2} \le 4k+2+(x+1)-(4k+2+y) \quad &\text{or} \quad 4k+2-\frac{x-y}{2} \ge x+1 \\ \Leftrightarrow \qquad 3x-3y \ge 8k+2 \quad &\text{or} \quad 3x-y \le 8k+2. \end{aligned}$$

By summarizing the results above, we have the lemma. $\square$

Theorem 9. *$\chi_{la}(C_3+C_3)=4$, $\chi_{la}(C_3+C_7)=4$ and $\chi_{la}(C_3+C_{11})=4$.*

Proof. By Theorem 8, $\chi_{la}(C_3+C_3)=4$. Suppose $G=C_3+C_{4k-1}$ for $k\ge 2$, and $\chi_{la}(G)=3$. Consider a local antimagic 3-coloring f of G given by Lemma 8.

For $k=2$, the only (x,y) satisfying all the conditions in Lemma 8 is $(9,3)$. The edge labels in C_3 are 4, 6, 7 with induced vertex labels 10, 11, 13. Consider the edge uv with $f(uv)=5$ which lies in C_{4k-1}. This means $10,11\notin\{f^+(u),f^+(v)\}$, a contradiction. Hence, $\chi_{la}(C_3+C_7)=4$.

For $k=3$, the only (x,y) satisfying all the conditions in Lemma 8 are $(13,3)$ and $(9,5)$.

(i) $(x,y)=(13,3)$. The edge labels in C_3 are $6,8,9$ with induced vertex labels $14,15,17$. Consider the edge uv with $f(uv)=7$ which lies in C_{4k-1}. This means $14,15\notin\{f^+(u),f^+(v)\}$, a contradiction.

(ii) $(x,y)=(9,5)$. The edge labels in C_3 are $3,7,12$ with induced vertex labels $10,15,19$. Consider the edge uv with $f(uv)=5$ which lies in C_{4k-1}. Thus $\{f^+(u),f^+(v)\}=\{15,19\}$. So the edge labels $10,5,14$ are in clockwise order in C_{4k-1}. This implies $9,10,5,14,1,9$ are in clockwise order in C_{4k-1}, a contradiction.

Hence, $\chi_{la}(C_3+C_{11})=4$. $\square$

Theorem 10. *For $k\ge 1$, suppose $G=C_5+C_{2k+3}$ and $\chi_{la}(G)=3$, then for any local antimagic 3-coloring of G, no edge in C_5 has label 1.*

Proof. Let f be a local antimagic 3-coloring of G. Suppose the edge with label 1 lies in $C_5=u_1u_2u_3u_4u_5u_1$. By Lemma 6, we can assume $f(u_1u_2)=1$, $f(u_2u_3)=2k+8$, $f(u_5u_1)=x$ and $f(u_3u_4)=y$. Thus, $f^+(u_1)=x+1$, $f^+(u_2)=2k+9$, $f^+(u_3)=2k+8+y$. There are three cases.

(1) Suppose $f^+(u_4)=x+1$ and $f^+(u_5)=2k+9$, then $f(u_4u_5)=x+1-y$. We have $x+1-y+x=2k+9$ implying $y=2x-2k-8$ and $x=k+4+\frac{y}{2}$. Hence, $k+4<x<2k+8$.

(2) Suppose $f^+(u_4)=x+1$ and $f^+(u_5)=2k+8+y$, then $f(u_4u_5)=x+1-y$. We have $x+1-y+x=2k+8+y$ implying $2x-2k-2y=7$ which is impossible.

(3) $f^+(u_4) = 2k+9$ and $f^+(u_5) = 2k+8+y$. By considering the local antimagic 3-coloring $2k+9-f$ of G, we have case (1).

Therefore, without loss of generality, assume $f(u_3u_4) = 2x-2k-8$ and $f(u_4u_5) = 2k+9-x$. Thus, $f^+(u_3) = 2x, f^+(u_4) = x+1, f^+(u_5) = 2k+9$. By applying Lemma 7(2) to u_3u_4, $(2k+8)+2-(x+1) \le 2x-2k-8 \le 2(2k+8)+1-2x$. This implies $\frac{4k+17}{3} \le x \le \frac{6k+25}{4}$. By applying Lemma 7(1) to u_4u_5, we have $2k+9-x \le 2x-(2k+8)-1$. This implies $x \ge \frac{4k+18}{3}$. Hence, $\frac{4k+18}{3} \le x \le \frac{6k+25}{4}$.

Suppose $x+1$ is even. Let uv be the edge with $f(uv) = \frac{x+1}{2}$. Note that $\{f^+(u), f^+(v)\} = \{2k+9, 2x\}$. By applying Lemma 7(3) to uv, we have $\frac{x+1}{2} \le 2k+8+(x+1)-2x$. This implies $x \le \frac{4k+17}{3}$, a contradiction.

Suppose $x+1$ is odd. Let uv be the edge with $f(uv) = k+5-\frac{x}{2}$, and st be the edge with $f(st) = k+4+\frac{x}{2}$. It is easy to check that $1 < k+5-\frac{x}{2} < x$, $k+5-\frac{x}{2} < 2k+9-x$ and $k+5-\frac{x}{2} < 2x-2k-8$. Also, $2k+9-x < k+4+\frac{x}{2} < 2k+8$, $x < k+4+\frac{x}{2}$ and $2x-2k-8 < k+4+\frac{x}{2}$. Therefore, both uv and st lies in C_{2k+3}, $\{f^+(u), f^+(v)\} = \{x+1, 2k+9\}$, and $\{f^+(s), f^+(t)\} = \{2k+9, 2x\}$. But then the edge labels $\frac{3x}{2}-k-4, k+5-\frac{x}{2}, k+4+\frac{x}{2}, \frac{3x}{2}-k-4$ are in clockwise order in C_{2k+3}, a contradiction.

Therefore, no edge in C_5 has label 1. □

As an immediate corollary, we have

Theorem 11. $\chi_{la}(C_5 + C_5) = 4$.

By using computer, we have checked that $\chi_{la}(C_3 + C_{4k-1}) = 4$ for $4 \le k \le 14$. We end this paper with the following conjecture.

Conjecture 4.1. For $k \ge 1$, $\chi_{la}(C_3 + C_{4k-1}) = 4$.

Conflict of interest. The authors declare that they have no conflict of interest.

Data Availability. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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