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Research Article

Nonlinear inclusion for thermo-electro-elastic: existence, dependence and optimal control

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Abstract: The objective of this paper is to examine a model of a thermo-electro-elastic body situated on a semi-insulator foundation. Friction is characterized by Tresca's friction law, and the contact is bilateral. The primary contribution is to derive the weak variational formulation of the model, constituting a system that couples three inclusions where the unknowns are the strain field, the electric field, and the temperature field. Subsequently, we demonstrate the unique solvability of the system, along with the continuous dependence of its solution under consideration. The secondary contribution involves the investigation of an associated optimal control problem, for which we establish the existence and convergence results. The proofs rely on arguments related to monotonicity, compactness, convex analysis, and lower semicontinuity.

Keywords: Thermo-electro-elastic materials, frictional contact problem, variational inequalities, stationary inclusion, continuous dependence, optimal control

AMS Subject classification: 74M15, 74Fxx, 74A45, 74M10, 47J22, 74S05

1. Introduction

The analysis of frictional contact problems has emerged as a prominent research focus, attracting growing interest due to its proven benefits in various domains, including technological and industrial applications. Consequently, the exploration of such problems has garnered significant attention in research circles. From a mathematical perspective, models describing thermo-electro-elastic phenomena are relatively recent and have been the subject of recent discussions in specific studies, as documented in references [4, 6, 7], and in other references [12, 16, 24, 25].

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The optimal control of the contact process is a topic of significant theoretical and practical importance. In many application scenarios, the primary focus is on investigating the observability properties of contact models and identifying their parameters. Furthermore, there is a necessity to examine the continuous dependence of solutions to contact problems concerning both data and parameters. This is crucial in addressing control and optimal design challenges associated with diverse mechanical structures. Nevertheless, optimal control problems related to stationary or differential inclusions and variational inequalities have been examined in diverse sources, encompassing references such as [3, 5, 8, 9, 11, 15, 20].

Nonlinear inclusions play a pivotal role in the exploration of various boundary value problems and find extensive applications across multiple disciplines, including Mechanics, Physics, Engineering, and Economics. Their solvability, when expressed through multivalued operators, requires insights derived from set-valued, convex, and nonsmooth analysis. Recent research has particularly focused on the variational formulation of contact models, often presented as inclusions or sweeping processes. For instance, recent work by the authors of [21] delves into a mathematical model describing frictionless contact between an elastic body and an obstacle. This study establishes that the model leads to a stationary inclusion for the strain field. These results are then applied in the investigation of an associated optimal control problem, where existence and convergence results are demonstrated. Similarly, the authors of [10] explore a mathematical model involving an electro-elastic body with a semi-insulator foundation. They derive a variational formulation in the form of a system that couples two inclusions, with unknowns encompassing the strain field and the electric field. These findings are employed in the consideration of an associated optimal control problem, for which the existence of optimal pairs is proven. Furthermore, the authors of [17] contribute to the field by examining the existence and uniqueness of a new class of time-dependent inclusions and sweeping processes in a real Hilbert space. Their study extends to the examination of mathematical models featuring a viscoelastic constitutive law. Numerous studies, detailed in [1, 2, 13, 14, 18, 19, 22], underscore the diversity of approaches and perspectives applied to tackle these challenging problems in this research area.

The objective of this article is to describe a new and nonstandard model for a complicated thermo-electro-elastic materials. We present a novel variational formulation using systems of three inclusions in which the unknowns are the strain field, the electric field, and the temperature field. The contact is bilateral and associated with Tresca's friction law. Additionally, we will address thermo-piezoelectric materials, for which the constitutive laws are given as follows.

$$\sigma = \mathcal{A}\varepsilon(u) + \mathcal{P}^T\nabla\varphi - \mathcal{M}\theta, \tag{1.1}$$

$$D = \mathcal{P}\varepsilon(u) - \beta\nabla\varphi - \mathcal{C}\theta, \tag{1.2}$$

$$q = \mathcal{K}\nabla\theta, \tag{1.3}$$

in which σ denotes the stress tensor, u is the displacement field, φ is the electric

potential field and θ is the temperature. The physics point of view of this constitutive laws can be found in [6, 25].
The purpose of our paper is to apply recent abstract results regarding the existence and uniqueness of stationary inclusions in Hilbert spaces, as demonstrated in [21]. The novelty of our work lies in introducing a new model for contact in thermo-electro-elastic materials. Our primary objective is twofold: firstly, to present a new and nonstandard model corresponding to the problem (1.1)-(1.3) and demonstrate its unique weak solvability; secondly, to establish the continuous dependence of the solution on the data and prove the solvability of an associated optimal control problem.
The paper is organized as follows. In Section 2, we present the mathematical model of a thermo-electro-elastic problem. Additionally, we etablish the variational formulation of this problem in the form of a system where the unknowns include the strain, electric field, and temperature field. Moving on to Section 3, we establish both the unique solvability of the model and the continuous dependence of the solution on the data. Section 4 is dedicated to examining an associated optimal control problem, for which we demonstrate the existence of optimal pairs.

2. Problem statement and variational formulation

In this section, we introduce a static contact problem for a thermo-electro-elastic body that occupies the domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with Lipschitz boundary Γ. Also, we suppose that $\Gamma = \partial\Omega$ is divided in three disjoint parts Γ_1, Γ_2 and Γ_3 such that $\overline{\Gamma}_1 \cup \overline{\Gamma}_2 \cup \overline{\Gamma}_3 = \Gamma$ and $meas(\Gamma_1) > 0$, on the one hand, and a partition of $\Gamma_1 \cup \Gamma_2$ into two open measurable parts Γ_a and Γ_b such that $meas(\Gamma_a) > 0$.
Let $\mathbb{S}^d$ denote the space of second order symmetric tensors on $\mathbb{R}^d$ while "$\cdot$" and "$\|\cdot\|$" represent both the inner product and the associated norm on $\mathbb{R}^d$ and $\mathbb{S}^d$, defined by

$$\begin{aligned} u \cdot v = u_i\, v_i \quad &, \quad \|v\| = (v \cdot v)^{1/2},\ \forall\, u,\, v \in \mathbb{R}^d, \\ \sigma \cdot \tau = \sigma_{ij}\, \tau_{ij} \quad &, \quad \|\tau\| = (\tau \cdot \tau)^{1/2},\ \forall\, \sigma,\, \tau \in \mathbb{S}^d. \end{aligned}$$

The normal and tangential components of the displacement vector $v \in \mathbb{R}^d$ and the stress tensor $\sigma \in \mathbb{S}^d$ on the boundary Γ are given by

$$\begin{aligned} v_\nu = v \cdot \nu \quad &, \quad v_\tau = v - v_\nu \nu, \\ \sigma_\nu = (\sigma\nu) \cdot \nu \quad &, \quad \sigma_\tau = \sigma\nu - \sigma_\nu \nu. \end{aligned}$$

From the orthogonality relations $v_\tau \cdot \nu = 0$ and $\sigma_\tau \cdot \nu = 0$, we derive the following useful equality

$$\sigma\nu \cdot v = \sigma_\nu v_\nu + \sigma_\tau \cdot v_\tau.$$

Then, the classical formulation of the frictional thermo-electro-elastic contact problem is as follows.

Problem (P). *Find a displacement* $u : \Omega \to \mathbb{R}^d$*, a stress field* $\sigma : \Omega \to \mathbb{S}^d$*, an electric potential* $\varphi : \Omega \to \mathbb{R}$ *, an electric displacement* $D : \Omega \to \mathbb{R}^d$ *and a temperature* $\theta : \Omega \to \mathbb{R}$ *:*

$$\sigma = \mathcal{A}\varepsilon(u) + \mathcal{P}^T\nabla\varphi - \mathcal{M}\theta \qquad \text{in} \quad \Omega, \tag{2.1}$$

$$D = \mathcal{P}\varepsilon(u) - \beta\nabla\varphi - \mathcal{C}\theta \qquad \text{in} \quad \Omega, \tag{2.2}$$

$$q = \mathcal{K}\nabla\theta \qquad \text{in} \quad \Omega, \tag{2.3}$$

$$\operatorname{Div}\sigma + f_0 = 0 \qquad \text{in} \quad \Omega, \tag{2.4}$$

$$\operatorname{div} D - q_0 = 0 \qquad \text{in} \quad \Omega, \tag{2.5}$$

$$\operatorname{div} q - h_0 = 0 \qquad \text{in} \quad \Omega, \tag{2.6}$$

$$u = 0 \qquad \text{on} \quad \Gamma_1, \tag{2.7}$$

$$\varphi = 0 \qquad \text{on} \quad \Gamma_a, \tag{2.8}$$

$$\theta = 0 \qquad \text{on} \quad \Gamma_1, \tag{2.9}$$

$$\sigma\nu = f_2 \qquad \text{on} \quad \Gamma_2, \tag{2.10}$$

$$D\cdot\nu = q_b \qquad \text{on} \quad \Gamma_b, \tag{2.11}$$

$$q\cdot\nu = h_n \qquad \text{on} \quad \Gamma_2, \tag{2.12}$$

$$u_\nu = 0 \qquad \text{on} \quad \Gamma_3, \tag{2.13}$$

$$\begin{cases} \|\sigma_\tau\| \le S_b, \\ \sigma_\tau = -S_b \dfrac{u_\tau}{\|u_\tau\|} \quad \text{if} \quad \|u_\tau\| \neq 0, \end{cases} \qquad \text{on} \quad \Gamma_3, \tag{2.14}$$

$$0 \le D\cdot\nu \le G, \quad D\cdot\nu = \begin{cases} 0 & \text{if} \quad \varphi < 0, \\ G & \text{if} \quad \varphi > 0, \end{cases} \qquad \text{on} \quad \Gamma_3, \tag{2.15}$$

$$0 \le q\cdot\nu \le H, \quad q\cdot\nu = \begin{cases} 0 & \text{if} \quad \theta < 0, \\ H & \text{if} \quad \theta > 0, \end{cases} \qquad \text{on} \quad \Gamma_3. \tag{2.16}$$

Here, conditions (2.1)-(2.3) represent the constitutive laws of thermo-electro-elastic, see [1–3] for more details, where $\mathcal{P} = (e_{ijk}) \in L^\infty(\Omega)$ is the piezoelectric tensor, $\beta = (\beta_{ij})$ is the symmetric and coercive electric permittivity tensors , $\mathcal{C} = (c_{ij})$ is the thermal expansion tensor and $\mathcal{K} = (k_{ij})$ is the thermal conductivity tensor. In addition, $(\mathcal{P})^T = (e_{kij})$ is the transpose tensor of $\mathcal{P}$. Equations (2.4)-(2.6) represent the equilibrium equations for displacement field, electric potential field and temperature field, respectively. Moreover, (2.7)-(2.13) are the mechanical, electrical and thermal boundary conditions. The condition (2.14) stands for Tresca law of dry friction, where S_b is the friction bound, see e.g., [23]. Equation (2.15) represents the electrical contact condition, where G represents a specified bound. This condition is applied under the assumption that the foundation is a semi-insulator. Refer to [10] for additional details regarding this equation. Finally, equation (2.16) delineates the thermal contact condition imposed at the boundary Γ_3, where H serves as an upper

limit. This condition is established under the assumption that the foundation is a semi-insulator. Suggests a scenariao where heat flux is present when the temperature is positive, but it be comes zero when the temerature is negative. Physically, this condition illustrates that, due to the semi-insulating nature of the foundation, heat is allowed to flow solely during temperature increases, while it is constrained or halted during temperature decreases.
The described scenario illustrates a condition in which the foundation serves as a perfect electric and thermal semi-insulator. This entails significant limitations on both electrical and thermal conductivity within the foundation material. Specifically, perfect electric semi-insulation hinders the flow of electric current, characterized by high electrical resistivity preventing easy electron passage. This limitation is crucial in applications like electronic devices or structural components where electrical insulation is essential. Similarly, perfect thermal semi-insulation imposes strong constraints on heat transfer, with low thermal conductivity resisting the flow of heat energy. This characteristic proves advantageous in scenarios requiring minimized heat transfer, such as in building materials to enhance insulation or specific manufacturing processes.
Next, to derive the variational formulation of Problem (**P**), according to the boundary conditions, we introduce the following variational subspaces

$$V = \{v \in H_1(\Omega)\ ;\ \ v = 0 \ \text{ on } \ \Gamma_1\,,\ \ v_\nu = 0 \ \text{ on } \ \Gamma_3\},$$

$$W = \{\psi \in H_1(\Omega)\ ;\ \ \psi = 0 \ \text{ on } \ \Gamma_a\},$$

$$Q = \{\theta \in H_1(\Omega)\ ;\ \ \theta = 0 \ \text{ on } \ \Gamma_1\},$$

$$\mathcal{H} = \big\{\tau = (\tau_{ij})\,;\ \tau_{ij} = \tau_{ji} \in L^2(\Omega),\ \forall\, 1 \le i,\, j \le d\big\}.$$

The spaces V, W and Q are Hilbert spaces for the following Euclidean norms

$$\|u\|_V = (u,u)_V^{1/2}\ ,\quad (u,v)_V = (\varepsilon(u),\varepsilon(v))_{\mathcal{H}}, \tag{2.17}$$

$$\|\varphi\|_W = (\varphi,\varphi)_W^{1/2}\ ,\quad (\varphi,\psi)_W = (\nabla\varphi,\nabla\psi)_{L^2(\Omega)}, \tag{2.18}$$

$$\|\theta\|_Q = (\theta,\theta)_Q^{1/2}\ ,\quad (\theta,\eta)_Q = (\nabla\theta,\nabla\eta)_{\mathcal{H}}. \tag{2.19}$$

It is known from Sobolev trace theorem, there exists $c_0 > 0$ depending only on Ω, Γ_3 and Γ_1 such that

$$\|v\|_{L^2(\Gamma)^d} \le c_0\,\|v\|_V,\quad \forall\, v \in V. \tag{2.20}$$

Furthermore, if σ belongs to the set $\mathcal{H}$, the subsequent Green-type formula is valid:

$$\int_\Omega \sigma\cdot\varepsilon(v)\,dx + \int_\Omega \operatorname{Div}\sigma\cdot v\,dx = \int_\Gamma \sigma\nu\cdot v\,da,\quad \forall\, v \in V. \tag{2.21}$$

In addition, the Sobolev trace theorem implies that there exists $c_1 > 0$ depending on Ω, Γ_a and Γ_3 such that

$$\|\psi\|_{L^2(\Gamma_3)} \leq c_1 \, \|\psi\|_W, \quad \forall \, \psi \in W. \tag{2.22}$$

Moreover, let ψ^+ represent the positive component of ψ in W. It's worth noting that, for a sufficiently smooth function $D \in L^2(\Omega)$, the Green-type formula is applicable:

$$(D, \nabla\psi)_{L^2(\Omega)^d} + (\operatorname{div} D, \psi)_{L^2(\Omega)} = \int_\Gamma D \cdot \nu\psi \, da, \quad \forall \, \psi \in W. \tag{2.23}$$

Also, the Sobolev trace theorem implies that there exists $c_2 > 0$ depending on Ω, Γ_1 and Γ_3 such that

$$\|\eta\|_{L^2(\Gamma_3)} \leq c_2 \, \|\eta\|_Q, \quad \forall \, \eta \in Q. \tag{2.24}$$

Moreover, let η^+ denote the positive component of η within the space Q. It's important to highlight that, for a sufficiently smooth function $q \in L^2(\Omega)$, the following Green-type formula holds.

$$(q, \nabla\eta)_{L^2(\Omega)^d} + (\operatorname{div} q, \eta)_{L^2(\Omega)} = \int_\Gamma q \cdot \nu\eta \, da, \quad \forall \, \eta \in Q. \tag{2.25}$$

Next, in the study of the solvability of Problem ($\mathbf{P}$), we need the following hypotheses.

($\mathcal{H}_1$) The tensor $\mathcal{A} : \Omega \times \mathbb{S}^d \longrightarrow \mathbb{S}^d$ is such that

(a) $\mathcal{A}(., \varepsilon)$ is measurable on Ω for all $\varepsilon \in \mathbb{S}^d$,

(b) there exist $L_\mathcal{A} > 0$ such that for all ε_1, $\varepsilon_2 \in \mathbb{S}^d$ and $x \in \Omega$, we have

$$\|\mathcal{A}(x, \varepsilon_1) - \mathcal{A}(x, \varepsilon_2)\| \leq L_\mathcal{A} \|\varepsilon_1 - \varepsilon_2\|, \tag{2.26}$$

(c) there exist $\alpha_\mathcal{A} > 0$ such that for all ε_1, $\varepsilon_2 \in \mathbb{S}^d$ and $x \in \Omega$, we have

$$(\mathcal{A}(x, \varepsilon_1) - \mathcal{A}(x, \varepsilon_2)) \cdot (\varepsilon_1 - \varepsilon_2) \geq \alpha_\mathcal{A} \|\varepsilon_1 - \varepsilon_2\|^2, \tag{2.27}$$

(d) $\mathcal{A}(x, 0) = 0$ for all $x \in \Omega$.

($\mathcal{H}_2$) The tensor of piezoelectric $\mathcal{P} = (e_{ijk}) : \Omega \times \mathbb{S}^d \longrightarrow \mathbb{R}^d$ is such that

(a) $e_{ijk} = e_{ikj} \in L^\infty(\Omega)$,

(b) there exist $L_\mathcal{P} > 0$ such that for all ε_1, $\varepsilon_2 \in \mathbb{S}^d$ and $x \in \Omega$, we have

$$\|\mathcal{P}(x, \varepsilon_1) - \mathcal{P}(x, \varepsilon_2)\| \leq L_\mathcal{P} \|\varepsilon_1 - \varepsilon_2\|, \tag{2.28}$$

(*c*) $\mathcal{P}(x,0) = 0$ for all $x \in \Omega$.

$(\mathcal{H}_3)$ The permittivity tensor $\beta = (\beta_{ij}) : \Omega \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is such that

(*a*) $\beta_{ij} = \beta_{ji} \in L^\infty(\Omega)$,

(*b*) there exist $L_\beta > 0$ such that for all $\xi_1, \xi_2 \in \mathbb{R}^d$ and $x \in \Omega$, we have

$$\|\beta(x,\xi_1) - \beta(x,\xi_2)\| \leq L_\beta \|\xi_1 - \xi_2\|, \tag{2.29}$$

(*c*) there exist $\alpha_\beta > 0$ such that for all ξ_1, $\xi_2 \in \mathbb{R}^d$ and $x \in \Omega$, we have

$$(\beta(x,\xi_1) - \beta(x,\xi_2)) \cdot (\xi_1 - \xi_2) \geq \alpha_\beta \|\xi_1 - \xi_2\|^2, \tag{2.30}$$

(*d*) $\beta(x,0) = 0$ for all $x \in \Omega$.

$(\mathcal{H}_4)$ The thermal operator $\mathcal{C} : \Omega \times \mathbb{R} \longrightarrow \mathbb{S}^d$ is such that

(*a*) $\mathcal{C}(.,r)$ is measurable on Ω for all $r \in \mathbb{R}$,

(*b*) there exist $L_\mathcal{C} > 0$ such that for all r_1, $r_2 \in \mathbb{R}$ and $x \in \Omega$, we have

$$\|\mathcal{C}(x,r_1) - \mathcal{C}(x,r_2)\| \leq L_\mathcal{C} |r_1 - r_2|, \tag{2.31}$$

(*c*) $\mathcal{C}(x,0) = 0$ for all $x \in \Omega$.

$(\mathcal{H}_5)$ The function $\mathcal{M} : \Omega \times \mathbb{R}^d \longrightarrow \mathbb{R}$ is such that

(*a*) $\mathcal{M}(.,\xi) \in L^\infty(\Omega)$,

(*b*) there exist $L_\mathcal{M} > 0$ such that for all ξ_1, $\xi_2 \in \mathbb{R}^d$ and $x \in \Omega$, we have

$$\|\mathcal{M}(x,\xi_1) - \mathcal{M}(x,\xi_2)\| \leq L_\mathcal{M} \|\xi_1 - \xi_2\|. \tag{2.32}$$

$(\mathcal{H}_6)$ The thermal conductivity operator $\mathcal{K} : \Omega \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is such that

(*a*) $\mathcal{K}(.,\xi)$ is measurable on Ω for all $\xi \in \mathbb{R}^d$,

(*b*) there exist $L_\mathcal{K} > 0$ such that for all $\xi_1, \xi_2 \in \mathbb{R}^d$ and $x \in \Omega$, we have

$$\|\mathcal{K}(x,\xi_1) - \mathcal{K}(x,\xi_2)\| \leq L_\mathcal{K} \|\xi_1 - \xi_2\|, \tag{2.33}$$

(*c*) there exist $\alpha_\mathcal{K} > 0$ such that for all ξ_1, $\xi_2 \in \mathbb{R}^d$ and $x \in \Omega$, we have

$$(\mathcal{K}(x,\xi_1) - \mathcal{K}(x,\xi_2)) \cdot (\xi_1 - \xi_2) \geq \alpha_\mathcal{K} \|\xi_1 - \xi_2\|^2, \tag{2.34}$$

(d) $\mathcal{K}(x,0)=0$ for all $x\in\Omega$.

$(\mathcal{H}_7)$ The forces, tractions, volume and the bounds S_b, G, and H satisfy

(i) $f_0\in L^2(\Omega)^d$, $f_2\in L^2(\Gamma_2)^d$, $q_0\in L^2(\Omega)$, $q_b\in L^2(\Gamma_b)$, $h_0\in L^2(\Omega)$,

(ii) S_b, G, $H\in L^2(\Gamma_3)$ and $S_b(x)$, $G(x)$, $H(x)\geq 0$ a.e. $x\in\Gamma_3$.

$(\mathcal{H}_0)$ $\min(\alpha_{\mathcal{A}},\alpha_\beta,\alpha_{\mathcal{K}})-(L_{\mathcal{M}}+L_{\mathcal{C}})>0.$

Using Riesz's representation theorem, we consider $f\in V$, $l_q\in W$ and $h\in Q$ defined by

$$\langle f,v\rangle_V=\langle f_0,v\rangle_{L^2(\Omega)^d}+\langle f_2,v\rangle_{L^2(\Gamma_2)^d},\quad\forall\, v\in V, \tag{2.35}$$

$$\langle l_q,\psi\rangle_W=\langle q_0,\psi\rangle_{L^2(\Omega)}-\langle q_b,\psi\rangle_{L^2(\Gamma_b)},\quad\forall\,\psi\in W, \tag{2.36}$$

$$\langle h,\xi\rangle_Q=\langle h_0,\xi\rangle_{L^2(\Omega)}-\langle h_n,\xi\rangle_{L^2(\Gamma_2)},\quad\forall\,\xi\in Q. \tag{2.37}$$

By employing Green's formula, we can formulate the following variational formulation of Problem (**P**) with respect to the displacement field, electric potential, and temperature.

Problem (PV). *Find a displacement $u\in V$, an electric potential $\varphi\in W$ and a temperature $\theta\in Q$ such that*

$$\begin{aligned}&\langle\mathcal{A}\varepsilon(u),\varepsilon(v-u)\rangle_{\mathcal{H}}+\langle\mathcal{P}^T\nabla\varphi-\mathcal{M}\theta,\varepsilon(v-u)\rangle_{\mathcal{H}}\\&\quad+\phi(v)-\phi(u)\geq\langle f,v-u\rangle_V,\quad\forall\, v\in V,\end{aligned} \tag{2.38}$$

$$\begin{aligned}&\langle\beta\nabla\varphi-\mathcal{P}\varepsilon(u)-\mathcal{C}\theta,\nabla(\psi-\varphi)\rangle_{L^2(\Omega)^d}\\&\quad+\lambda(\psi)-\lambda(\varphi)\geq\langle l_q,\psi-\varphi\rangle_W,\quad\forall\,\psi\in W,\end{aligned} \tag{2.39}$$

$$\begin{aligned}&\langle\mathcal{K}\nabla\theta,\nabla(\eta-\theta)\rangle_{L^2(\Omega)^d}+\gamma(\eta)-\gamma(\theta)\\&\quad\geq\langle h,\eta-\theta\rangle_Q,\quad\forall\,\eta\in Q,\end{aligned} \tag{2.40}$$

where the functions ϕ, λ and γ are defined by

$$\phi(v)=\int_{\Gamma_3}S_b\|v_\tau\|\,da,\quad\forall\, v\in V, \tag{2.41}$$

$$\lambda(\psi)=\int_{\Gamma_3}G\psi^+\,da,\quad\forall\,\psi\in W, \tag{2.42}$$

$$\gamma(\eta) = \int_{\Gamma_3} H\eta^+ \, da, \quad \forall\, \eta \in Q. \tag{2.43}$$

Note that Problem (**PV**) is called the primal variational formulation of the frictional contact Problem (**P**). Our focus in this section is on the investigation of a variational formulation of Problem (**P**) in the form of a system that couples three inclusions, where the unknowns are the strain field, the electric field, and the temperature field. We consider the product space $X = \mathcal{H} \times L^2(\Omega)^d \times L^2(\Omega)^d$ and $Y = V \times W \times Q$, endowed with their respective canonical inner products, defined as

$$\begin{aligned}(x_1, x_2)_X &= (\sigma_1, \sigma_2)_{\mathcal{H}} + (D_1, D_2)_{L^2(\Omega)^d} \\ &\quad + (q_1, q_2)_{L^2(\Omega)^d}, \quad \forall\, x_i = (\sigma_i, D_i, q_i) \in X \ \text{ for } \ i = 1, 2,\end{aligned} \tag{2.44}$$

$$\begin{aligned}(y_1, y_2)_Y &= (u_1, u_2)_V + (\varphi_1, \varphi_2)_W \\ &\quad + (\theta_1, \theta_2)_Q, \quad \forall\, y_i = (u_i, \varphi_i, \theta_i) \in Y \ \text{ for } \ i = 1, 2.\end{aligned} \tag{2.45}$$

Next, we introduce the mapping $\Pi : Y \to 2^X$ defined by

$$\Pi(g) = \Pi_1(f) \times \Pi_2(l_q) \times \Pi_3(h), \quad \forall\, g = (f, l_q, h) \in Y, \tag{2.46}$$

where $\Pi_1(f) \subset \mathcal{H}$, $\Pi_2(l_q) \subset L^2(\Omega)^d$ and $\Pi_3(h) \subset L^2(\Omega)^d$ are defined by

$$\Pi_1(f) = \big\{\tau \in \mathcal{H}\ ;\ \ (\tau, \varepsilon(v))_{\mathcal{H}} + \phi(v) \geq (f, v)_V, \quad \forall\, v \in V\big\}, \tag{2.47}$$

$$\Pi_2(l_q) = \big\{E \in L^2(\Omega)^d\ ;\ \ (E, \nabla\psi)_{L^2(\Omega)^d} + \lambda(\psi) \geq (l_q, \psi)_W, \quad \forall\, \psi \in W\big\}, \tag{2.48}$$

$$\Pi_3(h) = \big\{p \in L^2(\Omega)^d\ ;\ \ (p, \nabla\eta)_{L^2(\Omega)^d} + \gamma(\eta) \geq (h, \eta)_W, \quad \forall\, \eta \in Q\big\}. \tag{2.49}$$

Suppose that $(u, \sigma, \phi, D, \theta, q)$ are suitably smooth functions satisfying Problem (**P**). Using the same arguments as in [10], we get

$$\sigma \in \Pi_1(f)\ , \quad -\,\omega_1 = -\varepsilon(u) \in N_{\Pi_1(f)}(\sigma), \tag{2.50}$$

$$-\,D \in \Pi_2(l_q)\ , \quad -\,\omega_2 = -\nabla\varphi \in N_{\Pi_2(l_q)}(-D). \tag{2.51}$$

Note that N_K represents the outward normal cone of K in the sense of convex analysis. We recall that for all u, $\chi \in X$, the following equivalence hold.

$$\chi \in N_K(u) \iff u \in K,\ (\chi, v - u)_X \leq 0, \quad \forall\, v \in K. \tag{2.52}$$

Subsequently, we apply the Green's formula (2.25) along with equations (2.37) and (2.43) to observe that

$$(-q, \nabla\eta - \nabla\theta)_{L^2(\Omega)^d} + \gamma(\eta) - \gamma(\theta) \geq (h, \eta - \theta)_Q, \quad \forall\, \eta \in Q. \tag{2.53}$$

We choose $\eta = 2\theta$ and $\eta = 0_Q$ in (2.52) to drive

$$(-q, \nabla\theta)_{L^2(\Omega)^d} + \gamma(\theta) = (h, \theta)_Q. \tag{2.54}$$

Thus, by utilizing equations (2.52) and (2.53), we deduce that

$$(-q, \nabla\eta)_{L^2(\Omega)^d} + \gamma(\eta) \geq (h, \eta)_Q, \quad \forall\, \eta \in Q, \tag{2.55}$$

and using the definition of the set $\Pi_3(h)$ we find that

$$-\, q \in \Pi_3(h). \tag{2.56}$$

Subsequently, by employing (2.49) and (2.54), we obtain

$$\big(p - (-q), \nabla\theta\big)_{L^2(\Omega)} \geq 0, \quad \forall\, p \in \Pi_3(h), \tag{2.57}$$

and, using the notation $w_3 = \theta$, we find

$$(p - (-q), \nabla w_3)_{L^2(\Omega)} \geq 0, \quad \forall\, p \in \Pi_3(h). \tag{2.58}$$

Now, we combine equations (2.56) and (2.58), and then utilize equivalence (2.52) to observe that

$$-\, \nabla w_3 \in N_{\Pi_3(h)}(-q). \tag{2.59}$$

We utilize the inclusions (equations (2.50), (2.51) and (2.59)), along with the constitutive laws (2.1)-(2.3), to derive the following variational formulation of Problem $(\mathbf{P})$.

Problem (PVI). *Find* $(w_1, w_2, \nabla w_3) \in X$ *such that*

$$-\, w_1 \in N_{\Pi_1(f)}\left(\mathcal{A}w_1 + \mathcal{P}^T w_2 - \mathcal{M}w_3\right), \tag{2.60}$$

$$-\, w_2 \in N_{\Pi_2(l_q)}\left(\beta w_2 - \mathcal{P}w_1 - \mathcal{C}w_3\right), \tag{2.61}$$

$$-\, \nabla w_3 \in N_{\Pi_3(h)}\left(\mathcal{K}\nabla w_3\right). \tag{2.62}$$

Note that the Problem $(\mathbf{PVI})$ represents a variational formulation of the contact problem $(\mathbf{P})$ in terms of stress field σ , electric displacement fields D and the heat flux q and, therefore, this formulation is in the form of a system coupling three inclusions for the unknowns w_1, w_2 and ∇w_3.

3. Existence and continuous dependence results

We present here an existence and continuous dependence result corresponding to Problem (**PVI**).

Theorem 1. *Assume the hypotheses* $(\mathcal{H}_0)$-$(\mathcal{H}_7)$ *hold. Then, Problem* (**PVI**) *has a unique solution* $w = (w_1, w_2, \nabla w_3) \in X$*. Additionally, the following operators are Lipschitz continuous*

$$
\begin{aligned}
&g = (f, l_q, h) \mapsto w_1 = w_1(g) : Y \to \mathcal{H},\\
&g = (f, l_q, h) \mapsto w_2 = w_2(g) : Y \to L^2(\Omega)^d,\\
&g = (f, l_q, h) \mapsto \nabla w_3 = \nabla w_3(g) : Y \to L^2(\Omega)^d,
\end{aligned}
$$

Proof. The proof is based on the abstract result on stationary inclusion which has been discussed in [21].

Lemma 1. *For each* $g = (f, l_q, h) \in Y$*, the set* $\Pi(g) \subset X$ *defined by* (2.46) *is nonempty, closed and convex. Moreover, for each* $g = (f, l_q, h)$*,* $g' = (f', l_q', h') \in Y$ *and* $w = (w_1, w_2, \nabla w_3) \in X$*, one has*

$$
\|P_{\Pi(g)}w - P_{\Pi(g')}w\|_X \le \|g - g'\|_Y, \tag{3.1}
$$

where $P_K : X \to K$ *the projection operator on* K*, and* $P_{\Pi(g)}$ *defined by*

$$
P_{\Pi(g)}w = (P_{\Pi_1(f)}w_1, P_{\Pi_2(l_q)}w_2, P_{\Pi_3(h)}\nabla w_3). \tag{3.2}
$$

We recall that for all u*,* $\xi \in X$*, the following equivalence hold.*

$$
u = P_{\Pi(g)}\xi \Longleftrightarrow u \in \Pi(g), \quad (\xi - u, v - u)_X \le 0 \qquad \forall v \in \Pi(g). \tag{3.3}
$$

Proof. Using similar techniques as in [10, Lemma 3-4], we establish that the sets $\Pi_1(f)$ and $\Pi_2(l_q)$ are nonempty, closed and convex. Moreover, we prove that $\Pi_3(h)$ is nonempty, closed and convex set. Let $h \in Q$ be fixed. As the function $\eta \mapsto \gamma(\eta) : Q \to \mathbb{R}$ is subdifferentiable and attains zero at 0_Q, we can conclude the existence of an element $\lambda \in Q$ such that $\gamma(\eta) \ge (\lambda, \eta)_Q$ for all $\eta \in Q$. Additionally, recalling that $(\lambda, \eta)_Q = (\nabla\lambda, \nabla\eta)_{L^2(\Omega)^d}$. Hence, employing the notation $\delta = \nabla h - \nabla\lambda$, we find that

$$
(\delta, \nabla\eta)_{L^2(\Omega)^d} + \gamma(\eta) \ge (h, \eta)_Q, \quad \forall \eta \in Q. \tag{3.4}
$$

Combining equations (2.49) and (3.4), we observe that $\delta \in \Pi_3(h)$, establishing that $\Pi_3(h)$ is not empty. Conversely, it is evident that $\Pi_3(h)$ is a closed convex subset of $L^2(\Omega)^d$. Then, by the definition of $\Pi(g)$ it is clear that $\Pi(g)$ is a nonempty, closed and convex subset of X. This completes the demonstration of the first part of Lemma

1. Next, the rest of the proof will be divided in several claims.

Claim 1: Let ∇w_3, $q \in L^2(\Omega)^d$ and $h \in Q$ satisfying the condition:

$$-\nabla w_3 \in N_{\Pi_3(h)}(-q). \tag{3.5}$$

There exists a unique $\theta \in Q$ such that $\nabla w_3 = \nabla\theta$, and furthermore, the following inequality hold.

$$(-q, \nabla\eta - \nabla\theta)_{L^2(\Omega)^d} + \gamma(\eta) - \gamma(\theta) \geq (h, \eta - \theta)_Q, \quad \forall\, \eta \in Q. \tag{3.6}$$

Indeed, we note that the inclusion (3.5) implies that

$$-q \in \Pi_3(h)\ ; \quad (p - (-q), \nabla w_3)_{L^2(\Omega)^d} \geq 0,\ \forall\, p \in \Pi_3(h). \tag{3.7}$$

Let $r \in \nabla(Q)^{\perp}$, where, in the context here and subsequently, $M^{\perp}$ designates the orthogonal of of the subset $M \subset L^2(\Omega)^d$. Consequently, for all $\eta \in Q$, we have $(r, \nabla(\eta))_{L^2(\Omega)^d} = 0$. Then, utilizing equation (2.49), we establish that $-q \pm r \in \Pi_3(h)$. Subsequently, by testing with $p = -q \pm r$ in equation (3.7), we infer that $(r, \nabla\omega_3)_{L^2(\Omega)^d} = 0$, indicating that $\nabla\omega_3 \in \nabla(Q)^{\perp\perp} = \nabla(Q)$. It's noteworthy that the last equation follows from the fact that $\nabla(Q)$ is a closed subspace of $L^2(\Omega)^d$. Moving forward, the inclusion $\omega_3 \in \nabla(Q)$ implies the existence of an element $\theta \in Q$ such that

$$\nabla w_3 = \nabla\theta. \tag{3.8}$$

Furthermore, the uniqueness of θ is assured by equation (2.19). Subsequently, leveraging the subdifferentiability of the function γ at η, we ascertain the existence of an element λ such that

$$\gamma(\eta) - \gamma(\theta) \geq (\lambda, \eta - \theta)_Q = (\nabla\lambda, \nabla\eta - \nabla\theta)_{L^2(\Omega)^d}$$

and, setting $\rho := \nabla h - \nabla\lambda$, we infer that

$$(\rho, \nabla\eta - \nabla\theta)_{L^2(\Omega)^d} + \gamma(\eta) - \gamma(\theta) \geq (h, \eta - \theta)_Q, \quad \forall\, \eta \in Q. \tag{3.9}$$

We choose $\eta = 2\theta$ and $\eta = 0_Q$ in this equation to drive that

$$(\rho, \nabla\theta)_{L^2(\Omega)^d} + \gamma(\theta) = (h, \theta)_Q. \tag{3.10}$$

Hence, by combining equations (3.9) and (3.10), we deduce that

$$(\rho, \nabla\eta)_{L^2(\Omega)^d} + \gamma(\eta) \geq (h, \eta)_Q, \quad \forall\, \eta \in Q, \tag{3.11}$$

which implies that $\rho \in \Pi_3(h)$. This regularity, (3.7), (3.8) and (3.10) imply that

$$(-q, \nabla\theta)_{L^2(\Omega)^d} + \gamma(\theta) \leq (h, \theta)_Q. \tag{3.12}$$

Conversely, considering that $-q \in \Pi_3(h)$ and $\theta \in Q$ the inverse inequality holds, i.e.,

$$(-q, \nabla\theta)_{L^2(\Omega)^d} + \gamma(\theta) \geq (h, \theta)_Q. \tag{3.13}$$

Thus, we conclud that

$$(-q, \nabla\theta)_{L^2(\Omega)^d} + \gamma(\theta) = (h, \theta)_Q. \tag{3.14}$$

Finally, using (3.14) and inclusion $-q \in \Pi_3(h)$, we conclude that the estimation (3.6) is verified.

Claim 2: By employing the same techniques as presented in [10, Lemma 1-2] and Claim 1, we establish that for (w_1, w_2), $(\sigma, D) \in \mathcal{H} \times L^2(\Omega)^d$, and $(f, l_q) \in V \times W$, satisfying the conditions

$$-w_1 \in N_{\Pi_1(f)}(\sigma), \tag{3.15}$$

$$-w_2 \in N_{\Pi_2(l_q)}(-D), \tag{3.16}$$

there exists a unique element $(u, \varphi) \in V \times W$ such that $(w_1, w_2) = (\varepsilon(u), \nabla\varphi)$, and furthermore, the following inequalities hold.

$$(\sigma, \varepsilon(v) - \varepsilon(u))_{\mathcal{H}} + \phi(v) - \phi(u) \geq (f, v - u)_V, \quad \forall\, v \in V, \tag{3.17}$$

$$(-D, \nabla\psi - \nabla\varphi)_{L^2(\Omega)^d} + \lambda(\psi) - \lambda(\varphi) \geq (l_q, \psi - \varphi)_W, \quad \forall\, \psi \in W. \tag{3.18}$$

Claim 3: Suppose $g = (f, l_q, h)$, $g' = (f', l'_q, h') \in Y$, and $w = (w_1, w_2, \nabla w_3)$. Let's denote

$$\begin{cases} \sigma = P_{\Pi_1(f)} w_1 \;\; ; \quad -D = P_{\Pi_2(l_q)} w_2 \;\; ; \quad -q = P_{\Pi_3(h)} \nabla w_3, \\ \sigma' = P_{\Pi_1(f')} w_1 \;\; ; \quad -D' = P_{\Pi_2(l'_q)} w_2 \;\; ; \quad -q' = P_{\Pi_3(h')} \nabla w_3. \end{cases} \tag{3.19}$$

Through the equivalence (3.3), we obtain that

$$\begin{cases} \sigma \in \Pi_1(f) \;, \quad (w_1 - \sigma, \tau - \sigma)_{\mathcal{H}} \leq 0, \quad \forall\, \tau \in \Pi_1(f), \\ -D \in \Pi_2(l_q) \;, \quad (w_2 - (-D), E - (-D))_{L^2(\Omega)^d} \leq 0, \quad \forall\, E \in \Pi_2(l_q), \\ -q \in \Pi_3(h) \;, \quad (\nabla w_3 - (-q), p - (-q))_{L^2(\Omega)^d} \leq 0, \quad \forall\, p \in \Pi_3(h). \end{cases}$$

Consequently, from equation (2.52), it follows that

$$w_1 - \sigma \in N_{\Pi_1(f)}(\sigma) \;; \quad w_2 - (-D) \in N_{\Pi_2(l_q)}(-D) \;; \quad \nabla w_3 - (-q) \in N_{\Pi_3(h)}(-q).$$

According to Claim 1-2, i.e., (3.6), (3.17) and (3.18), there exists a singular element $(u, \varphi, \theta) \in Y$ such that

$$\sigma - w_1 = \varepsilon(u)\,, \quad -D - w_2 = \nabla\varphi\,, \quad -q - \nabla w_3 = \nabla\theta, \tag{3.20}$$

$$(\sigma, \varepsilon(v) - \varepsilon(u))_{\mathcal{H}} + \phi(v) - \phi(u) \geq (f, v - u)_V, \quad \forall\, v \in V, \tag{3.21}$$

$$(-D, \nabla\psi - \nabla\varphi)_{L^2(\Omega)^d} + \lambda(\psi) - \lambda(\varphi) \geq (l_q, \psi - \varphi)_W, \quad \forall\, \psi \in W, \tag{3.22}$$

$$(-q, \nabla\eta - \nabla\theta)_{L^2(\Omega)^d} + \gamma(\eta) - \gamma(\theta) \geq (h, \eta - \theta)_Q, \quad \forall\, \eta \in Q. \tag{3.23}$$

Analogous reasoning demonstrates the existence of a singular element $(u', \varphi', \theta') \in Y$ such that

$$\sigma' - w_1 = \varepsilon(u')\,, \quad -D' - w_2 = \nabla\varphi'\,, \quad -q' - \nabla w_3 = \nabla\theta', \tag{3.24}$$

$$(\sigma', \varepsilon(v) - \varepsilon(u'))_{\mathcal{H}} + \phi(v) - \phi(u') \geq (f', v - u')_V, \quad \forall\, v \in V, \tag{3.25}$$

$$(-D', \nabla\psi - \nabla\varphi')_{L^2(\Omega)^d} + \lambda(\psi) - \lambda(\varphi') \geq (l'_q, \psi - \varphi')_W, \quad \forall\, \psi \in W, \tag{3.26}$$

$$(-q', \nabla\eta - \nabla\theta')_{L^2(\Omega)^d} + \gamma(\eta) - \gamma(\theta') \geq (h', \eta - \theta')_Q, \quad \forall \eta \in Q. \tag{3.27}$$

We take $(v, \psi, \eta) = (u', \varphi', \theta')$ in (3.21)-(3.23), $(v, \psi, \eta) = (u, \varphi, \theta)$ in (3.25)-(3.27) and add the obtained inequalities to deduce

$$(\sigma - \sigma', \varepsilon(u) - \varepsilon(u'))_{\mathcal{H}} \leq (f - f', u - u')_V, \tag{3.28}$$

$$(-D - (-D'), \nabla\varphi - \nabla\varphi')_{L^2(\Omega)^d} \leq (l_q - l'_q, \varphi - \varphi')_W, \tag{3.29}$$

$$(-q - (-q'), \nabla\theta - \nabla\theta')_{L^2(\Omega)^d} \leq (h - h', \theta - \theta')_Q. \tag{3.30}$$

Then, using the relations (3.20) and (3.24) , we conclude that

$$\begin{cases} \varepsilon(u) - \varepsilon(u') = \sigma - \sigma', \\ \nabla\varphi - \nabla\varphi' = -D - (-D'), \\ \nabla\theta - \nabla\theta' = -q - (-q'). \end{cases} \tag{3.31}$$

Hence, we combine the inequalities (3.28)-(3.30) and the relations (3.31) to find

$$\begin{cases} \|\sigma - \sigma'\|^2_{\mathcal{H}} \leq \|f - f'\|_V \|u - u'\|_V, \\ \| - D - (-D')\|^2_{L^2(\Omega)^d} \leq \|l_q - l'_q\|_W \|\varphi - \varphi'\|_W, \\ \| - q - (-q')\|^2_{L^2(\Omega)^d} \leq \|h - h'\|_Q \|\theta - \theta'\|_Q. \end{cases} \tag{3.32}$$

Afterward, as

$$\begin{cases} \|u-u'\|_V = \|\varepsilon(u)-\varepsilon(u')\|_{\mathcal{H}} = \|\sigma-\sigma'\|_{\mathcal{H}}, \\ \|\varphi-\varphi'\|_W = \|\nabla\varphi-\nabla\varphi'\|_{L^2(\Omega)^d} = \|-D-(-D')\|_{L^2(\Omega)^d}, \\ \|\theta-\theta'\|_Q = \|\nabla\theta-\nabla\theta'\|_{L^2(\Omega)^d} = \|-q-(-q')\|_{L^2(\Omega)^d}, \end{cases}$$

we can conclude that

$$\begin{cases} \|\sigma-\sigma'\|_{\mathcal{H}} \le \|f-f'\|_V, \\ \|-D-(-D')\|_{L^2(\Omega)^d} \le \|l_q-l'_q\|_W, \\ \|-q-(-q')\|_{L^2(\Omega)^d} \le \|h-h'\|_Q. \end{cases} \tag{3.33}$$

Then, by utilizing (2.44), (2.45), (3.19) and (3.33), we obtain that

$$\begin{aligned} \|P_{\Pi(g)}w-P_{\Pi(g')}w\|_X &= \|\sigma-\sigma'\|_{\mathcal{H}} + \|-D-(-D')\|_{L^2(\Omega)^d} + \|-q-(-q')\|_{L^2(\Omega)^d} \\ &\le \|f-f'\|_V + \|l_q-l'_q\|_W + \|h-h'\|_Q \\ &\le \|g-g'\|_Y. \end{aligned} \tag{3.34}$$

This concludes the proof of Lemma 1. □

Lemma 2. *Under the assumptions $\mathcal{H}_0$-$\mathcal{H}_6$. The operator $\mathbb{A}: X \to X$ defined by*

$$\begin{aligned} \mathbb{A}(w,w')_X = {} & (\mathcal{A}w_1+\mathcal{P}^T w_2-\mathcal{M}w_3, w'_1)_{\mathcal{H}} + (\beta w_2-\mathcal{P}w_1-\mathcal{C}w_2, w'_2)_{L^2(\Omega)^d} \\ & + (\mathcal{K}\nabla w_3, \nabla w'_3)_{L^2(\Omega)^d}, \quad \forall\, w=(w_1,w_2,\nabla w_3),\ w'=(w'_1,w'_2,\nabla w'_3) \in X, \end{aligned} \tag{3.35}$$

satisfies the following properties

(i) there exists $\alpha_{\mathbb{A}}>0$ such that for all $u,v\in X$, it yields

$$(\mathbb{A}u-\mathbb{A}v, u-v)_X \ge \alpha_{\mathbb{A}}\|u-v\|_X^2, \tag{3.36}$$

(ii) there exists $L_{\mathbb{A}}>0$ such that for all $u,v\in X$, it yields

$$\|\mathbb{A}u-\mathbb{A}v\|_X \le L_{\mathbb{A}}\|u-v\|_X. \tag{3.37}$$

Proof. First, for all $w=(w_1,w_2,\nabla w_3)$, $w'=(w_1',w_2',\nabla w_3')\in X$, we have

$$\begin{aligned}(\mathbb{A}w-\mathbb{A}w',w-w')_X &= (\mathcal{A}w_1-\mathcal{A}w_1',w_1-w_1')_{\mathcal{H}}+(\beta w_2-\beta w_2',w_2-w_2')_{L^2(\Omega)^d}\\ &+(\mathcal{K}\nabla w_3-\mathcal{K}\nabla w_3',\nabla w_3-\nabla w_3')_{L^2(\Omega)^d}+(\mathcal{P}^T w_2-\mathcal{P}^T w_2',w_1-w_1')_{\mathcal{H}}\\ &-(\mathcal{P}w_1-\mathcal{P}w_1',w_2-w_2')_{L^2(\Omega)^d}-(\mathcal{M}w_3-\mathcal{M}w_3',w_1-w_1')_{\mathcal{H}}\\ &-(\mathcal{C}w_3-\mathcal{C}w_3',w_2-w_2')_{L^2(\Omega)^d}.\end{aligned}$$

Given that $(\mathcal{P}^T y_2,y_1)_{\mathcal{H}}=(\mathcal{P}y_1,y_2)_{L^2(\Omega)^d}$ for all $y=(y_1,y_2)\in X$, we can infer that

$$\begin{aligned}(\mathbb{A}w-\mathbb{A}w',w-w')_X &= (\mathcal{A}w_1-\mathcal{A}w_1',w_1-w_1')_{\mathcal{H}}+(\beta w_2-\beta w_2',w_2-w_2')_{L^2(\Omega)^d}\\ &+(\mathcal{K}\nabla w_3-\mathcal{K}\nabla w_3',\nabla w_3-\nabla w_3')_{L^2(\Omega)^d}-(\mathcal{M}w_3-\mathcal{M}w_3',w_1-w_1')_{\mathcal{H}}\\ &-(\mathcal{C}w_3-\mathcal{C}w_3',w_2-w_2')_{L^2(\Omega)^d}.\end{aligned}$$

Thus by assumptions $\mathcal{H}_1(c),\mathcal{H}_3(c)$ and $\mathcal{H}_6(c)$, we find that all $y=(y_1,y_2)\in X$, we can infer that

$$\begin{aligned}&(\mathbb{A}w-\mathbb{A}w',w-w')_X\\ &\geq \min(\alpha_{\mathcal{A}},\alpha_\beta,\alpha_{\mathcal{K}})\left\{\|w_1-w_1'\|_{\mathcal{H}}^2+\|w_2-w_2'\|_{L^2(\Omega)^d}^2+\|\nabla w_3-\nabla w_3'\|_{L^2(\Omega)^d}^2\right\}\\ &-(\mathcal{M}w_3-\mathcal{M}w_3',w_1-w_1')_{\mathcal{H}}-(\mathcal{C}w_3-\mathcal{C}w_3',w_2-w_2')_{L^2(\Omega)^d}.\end{aligned}$$

We now use the assumptions $\mathcal{H}_4(b)$ and $\mathcal{H}_5(b)$ to see that

$$(\mathbb{A}w-\mathbb{A}w',w-w')_X\geq \min(\alpha_{\mathcal{A}},\alpha_\beta,\alpha_{\mathcal{K}})\|w-w'\|_X^2-(L_{\mathcal{M}}+L_{\mathcal{C}})\|w-w'\|_X^2,$$

which implies the inequality (3.36) holds for the constant $\alpha_{\mathbb{A}}=\min(\alpha_{\mathcal{A}},\alpha_\beta,\alpha_{\mathcal{K}})-(L_{\mathcal{M}}+L_{\mathcal{C}})>0$. In a similar manner, the assumptions $\mathcal{H}_1(b)$-$\mathcal{H}_6(b)$ imply that

$$\begin{aligned}(\mathbb{A}w-\mathbb{A}w',w'')_X \leq\, & L_{\mathcal{A}}\,\|w_1-w_1'\|_{\mathcal{H}}\|w_1''\|_{\mathcal{H}}+L_\beta\,\|w_2-w_2'\|_{L^2(\Omega)^d}\|w_2''\|_{L^2(\Omega)^d}\\ &+L_{\mathcal{K}}\,\|\nabla w_3-\nabla w_3'\|_{L^2(\Omega)^d}\|\nabla w_3''\|_{L^2(\Omega)^d}+L_{\mathcal{P}}\,\|w_2-w_2'\|_{L^2(\Omega)^d}\|w_1''\|_{\mathcal{H}}\\ &+L_{\mathcal{P}}\,\|w_1-w_1'\|_{\mathcal{H}}\|w_2''\|_{L^2(\Omega)^d}+L_{\mathcal{M}}\,\|\nabla w_3-\nabla w_3'\|_{L^2(\Omega)^d}\|\nabla w_1''\|_{\mathcal{H}}\\ &+L_{\mathcal{C}}\,\|\nabla w_3-\nabla w_3'\|_{L^2(\Omega)^d}\|\nabla w_2''\|_{L^2(\Omega)^d}.\end{aligned}$$

Then, assumption (3.7) holds with $L_{\mathbb{A}}=\max(L_{\mathcal{A}}+L_\beta+L_{\mathcal{K}}+2L_{\mathcal{P}}+L_{\mathcal{M}}+L_{\mathcal{C}})$. □

Now, utilizing Lemma 1, Lemma 2, and the stationary inclusion result [21, Theorem 4.1], it follows that Problem (**PVI**) has a unique solution $w=(w_1,w_2,\nabla w_3)\in X$. □

Remark 1. Consider a 6-tuple of functions $(u, \sigma, \phi, D, \theta, q) \in V \times \mathcal{H} \times W \times L^2(\mathbb{R}^d) \times Q \times L^2(\mathbb{R}^d)$ that satisfies equations (1.1)-(1.3) and the inclusions

$$-\varepsilon(u) \in N_{\Pi_1(f)}(\sigma), \tag{3.38}$$

$$-\nabla\varphi \in N_{\Pi_2(l_q)}(-D), \tag{3.39}$$

$$-\nabla\theta \in N_{\Pi_L(h)}(-q). \tag{3.40}$$

This 6-tuple is referred to as a weak solution to the thermo-electro-elastic contact problem (2.1)-(2.16).

Based on this remark, we derive the following implication from Theorem 1.

Corollary 1. *Assume hypotheses* $(\mathcal{H}_0)$*-*$(\mathcal{H}_7)$*. Then, Problem* $(\mathbf{P})$ *has a unique weak solution* $(u, \sigma, \varphi, D, \theta, q)$*. Additionally, the operators*

$$L^2(\Omega)^d \times L^2(\Gamma_2)^d \times L^2(\Omega) \times L^2(\Gamma_b) \times L^2(\Omega) \times L^2(\Gamma) \to V \times \mathcal{H} \times W \times L^2(\Omega)^d \times Q \times L^2(\Omega)^d$$

$$(f_0, f_2, q_0, q_b, h_0, h_n) \mapsto (u, \sigma, \varphi, D, \theta, q)$$

is Lipschitz continuous.

Proof. Consider the solution $w = (w_1, w_2, \nabla w_3)$ to Problem $(\mathbf{PVI})$, as established in Theorem 1. Drawing upon Claim 1 and Claim 2, we confirm the existence of a unique triple of functions (u, φ, θ) such that $(w_1, w_2, \nabla w_3) = (\varepsilon(u), \nabla\varphi, \nabla\theta)$. Next, we define the functions σ, D and q, using equations (1.1), (1.2), and (1.3), respectively. This leads to the observation that $(u, \sigma, \varphi, D, \theta, q)$ constitutes the unique weak solution to Problem $(\mathbf{P})$. Additionally, it is important to recalling that

$$\begin{aligned}
\|u\|_V &= \|\varepsilon(u)\|_{\mathcal{H}} = \|w_1\|_{\mathcal{H}}, \\
\|\varphi\|_W &= \|\nabla\varphi\|_{L^2(\Omega)^d} = \|w_2\|_{L^2(\Omega)^d}, \\
\|\theta\|_Q &= \|\nabla\theta\|_{L^2(\Omega)^d} = \|\nabla w_3\|_{L^2(\Omega)^d}.
\end{aligned}$$

Furthermore, Theorem 1 ensures that the operators

$$\begin{aligned}
&g = (f, l_q, h) \mapsto w_1 = w_1(fg) : V \times W \times Q \to \mathcal{H}, \\
&g = (f, l_q, h) \mapsto w_2 = w_2(g) : V \times W \times Q \to L^2(\Omega)^d, \\
&g = (f, l_q, h) \mapsto \nabla w_3 = \nabla w_3(g) : V \times W \times Q \to L^2(\Omega)^d
\end{aligned}$$

are Lipschitz continuous. Therefore, we can infer that the operators

$$\begin{aligned}
&g \mapsto u = u(g) : V \times W \times Q \to V,\\
&g \mapsto \varphi = \varphi(g) : V \times W \times Q \to W,\\
&g \mapsto \nabla\theta = \nabla\theta(g) : V \times W \times Q \to Q
\end{aligned}$$

are also Lipschitz continuous. Subsequently, employing equations (1.1), (1.2), and (1.3), and taking into account the properties of $\mathcal{A}$, $\mathcal{P}$, $\mathcal{C}$, $\mathcal{M}$, β, and $\mathcal{K}$, we can establish that the operators

$$\begin{aligned}
&g \mapsto \sigma = \sigma(g) : V \times W \times Q \to \mathcal{H},\\
&g \mapsto D = D(g) : V \times W \times Q \to L^2(\Omega)^d,\\
&g \mapsto q = q(g) : V \times W \times Q \to L^2(\Omega)^d
\end{aligned}$$

are Lipschitz continuous. Finally, equations (2.35)-(2.37) imply that the operators

$$\begin{aligned}
&(f_0, f_2) \mapsto f : L^2(\Omega)^d \times L^2(\Gamma_2)^d \to V,\\
&(q_0, q_b) \mapsto l_q : L^2(\Omega) \times L^2(\Gamma_b) \to W,\\
&(h_0, h_n) \mapsto h : L^2(\Omega) \times L^2(\Gamma_2) \to Q,
\end{aligned}$$

are Lipschitz continuous. Then, we can see that the operator $(f_0, f_2, q_0, q_b, h_0, h_n) \mapsto (u, \sigma, \varphi, D, \theta, q)$, which associates the input data $(f_0, f_2, q_0, q_b, h_0, h_n)$ with the weak solution of Problem (**P**), is formed as a composition of Lipschitz continuous operators. This completes the proof. □

The results elucidated in Theorem 1 can be mechanically understood as follows: if the triplet (σ, D, q) associated with (u, φ, θ) through the thermo-electro-elastic constitutive law (1.1)-(1.3) a solution to the variational formulation presented as a system coupling three inclusions of the thermo-piezoelectric contact Problem (**PVI**) given by (2.1)-(2.16). Then, the triplet (u, φ, θ) constitutes a solution to the variational formulation of the thermo-piezoelectric contact problem (2.1)-(2.16) (referred to as Problem (**PV**)).

4. Optimal control problem

In this section, we study an optimal control problem associated with Problem (**PVI**). From Theorem 1, we observe that the solution to Problem (**PVI**) depends on the data f_0, f_2, q_0, q_b, h_0, and h_n. Each of these quantities could play a role in controlling the variational problem (**PVI**). We now assume that $f_0 \in L^2(\Omega)^d$, $q_0 \in L^2(\Omega)$, and

$h_0 \in L^2(\Omega)$ are given functions. Next, we consider the operator $\Lambda : L^2(\Gamma_2)^d \times L^2(\Gamma_b) \times L^2(\Gamma_2) \to Y$ defined by

$$\Lambda(f_2, q_b, h_n) = (f, l_q, h) \text{ such that (2.35)-(2.37) hold.} \tag{4.1}$$

We note that for each $(f_2, q_b, h_n) \in L^2(\Gamma_2)^d \times L^2(\Gamma_b) \times L^2(\Gamma_2)$, it follows from Theorem 1 that the Problem (**PVI**) has a unique solution $w = w(\Lambda(f_2, q_b, h_n))$. We now establish that the admissible set for Problem (**PVI**) is given by

$$\mathcal{U}_{\text{ad}} = \big\{(w, f_2, q_b, h_n) \,:\, w = w(\Lambda(f_2, q_b, h_n)) \text{ is a solution of Problem } (\mathbf{PVI})\big\}. \tag{4.2}$$

Next, we consider the cost functional $F : X \times L^2(\Gamma_2)^d \times L^2(\Gamma_b) \times L^2(\Gamma_2) \to \mathbb{R}$ and the following optimal control problem.

Problem (Q). *Find* $(w^*, f_2^*, q_b^*, h_n^*) \in \mathcal{U}_{ad}$ *such that*

$$F(w^*, f_2^*, q_b^*, h_n^*) = \min_{(w, f_2, q_b, h_n) \in \mathcal{U}_{\text{ad}}} F(w, f_2, q_b, h_n), \tag{4.3}$$

where w denotes the unique solution of Problem (**PVI**) with $(f, l_q, h) = \Lambda(f_2, q_b, h_n)$.

In order to state the main existence result of Problem ($\mathcal{Q}$), we consider the hypotheses below

($\mathcal{A}_1$) For all sequences $\{w_n\} \subset X$ and $\{g_n\} \subset L^2(\Gamma_2)^d \times L^2(\Gamma_b) \times L^2(\Gamma_2)$ such that $w_n \to w$ in X and $g_n \rightharpoonup g$ in $L^2(\Gamma_2)^d \times L^2(\Gamma_b) \times L^2(\Gamma_2)$, we have

$$\liminf_{n \to \infty} F(w_n, g_n) \geq F(w, g). \tag{4.4}$$

($\mathcal{A}_2$) There exists $L : L^2(\Gamma_2)^d \times L^2(\Gamma_b) \times L^2(\Gamma_2) \to \mathbb{R}$ such that

(i) $F(w, f_2, q_b, h_n) \geq L(f_2, q_b, h_n), \quad \forall\, w \in X, \forall\, (f_2, q_b, h_n) \in L^2(\Gamma_2)^d \times L^2(\Gamma_b) \times L^2(\Gamma_2)$,

(ii) $\|y_n\| \to +\infty \implies L(y_n) \to \infty, \quad \forall\, y_n \in L^2(\Gamma_2)^d \times L^2(\Gamma_b) \times L^2(\Gamma_2)$.

Then we have the following theorem.

Theorem 2. *Assume the hypotheses of Theorem 1 and* ($\mathcal{A}_1$)*-*($\mathcal{A}_2$) *hold. Then, Problem* ($\mathcal{Q}$) *has at least one solution* $(w^*, f_2^*, q_b^*, h_n^*) \in \mathcal{U}_{ad}$.

The proof relies on the utilization of a specific version of the Weierstrass theorem. Before commencing the proof of this theorem, we require the following two crucial lemmas.

Lemma 3. *Consider a sequence* $g_n = (f_{2n}, q_{bn}, h_{nn}) \in L^2(\Gamma_2)^d \times L^2(\Gamma_b) \times L^2(\Gamma_2)$ *such that*

$$g_n \rightharpoonup g \quad in \;\; L^2(\Gamma_2)^d \times L^2(\Gamma_b) \times L^2(\Gamma_2) \tag{4.5}$$

Then, we have the following convergence results

$$\Lambda g_n \to \Lambda g \quad in \;\; Y \quad where \;\; g = (f_2, q_b, h_n). \tag{4.6}$$

Proof. Lemma 3 can be proved using similar techniques as in [10, Lemma 5]. □

Lemma 4. *Consider the function* $\mathcal{J} : L^2(\Gamma_2)^d \times L^2(\Gamma_b) \times L^2(\Gamma_2) \to \mathbb{R}$ *defined by*

$$\mathcal{J}(g) = F(w(\Lambda g), g), \quad \forall\, g = (f_2, q_b, h_n) \in L^2(\Gamma_2)^d \times L^2(\Gamma_b) \times L^2(\Gamma_2). \tag{4.7}$$

Next, we examine the following auxiliary problem

$$Find\; g^* \in \mathbb{L} \;\; such\; that \;\; \mathcal{J}(g^*) = \min_{g \in \mathbb{L}} \mathcal{J}(g), \tag{4.8}$$

where $g = (f_2, q_b, h_n)$, $g^* = (f_2^*, q_b^*, h_n^*)$, *and* $\mathbb{L} = L^2(\Gamma_2)^d \times L^2(\Gamma_b) \times L^2(\Gamma_2)$. *Then, Problem 4.8 has at least one solution* g^*.

Proof. In the proof, we will verify that the function $\mathcal{J}$ satisfies all the conditions of the Weierstrass Theorem [21, Theorem 5.1]. Firstly, we show that $\mathcal{J}$ is a weakly lower semicontinuous function. Consider a sequence $\{g_n\} \subset \mathbb{L}$ such that $g_n \rightharpoonup g$ in $\mathbb{L}$, then, according to Lemma 3, it is demonstrated that $\Lambda g_n \to \lambda g$ in Y. Consequently, by Theorem 1, we infer that $w(\Lambda g_n) \to w(\Lambda g)$ in X. We employ equations (4.4) and (4.7) to observe that

$$\liminf_{n \to \infty} \mathcal{J}(g_n) \geq \mathcal{J}(g). \tag{4.9}$$

Furthermore, utilizing assumption $\mathcal{A}(i)$, for any sequence $\{g_n\} \subset \mathbb{L}$, it follows that

$$\mathcal{J}(g_n) = F(w(\Lambda g_n), g_n) \geq L(g_n), \quad \forall\, n \in \mathbb{N}.$$

Hence, if $\|g_n\|_{\mathbb{L}} \to \infty$, as indicated by assumption $\mathcal{A}(ii)$, we conclude that $\mathcal{J}(g_n) \to \infty$, demonstrating the coerciveness of the function $\mathcal{J}$. Also, taking into account the reflexivity of the space $\mathbb{L}$. Hence the Weierstrass theorem implies the existence of a solution to Problem 4.8. □

We now have all the ingredients to provide the proof of Theorem 2.

Proof. We proceed to establish the proof of Theorem 2, utilizing equations (4.2), (4.3), and (4.7). This allows us to observe that $(w^*, f_2^*, q_b^*, h_n^*)$ is a solution to Problem $(\mathcal{Q})$ if and only if (f_2^*, q_b^*, h_n^*) is a solution to (4.8), and $w^* = w(\Lambda(f_2^*, q_b^*, h_n^*))$. Finally, by utilizing this equivalence and Lemma 4, we establish that (4.3) holds, thus concluding the proof. □

We conclude this section by presenting a typical example of the cost function F and an illustration of optimal control problems where the results established in Theorem 2 apply.

Example 1. Let $F : X \times L^2(\Gamma_2)^d \times L^2(\Gamma_b) \times L^2(\Gamma_2) \to \mathbb{R}$ be defined by

$$F(w, f_2, q_b, h_n) = Z(w) + L(f_2, q_b, h_n).$$

Assume the following hypotheses hold.

(i) $Z : X \to \mathbb{R}$ is a continuous positive function,

(ii) $L : L^2(\Gamma_2)^d \times L^2(\Gamma_b) \times L^2(\Gamma_2) \to \mathbb{R}$ is a weakly semicontinuous coercive function.

Then, the function F satisfies assumptions $\mathcal{A}_1$ and $\mathcal{A}_2$.

Let's now consider control problem example, for which the existence result is given by Theorem 2.

Example 2. Consider $\mathbb{L} = L^2(\Gamma_2)^d \times L^2(\Gamma_b) \times L^2(\Gamma_2)$, $g = (f_2, q_b, h_n)$, $g^* = (f_2^*, q_b^*, h_n^*)$ and let consider $(u_g, \sigma_g, \varphi_g, D_g, \theta_g, q_n)$ the weak solution of Problem $(\mathcal{P})$. Then, an example of Problem $(\mathcal{Q})$ is as follows

Find $g^* \in \mathbb{L}$ such that

$$\begin{aligned} &\alpha_3 \int_{\Gamma_2} (\theta_{g^*} - \theta_R)^2 \, da + \alpha_2 \int_{\Gamma_2} (f_2^*)^2 \, da + \alpha_1 \int_{\Gamma_b} (q_b^*)^2 \, da + \alpha_0 \int_{\Gamma_2} (h_n^*)^2 \, da \\ &\leq \alpha_3 \int_{\Gamma_2} (\theta_g - \theta_R)^2 \, da + \alpha_2 \int_{\Gamma_2} (f_2)^2 \, da + \alpha_1 \int_{\Gamma_b} (q_b)^2 \, da + \alpha_0 \int_{\Gamma_2} (h_n)^2 \, da, \end{aligned} \tag{4.10}$$

where $\theta_R \in L^2(\Gamma_2)$ and α_0, α_1, α_2 and α_3 are given positive constants. Next, we consider the operator

$$\Lambda : X \to W, \quad \Lambda(w_1, w_2, \nabla w_3) = \nabla w_3 \;\text{ such that }\; \theta = w_3.$$

The mechanical interpretation is as follows: Given a contact process in the form of (2.1)-(2.16), we are seeking tractions f_2, a density of electric charges q_b, and the strength of the heat source h_n such that the temperature on the contact surface Γ_3 is as close as possible to the temperature θ_f. Furthermore, the associated cost functional $F : X \times \mathbb{L} \to \mathbb{R}$, defined by

$$F(w, g) = \alpha_3 \int_{\Gamma_2} (\Lambda w - \theta_R)^2 \, da + \alpha_2 \int_{\Gamma_2} f_2^2 \, da + \alpha_1 \int_{\Gamma_b} q_b^2 \, da + \alpha_0 \int_{\Gamma_2} h_n^2 \, da.$$

We observe that the operator $\Lambda : X \to W$ exhibits continuity. Leveraging this property, it becomes evident that the cost functional meets the criteria of assumptions $\mathcal{A}_1$ and $\mathcal{A}_2$. Consequently, Theorem 2 ensures the existence of solutions for the optimal control problem 4.10.

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Data Availability. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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