

## 2-semi equivelar maps on the torus and the Klein bottle with few vertices

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**Abstract:** The  $k$ -semi equivelar maps, for  $k \geq 2$ , are generalizations of maps on the surfaces of Johnson solids to closed surfaces other than the 2-sphere. In the present study, we determine 2-semi equivelar maps of curvature 0 exhaustively on the torus and the Klein bottle. Furthermore, we classify (up to isomorphism) all these 2-semi equivelar maps on the surfaces with up to 12 vertices.

**Keywords:** 2-Semi equivelar maps, face-sequence, torus, Klein bottle.

**AMS Subject classification:** 52B70, 52C10, 05C30, 05C38.

### 1. Introduction

A map  $M$  on a surface  $S$  is an embedding of a graph  $G$  into  $S$  such that: (i) the closure of each component of  $S \setminus G$  is topologically a  $p$ -gonal 2-disk  $D_p$  ( $p \geq 3$ ), which is called a face of  $M$  and (ii) the non-empty intersection of any two distinct faces is either a vertex or an edge, see [1]. The vertices and edges of the underlying graph  $G$  in a map  $M$  are called the vertices and edges of the map. Let  $M_1$  and  $M_2$  be two maps with the vertex sets  $V(M_1)$  and  $V(M_2)$ . Then  $M_1$  and  $M_2$  are isomorphic, denoted as  $M_1 \cong M_2$ , if there is a bijective map between  $V(M_1)$  and  $V(M_2)$  that

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sends vertices to vertices, edges to edges and faces to faces, and preserves incidence, see [11].

The face-sequence of a vertex  $v \in V(M)$ , denoted as  $f_{seq}(v)$ , is  $f_{seq}(v) = (p_1^{n_1}.p_2^{n_2} \dots p_k^{n_k})$  if the consecutive  $n_1$  numbers of  $p_1$ -gon,  $n_2$  numbers of  $p_2$ -gon,  $\dots$ ,  $n_k$  numbers of  $p_k$ -gon are incident at  $v$  in the given cyclic order. The curvature of the vertex  $v$ , denoted as  $\phi(v)$ , is then defined as  $\phi(v) = 1 - (\sum_{i=1}^k n_i)/2 + \sum_{i=1}^k (n_i/p_i)$ . We say that a map  $M$  has the combinatorial curvature  $k$  if  $\phi(v) = k$  for every  $v \in V(M)$ . Some maps of positive curvature are discussed in [12, 19].

A map  $M$  having  $k$  distinct face-sequences, say  $f_1, f_2, \dots, f_k$ , is called a  $k$ -semi equivelar map of type  $[f_1 : f_2 : \dots : f_k]$ . The 11 Archimedean tilings, 20 2-uniform tilings, 61 3-uniform tilings, 151 4-uniform tilings, 332 5-uniform tilings, and 673 6-uniform tilings on the Euclidean plane provide 1-, 2-, 3-, 4-, 5-, and 6-semi equivelar maps on the plane respectively, see [3, 8].

In the case when  $k = 1$ , the map is referred to as a semi equivelar map. Datta and Maity [5] described all types of semi equivelar maps on the surfaces of Euler characteristic 2 (2-sphere  $S^2$ ) and 1 (projective plane  $\mathbb{R}P^2$ ). The first author with Maity and Upadhyay classified some semi equivelar maps on the surface of Euler characteristic  $-1$ , [17, 18]. Karabáš and Nedela [9, 10] have described some semi equivelar maps on the orientable surfaces of Euler characteristic  $-2$  (double torus),  $-4$ , and  $-6$ . Semi equivelar maps have been studied extensively for the surfaces of Euler characteristic 0, that is, on the torus and Klein bottle. Kurth [11] as well as Brehm and Kühnel [2] have given a technique to enumerate semi equivelar maps of type  $[3^6]$ ,  $[4^4]$  and  $[6^3]$  on the torus. Datta and Nilakantan [6] have classified these types of maps on the torus and Klein bottle for  $\leq 11$  vertices. Further, Datta and upadhyay [7] have extended this classification for  $\leq 15$  vertices. In [16], the first author with upadhyay have classified 1-semi equivelar maps of types  $[3^4.6]$ ,  $[3^3.4^2]$ ,  $[3^2.4.3.4]$ ,  $[3.4.6.4]$ ,  $[3.6.3.6]$ ,  $[3.12^2]$ ,  $[4.6.12]$ ,  $[4.8^2]$  on the torus and Klein bottle on at most 20 vertices. Recently  $k$ -semi-equivelar maps have been also studied for  $k \geq 2$ . Some 2-semi equivelar maps have been described on the torus, Klein bottle, and plane, see [13, 14]. In this article, we show:

**Theorem 1.** *If  $T$  denotes the type of a 2-semi equivelar map of curvature 0 on the torus or Klein bottle, then  $T \in \{[3^6 : 3^4.6], [3^6 : 3^3.4^2], [3^6 : 3^2.4.3.4], [3^6 : 3^2.4.12], [3^6 : 3^2.6^2], [3^4.6 : 3.6.3.6], [3^3.4^2 : 3^2.4.3.4], [3^3.4^2 : 3.4.6.4], [3^3.4^2 : 4^4], [3^2.4.3.4 : 3.4.6.4], [3^2.6^2 : 3^4.6], [3^2.6^2 : 3.6.3.6], [3.4^2.6 : 3.4.6.4], [3.4^2.6 : 3.6.3.6], [3.4.6.4 : 4.6.12], [3.12^2 : 3.4.3.12]\}$ .*

Further, we enumerate all the above types of 2-semi equivelar maps on  $\leq 12$  vertices and show:

**Theorem 2.** *There are exactly 31 2-semi equivelar maps of curvature 0 on the surfaces of Euler characteristic 0 on at most 12 vertices; 18 of which are on the torus and 13 are on the Klein bottle. These 18 are  $\mathcal{A}_2(T)$ ,  $\mathcal{A}_3(T)$ ,  $\mathcal{B}_2(T)$ ,  $\mathcal{C}_2(T)$ ,  $\mathcal{E}_3(T)$ ,  $\mathcal{E}_4(T)$ ,  $\mathcal{E}_6(T)$ ,  $\mathcal{E}_8(T)$ ,  $\mathcal{E}_{11}(T)$ ,  $\mathcal{E}_{12}(T)$ ,  $\mathcal{E}_{13}(T)$ ,  $\mathcal{E}_{14}(T)$ ,  $\mathcal{F}_2(T)$ ,  $\mathcal{F}_3(T)$ ,  $\mathcal{F}_5(T)$ ,  $\mathcal{F}_6(T)$ ,  $\mathcal{F}_7(T)$ ,  $\mathcal{F}_9(T)$ , and 13 are*

$\mathcal{A}_1(K), \mathcal{B}_1(K), \mathcal{C}_1(K), \mathcal{D}_1(K), \mathcal{E}_1(K), \mathcal{E}_2(K), \mathcal{E}_5(K), \mathcal{E}_7(K), \mathcal{E}_9(K), \mathcal{E}_{10}(K), \mathcal{F}_1(K), \mathcal{F}_4(K), \mathcal{F}_8(K)$ , given in example Section 3.

This article is organized in the following manner. In Section 2, we give some definitions and notations that are used in the further sections. In Section 3, we present examples of 2 semi-equivelar maps on the torus and the Klein bottle. Further, in Section 4, we describe classification of the above types 2-semi-equivelar maps. We end up this article by presenting some concluding remarks in Section 5.

## 2. Definitions and notations

Let  $M$  be a map with the vertex set  $V(M)$ , edge set  $E(M)$ , and face set  $F(M)$ . For  $v \in V(M)$ , consider  $K_v = \{f \in F(M) : v \in f\}$ . Then the geometric carrier  $|K_v|$  (the union of all the elements in  $K_v$ ) is a 2-disk with the boundary cycle  $C_n = C_n(v_1, v_2, \dots, v_n)$ . The cycle  $C_n$  is called the link of the vertex  $v$  and is denoted as  $\text{lk}(v)$ . For example, if  $v$  is a vertex with  $f_{seq}(v) = (3^6)$ ,  $(3^3.4^2)$  or  $(3^2.4.3.4)$ , then  $\text{lk}(v) = C_6(v_1, v_2, v_3, v_4, v_5, v_6)$ ,  $\text{lk}(v) = C_7(v_1, \mathbf{v_2}, v_3, v_4, v_5, v_6, \mathbf{v_7})$  or  $\text{lk}(v) = C_7(v_1, v_2, v_3, \mathbf{v_4}, v_5, v_6, \mathbf{v_7})$ , respectively (see Figure 1). Here, the bold appearance of some  $v_i$ 's means  $v$  is not adjacent with these  $v_i$ 's. We use a similar notation frequently in Section 4 to express the link of a vertex with a specific face-sequence.

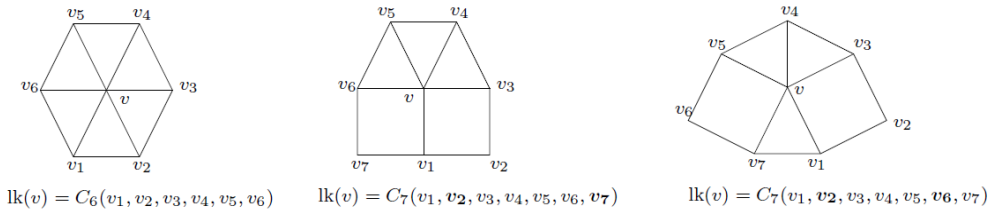


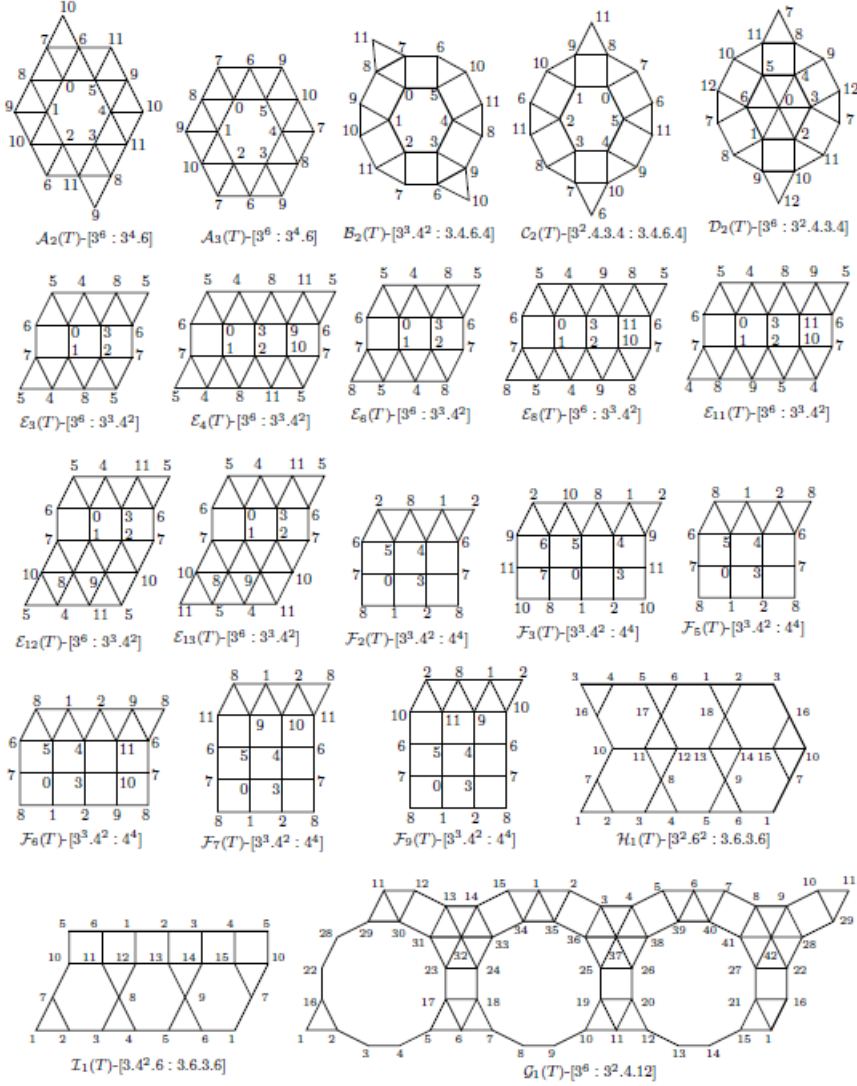
Figure 1: vertex  $v$  with face-sequences  $(3^6)$ ,  $(3^3.4^2)$  and  $(3^2.4.3.4)$  (see from left)

## 3. Example: 2-semi-equivelar maps on the torus and Klein bottle

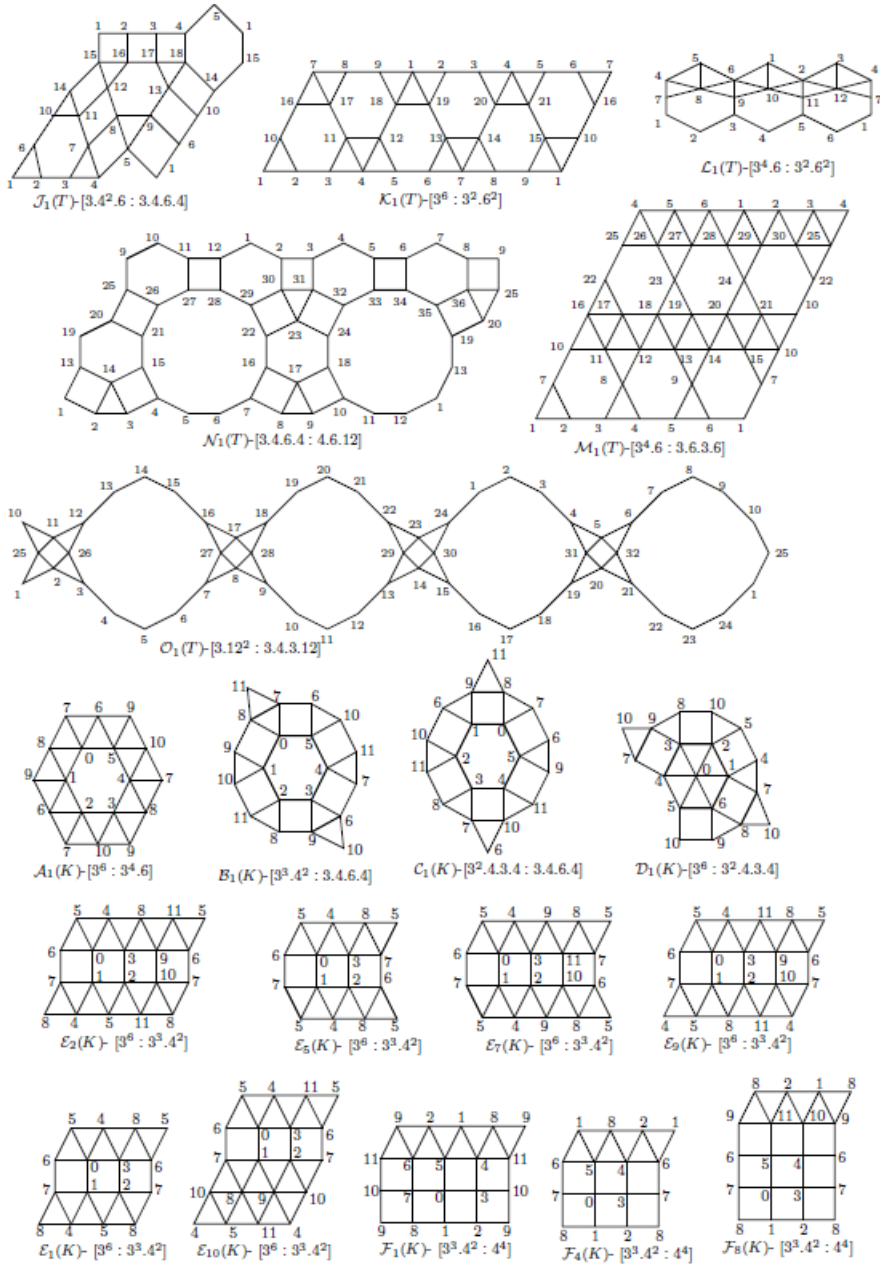
Here, we present examples of 2-semi equivelar maps on the surfaces of Euler characteristic 0. The notations  $\mathcal{A}_i(T)-[f_1 : f_2]$  to  $\mathcal{O}_i(T)-[g_1 : g_2]$  represent maps on the torus of type  $[f_1 : f_2]$  to  $[g_1 : g_2]$  respectively. Similarly, the notations  $\mathcal{A}_i(K)-[f_1 : f_2]$  to  $\mathcal{O}_i(K)-[g_1 : g_2]$  represent maps on the Klein bottle of type  $[f_1 : f_2]$  to  $[g_1 : g_2]$  respectively.

Now, we show the following.

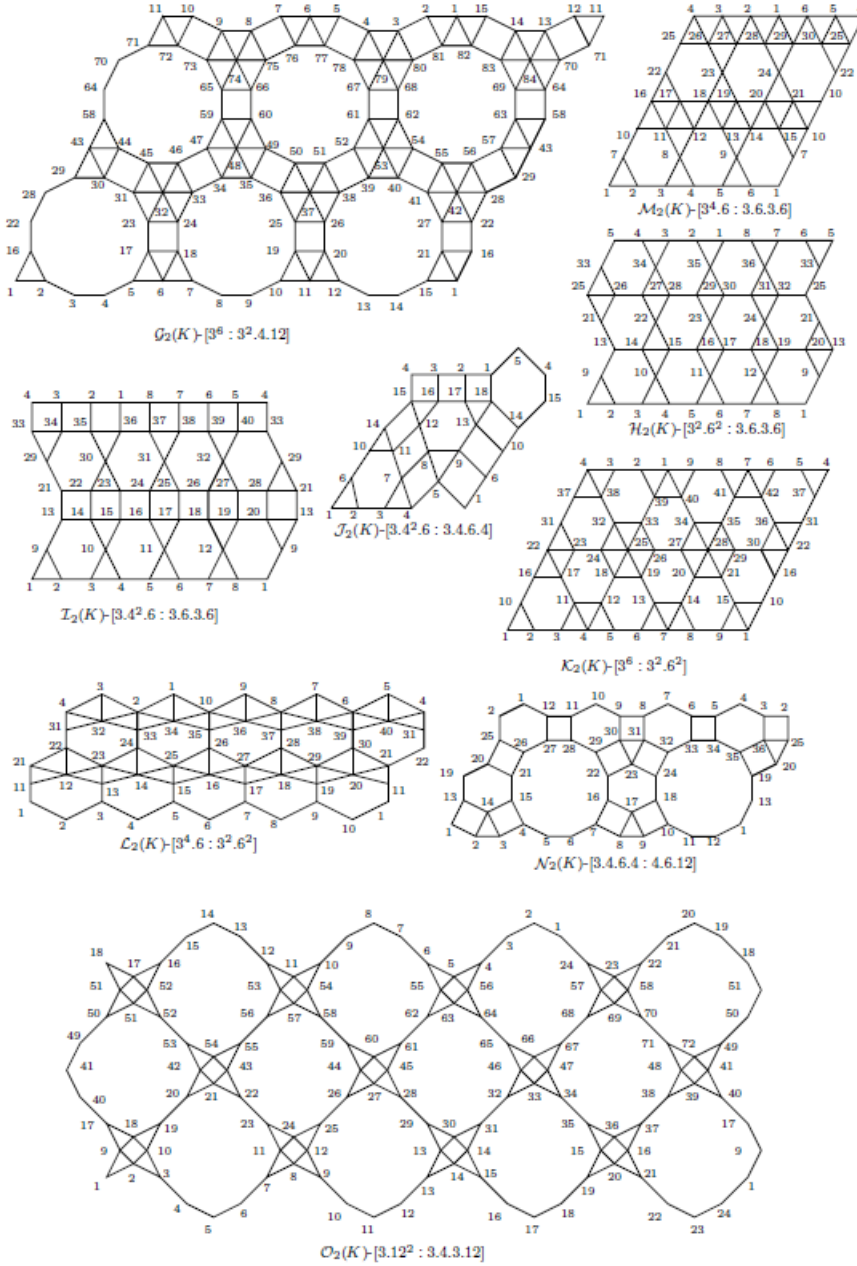
**Lemma 1.** *For the maps given in Section 3, we have:*



- a:  $\mathcal{E}_3(T) \not\cong \mathcal{E}_6(T)$ ,
- b:  $\mathcal{E}_i(T) \not\cong \mathcal{E}_j(T)$ , for  $i, j \in \{4, 8, 11, 12, 13\}$  and  $i \neq j$ ,
- c:  $\mathcal{F}_2(T) \not\cong \mathcal{F}_5(T)$ ,
- d:  $\mathcal{F}_i(T) \not\cong \mathcal{F}_j(T)$ , for  $i, j \in \{3, 6, 7, 9\}$  and  $i \neq j$ ,
- e:  $\mathcal{E}_1(K) \not\cong \mathcal{E}_5(K)$ ,
- f:  $\mathcal{E}_i(K) \not\cong \mathcal{E}_j(K)$ , for  $i, j \in \{2, 7, 9, 10\}$  and  $i \neq j$ ,
- i:  $\mathcal{F}_1(K) \not\cong \mathcal{F}_8(K)$ .



*Proof.* A cycle  $C$  is called contractible if it bounds a 2-disk, otherwise called non-contractible. Note that in  $\mathcal{E}_3(T)$ , we see exactly one vertical non-contractible cycle of length 3 at each vertex (for example, we see  $C_3(0, 1, 4)$  at vertex 4), while in  $\mathcal{E}_6(T)$ , there are two non-contractible cycles at each vertex (for example, we have  $C_3(0, 1, 4)$  and  $C_3(2, 3, 4)$  at vertex 4). This proves *a*. Following a similar argument, we see



$\mathcal{E}_1(K) \not\cong \mathcal{E}_5(K)$ . This proves *e*.

In  $\mathcal{F}_5(K)$ , at each vertex, we have a vertical non-contractible cycle of length 3 (for example, we see  $C_5(0, 1, 5)$  at vertex 1), which is not true in  $\mathcal{F}_2(K)$ . This proves *c*. Similarly, we get  $\mathcal{F}_1(K) \not\cong \mathcal{F}_8(K)$ . This proves *i*.

The following polynomials  $p_{G(M)}(a)$  denote the characteristic polynomial (computed from MATLAB) of the adjacency matrix associated with the underlying graph  $G$  of map  $M$ . We know if two maps have distinct characteristic polynomials, they are non-isomorphic. This proves  $b$ ,  $d$ , and  $f$ .

$$p_{G(\mathcal{E}_4(T))}(a) = a^{12} - 32a^{10} - 40a^9 + 254a^8 + 440a^7 - 628a^6 - 1400a^5 + 105a^4 + 1000a^3 + 300a^2,$$

$$p_{G(\mathcal{E}_8(T))}(a) = a^{12} - 32a^{10} - 48a^9 + 254a^8 + 656a^7 - 292a^6 - 2352a^5 - 2167a^4 + 624a^3 + 2044a^2 + 1120a + 192,$$

$$p_{G(\mathcal{E}_{11}(T))}(a) = a^{12} - 31a^{10} - 32a^9 + 222a^8 + 180a^7 - 746a^6 - 220a^5 + 1201a^4 - 228a^3 - 647a^2 + 322a,$$

$$p_{G(\mathcal{E}_{12}(T))}(a) = a^{12} - 33a^{10} - 44a^9 + 258a^8 + 432a^7 - 682a^6 - 1032a^5 + 957a^4 + 560a^3 - 789a^2 + 276a - 32,$$

$$p_{G(\mathcal{E}_{13}(T))}(a) = a^{12} - 33a^{10} - 44a^9 + 252a^8 + 456a^7 - 568a^6 - 1296a^5 + 348a^4 + 1328a^3 + 108a^2 - 432a - 128,$$

$$p_{G(\mathcal{F}_3(T))}(a) = a^{12} - 26a^{10} - 17a^9 + 176a^8 + 91a^7 - 505a^6 - 95a^5 + 590a^4 - 90a^3 - 118a^2 + 24a,$$

$$p_{G(\mathcal{F}_6(T))}(a) = a^{12} - 28a^{10} - 24a^9 + 212a^8 + 280a^7 - 524a^6 - 976a^5 + 80a^4 + 860a^3 + 528a^2 + 96a,$$

$$p_{G(\mathcal{F}_7(T))}(a) = a^{12} - 27a^{10} - 20a^9 + 201a^8 + 192a^7 - 532a^6 - 552a^5 + 492a^4 + 560a^3 - 84a^2 - 192a - 44,$$

$$p_{G(\mathcal{F}_9(T))}(a) = a^{12} - 27a^{10} - 20a^9 + 207a^8 + 168a^7 - 610a^6 - 288a^5 + 723a^4 - 136a^3 - 171a^2 + 84a - 11,$$

$$p_{G(\mathcal{E}_2(K))}(a) = a^{12} - 32a^{10} - 40a^9 + 254a^8 + 440a^7 - 644a^6 - 1400a^5 + 457a^4 + 1640a^3 + 156a^2 - 640a - 192,$$

$$p_{G(\mathcal{E}_7(K))}(a) = a^{12} - 32a^{10} - 48a^9 + 258a^8 + 640a^7 - 364a^6 - 220a^5 - 1635a^4 + 496a^3 + 684a^2 - 32a - 64,$$

$$p_{G(\mathcal{E}_9(K))}(a) = a^{12} - 31a^{10} - 39a^9 + 227a^8 + 377a^7 - 561a^6 - 1129a^5 + 416a^4 + 1283a^3 + 92a^2 - 492a - 144.$$

This proves the lemma. □

#### 4. Proofs: Classification of 2-semi equivelar maps

In this section, we prove our main results Theorems 1 and 2. In our earlier study [13], we have shown the following:

**Proposition 1.** *Let  $v$  be a vertex with the face-sequence  $f$  such that  $\phi(v) = 0$ . Then  $f \in S$ , where  $S = \{(3^3.4^2), (3^6), (3.4^2.6), (3^2.6^2), (3^4.6), (3^2.4.3.4), (3.6.3.6)\}$ ,*

$(4^4), (3.4.6.4), (3^2.4.12), (4.8^2), (3.12^2), (6^3), (5^2.10), (3.8.24), (3.9.18), (3.10.15), (4.5.20), (3.7.42), (4.6.12), (3.4.3.12)\}$ .

From the above proposition and the compatibility of face-sequences, it is easy to see that if a 2-semi equivelar map exists, then its possible type  $T \in S_1 = A \cup B$ , where  $A = \{[3^3.4^2 : 3^2.6^2], [3^3.4^2 : 3.4^2.6], [3^3.4^2 : 3^4.6], [3^3.4^2 : 3^2.4.12], [3^3.4^2 : 4.8^2], [3^3.4^2 : 4.5.20], [3^3.4^2 : 4.6.12], [3^3.4^2 : 3.4.3.12], [3.4^2.6 : 3^2.6^2], [3.4^2.6 : 3^4.6], [3.4^2.6 : 3^2.4.3.4], [3.4^2.6 : 4^4], [3.4^2.6 : 3^2.4.12], [3.4^2.6 : 4.8^2], [3.4^2.6 : 6^3], [3.4^2.6 : 4.5.20], 3.4^2.6 : 4.6.12], [3.4^2.6 : 3.4.3.12], [3^2.6^2 : 3^2.4.3.4], [3^2.6^2 : 3^2.4.12], [3^2.6^2 : 6^3], [3^4.6 : 3^2.4.3.4], [3^4.6 : 3^2.4.12], [3^4.6 : 6^3], [3^4.6 : 4.6.12], [3^2.4.3.4 : 4^4], [3^2.4.3.4 : 3^2.4.12], [3^2.4.3.4 : 3.4.3.12], [3^2.4.3.4 : 4.8^2], [3.6.3.6 : 4.6.12], [3.6.3.6 : 6^3], [4^4 : 3.4.6.4], [4^4 : 3^2.4.12], [4^4 : 4.8^2], [4^4 : 4.5.20], [4^4 : 4.6.12], [4^4 : 3.4.3.12], [3.4.6.4 : 3^2.4.12], [3.4.6.4 : 6^3], [3.4.6.4 : 4.8^2], [3.4.6.4 : 4.5.20], [3.4.6.4 : 3.4.3.12], [3^2.4.12 : 3.12^2], [3^2.4.12 : 3.4.3.12], [3^2.4.12 : 4.5.20], [3^2.4.12 : 4.6.12], [3^2.4.12 : 4.8^2], [4.8^2 : 4.5.20], [4.8^2 : 4.6.12], [4.8^2 : 3.4.3.12], [3.12^2 : 3.4.3.12], [6^3 : 4.6.12], [5^2.10 : 4.5.20]\}$  and  $B = \{[3^6 : 3^4.6], [3^6 : 3^3.4^2], [3^6 : 3^2.4.3.4], [3^6 : 3^2.4.12], [3^6 : 3^2.6^2], [3^4.6 : 3.6.3.6], [3^3.4^2 : 3^2.4.3.4], [3^3.4^2 : 3.4.6.4], [3^3.4^2 : 4^4], [3^2.4.3.4 : 3.4.6.4], [3^2.6^2 : 3^4.6], [3^2.6^2 : 3.6.3.6], [3.4^2.6 : 3.4.6.4], [3.4^2.6 : 3.6.3.6], [3.4.6.4 : 4.6.12], [3.12^2 : 3.4.3.12]\}$ .

We note the following:

**Remark.** Let  $M$  be a 2-semi equivelar map of type  $[f_1 : f_2]$  with the vertex set  $V(M)$ . Then for  $f_1$  (or  $f_2$ ), there is a vertex  $v$  in  $V(M)$  with the face-sequence  $f_1$  (resp.  $f_2$ ) such that  $\text{lk}(v)$  contains a vertex  $u$  with the face-sequence  $f_2$  (resp.  $f_1$ ). Such a vertex  $v$  is called a critical vertex. Clearly,  $M$  does not exist if it has no critical vertex.

Now we prove the following:

**Proof of Theorem 1.** As given above, a 2-semi equivelar map  $M$  has possible type  $T \in S_1$ . Note that for each  $T \in B$ , there exists a 2-semi equivelar map of the type  $T$ , see examples in Section 3. For  $T \in A$ , we show:

**Claim 1.** There exist no 2-semi equivelar maps of type  $T \in A$ .

Consider a map  $M$  of type  $T \in A$ , where  $T = [f_1 = 3^3.4^2 : f_2]$ . Let  $x_0$  be a critical vertex with the face-sequence  $(3^3.4^2)$  such that  $\text{lk}(x_0) = C_7(x_1, \mathbf{x}_2, x_3, x_4, x_5, x_6, \mathbf{x}_7)$ . Then we have the following cases for  $f_2$ :

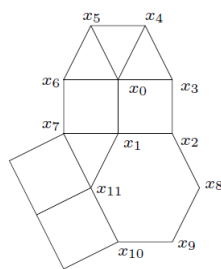
**Case 1.**  $f_2 = (3^2.6^2)$ , i.e.,  $T = [f_1 = 3^3.4^2 : f_2 = 3^2.6^2]$ .

Then at least one vertex in  $\{x_4, x_5\}$  has the face-sequence  $(3^2.6^2)$ . If  $f_{\text{seq}}(x_4) = (3^2.6^2)$  (or  $f_{\text{seq}}(x_5) = (3^2.6^2)$ ), then  $x_3$  (resp.  $x_6$ ) has the face-sequence  $f_3 \neq f_1, f_2$ . This means  $x_0$  is not a critical vertex.

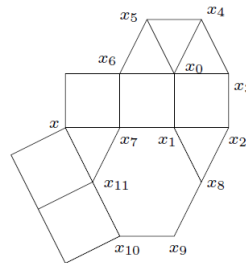
**Case 2.**  $f_2 = (3.4^2.6)$ , i.e.,  $T = [f_1 = 3^3.4^2 : f_2 = 3.4^2.6]$ .



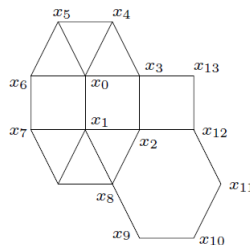
Then at least one vertex in  $\{x_1, x_2, x_3, x_6, x_7\}$  has the face-sequence  $(3.4^2.6)$ . If  $f_{seq}(x_3) = (3.4^2.6)$ , then we see consecutive 2 triangular faces and one hexagonal face at  $x_4$ , which implies  $f_{seq}(x_4) \neq f_1, f_2$ . Similarly, if  $f_{seq}(x_6) = (3.4^2.6)$ , then  $f_{seq}(x_5) \neq f_1, f_2$ . If  $f_{seq}(x_1) = (3.4^2.6)$ , then  $\text{lk}(x_1) = C_9(x_0, \mathbf{x}_3, x_2, \mathbf{x}_8, \mathbf{x}_9, \mathbf{x}_{10}, x_{11}, x_7, \mathbf{x}_6)$  or  $\text{lk}(x_1) = C_9(x_0, \mathbf{x}_6, x_7, \mathbf{x}_{11}, \mathbf{x}_{10}, \mathbf{x}_9, x_8, x_2, \mathbf{x}_3)$ . The first case of  $\text{lk}(x_1)$  implies  $f_{seq}(x_{11}) = (3.4^2.6)$ . Now considering  $\text{lk}(x_{11})$ , we get one triangular face adjacent with two quadrangular faces at  $x_7$  (see Figure 4.1), which shows  $f_{seq}(x_7) \neq f_1, f_2$ . Similarly, for the later case of  $\text{lk}(x_1)$ , we get a vertex  $x$  such that  $f_{seq}(x) \neq f_1, f_2$ , (see Figure 4.2). So,  $f_{seq}(x_1) \neq (3.4^2.6)$ . If  $f_{seq}(x_2) = (3.4^2.6)$ , then  $\text{lk}(x_2) = C_9(x_3, \mathbf{x}_{13}, x_{12}, \mathbf{x}_{11}, \mathbf{x}_{10}, \mathbf{x}_9, x_8, x_1, \mathbf{x}_0)$ , which shows  $f_{seq}(x_1) = (3^3.4^2)$  and then we get consecutive two triangular faces and one hexagonal face at  $x_8$  (see Figure 4.3), which implies  $f_{seq}(x_8) \neq f_1, f_2$ . So,  $f_{seq}(x_2) \neq f_2$ . Similarly,  $f_{seq}(x_7) \neq f_2$ . Thus  $x_0$  is not a critical vertex.



(a)



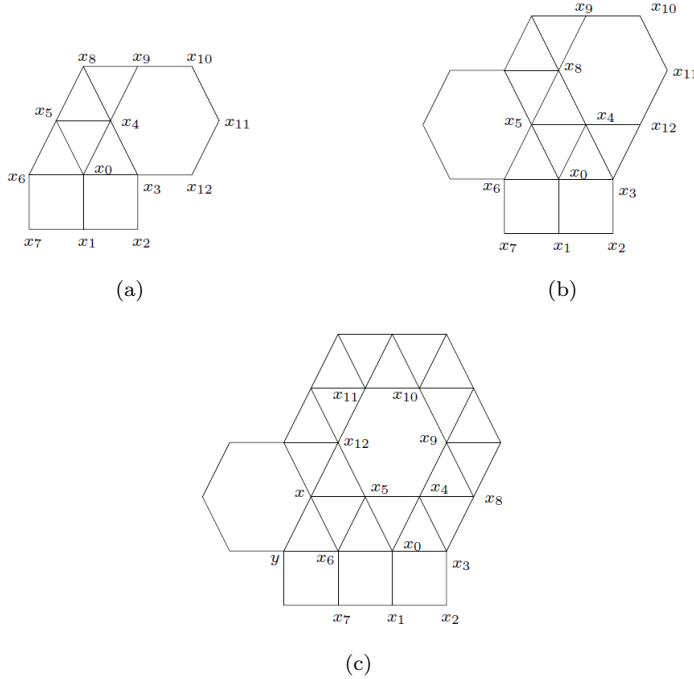
(b)



(c)

**Case 3.**  $f_2 = (3^4.6)$ , i.e.,  $T = [f_1 = 3^3.4^2 : f_2 = 3^4.6]$ .

Then at least one vertex in  $\{x_4, x_5\}$  has the face-sequence  $(3^4.6)$ . Without loss of generality let  $f_{seq}(x_4) = (3^4.6)$ . Then  $\text{lk}(x_4) = C_8(x_3, x_0, x_5, x_8, x_9, \mathbf{x}_{10}, \mathbf{x}_{11}, \mathbf{x}_{12})$  or  $\text{lk}(x_4) = C_8(x_{12}, x_3, x_0, x_5, x_8, \mathbf{x}_9, \mathbf{x}_{10}, \mathbf{x}_{11})$  or  $\text{lk}(x_4) = C_8(x_5, x_0, x_3, x_8, x_9, \mathbf{x}_{10}, \mathbf{x}_{11}, \mathbf{x}_{12})$ . In the first case of  $\text{lk}(x_4)$ , we see that  $f_{seq}(x_3) \neq f_1, f_2$  (see Figure 4.4). In the second case of  $\text{lk}(x_4)$ , considering successively  $\text{lk}(x_8)$  and



$\text{lk}(x_5)$ , we see that  $f_{seq}(x_6) \neq f_1, f_2$  (see Figure 4.5). In the last case of  $\text{lk}(x_4)$ , considering successively  $\text{lk}(x_9)$ ,  $\text{lk}(x_{10})$ ,  $\text{lk}(x_{11})$ ,  $\text{lk}(x_{12})$ ,  $\text{lk}(x_5)$ ,  $\text{lk}(x_6)$ , and then  $\text{lk}(x)$ , we get a vertex  $y$  in  $\text{lk}(6)$  such that  $f_{seq}(y) \neq f_1, f_2$  (see Figure 4.6). Thus  $x_0$  is not a critical vertex.

**Case 4.**  $f_2 = (3^2.4.12)$ , i.e.,  $T = [f_1 = 3^3.4^2 : f_2 = 3^2.4.12]$ .

Then at least one vertex in  $\{x_2, x_3, x_4, x_5, x_6, x_7\}$  has the face-sequence  $(3^2.4.12)$ . Note that, the case  $f_{seq}(x_2) = (3^2.4.12)$  is similar to the case  $f_{seq}(x_7) = (3^2.4.12)$ , the case  $f_{seq}(x_3) = (3^2.4.12)$  is similar to the case  $f_{seq}(x_6) = (3^2.4.12)$ , and the case  $f_{seq}(x_4) = (3^2.4.12)$  is similar to the case  $f_{seq}(x_5) = (3^2.4.12)$ . So, it is enough to see the cases when  $f_{seq}(x_2) = (3^2.4.12)$ ,  $f_{seq}(x_3) = (3^2.4.12)$ , and  $f_{seq}(x_4) = (3^2.4.12)$ . For  $f_{seq}(x_2) = (3^2.4.12)$ , let  $\text{lk}(x_2) = C_{14}(x_3, x_0, x_1, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{17}, x_{18})$ . This implies  $f_{seq}(x_3) = (3^2.4.12)$  and  $f_{seq}(x_i) = (3^2.4.12)$ , for  $9 \leq i \leq 18$ , which further gives  $f_{seq}(x_4) \neq f_1, f_2$ . If  $f_{seq}(x_3) = (3^2.4.12)$ , then  $f_{seq}(x_4) \neq f_1, f_2$ . If  $f_{seq}(x_4) = (3^2.4.12)$ , then  $f_{seq}(x_3) \neq f_1, f_2$ . This means  $x_0$  is not a critical vertex.

**Case 5.**  $f_2 = (4.8^2)$ , i.e.,  $T = [f_1 = 3^3.4^2 : f_2 = 4.8^2]$ .

Then at least one vertex in  $\{x_2, x_7\}$  has the face-sequence  $(4.8^2)$ . Note that, the case

$f_{seq}(x_2) = (4.8^2)$  is similar to the case  $f_{seq}(x_7) = (4.8^2)$ . If  $f_{seq}(x_2) = (4.8^2)$ , then  $f_{seq}(x_3) \neq f_1, f_2$ . This means  $x_0$  is not a critical vertex.

**Case 6.**  $f_2 = (4.5.20)$ , i.e.,  $T = [f_1 = 3^3.4^2 : f_2 = 4.5.20]$ .

Then at least one vertex in  $\{x_2, x_7\}$  has the face-sequence  $(4.8^2)$ . Note that, the case  $f_{seq}(x_2) = (4.5.20)$  is similar to the case  $f_{seq}(x_7) = (4.5.20)$ . If  $f_{seq}(x_2) = (4.5.20)$ , then  $f_{seq}(x_1) \neq f_1, f_2$ . This means  $x_0$  is not a critical vertex.

**Case 7.**  $f_2 = (4.6.12)$ , i.e.,  $T = [f_1 = 3^3.4^2 : f_2 = 4.6.12]$ .

Then at least one vertex in  $\{x_2, x_7\}$  has the face-sequence  $(4.6.12)$ . The case  $f_{seq}(x_2) = (4.6.12)$  is similar to the case  $f_{seq}(x_7) = (4.6.12)$ . If  $f_{seq}(x_2) = (4.6.12)$ , then  $f_{seq}(x_1) \neq f_1, f_2$ . This means  $x_0$  is not a critical vertex.

**Case 8.**  $f_2 = (3.4.3.12)$ , i.e.,  $T = [f_1 = 3^3.4^2 : f_2 = 3.4.3.12]$ .

Then at least one vertex in  $\{x_2, x_3, x_6, x_7\}$  has the face-sequence  $(3.4.3.12)$ . Note that, the cases  $f_{seq}(x_2) = (3.4.3.12)$  and  $f_{seq}(x_3) = (3.4.3.12)$  are similar to the cases  $f_{seq}(x_7) = (3.4.3.12)$  and  $f_{seq}(x_6) = (3.4.3.12)$  respectively. If  $f_{seq}(x_2) = (3.4.3.12)$ , then  $f_{seq}(x_3) = (3.4.3.12)$ . This gives  $f_{seq}(x_4) \neq f_1, f_2$ . Similarly, we see  $f_{seq}(x_3) \neq f_1, f_2$ . This means  $x_0$  is not a critical vertex.

Hence, in all the above cases for  $f_2$ ,  $x_0$  is not a critical vertex. Thus the map  $M$  of type  $T = [f_1 = 3^3.4^2 : f_2]$  does not exist. By a similar computation, it can be shown easily that  $M$  does not exist for the remaining type  $T \in A$ . Thus the claim and hence the lemma.  $\square$

Further, we enumerate and classify 2-semi equivelar map  $M$  of type  $T \in B$  for the number of vertices  $|V(M)| \leq 12$ , by the Lemmas 2-B. The classification is exhaustive search for all possible cases. Let the vertex set  $V(M) = \{0, 1, \dots, 11\}$ . Then:

**Lemma 2.** *There exists no 2-semi equivelar map of type  $T \in S_2 = \{[3^6 : 3^2.4.12], [3^6 : 3^2.6^2], [3^4.6 : 3.6.3.6], [3^2.6^2 : 3^4.6], [3^2.6^2 : 3.6.3.6], [3.4^2.6 : 3.6.3.6], [3.4^2.6 : 3.4.6.4], [3.4.6.4 : 4.6.12], [3.12^2 : 3.4.3.12]\}$  for the number of vertices  $\leq 12$ .*

*Proof.* Since one requires more than 12 vertices to complete the link of a vertex with the face-sequence  $(3^2.4.12)$ ,  $(4.6.12)$  or  $(3.4.3.12)$ ,  $M$  of type  $[3^6 : 3^2.4.12]$ ,  $[3.4.6.4 : 4.6.12]$  or  $[3.12^2 : 3.4.3.12]$  does not exist on the given  $V(M)$ . For the remaining types, we have following cases:

**Case 1.** If  $M$  is of the type  $[3^6 : 3^2.6^2]$ , then without loss of generality let 0 be a critical vertex with the face-sequence  $(3^2.6^2)$  and let  $\text{lk}(0) = C_{10}(1, \mathbf{2}, \mathbf{3}, \mathbf{4}, 5, 6, 7, \mathbf{8}, \mathbf{9}, \mathbf{10})$ . Then the only vertex in  $\text{lk}(0)$  which can have face-sequence  $(3^6)$  is 6. This gives  $\text{lk}(6) = C_6(5, 0, 7, x_1, x_2, x_3)$ , where  $(x_1, x_2, x_3) \in$

$S_3 = \{(2, 1, 10), (2, 3, 11), (3, 2, 11), (3, 4, 11), (4, 3, 2), (4, 3, 11), (11, 4, 3)\}$ . If  $(x_1, x_2, x_3) = (2, 1, 10)$ , then  $\text{lk}(6) = C_6(5, 0, 7, 2, 1, 10)$ . This implies  $\text{lk}(2) = C_{10}(3, \mathbf{4}, \mathbf{5}, \mathbf{0}, 1, 6, 7, \mathbf{8}, \mathbf{x}_5, \mathbf{x}_4)$ . Now by the fact that two distinct hexagonal faces can not share more than 2 vertices, we get  $x_4 = 11$  and then  $x_5$  has no suitable value in  $V(M)$  so that  $\text{lk}(2)$  can be completed. So,  $(x_1, x_2, x_3) \neq (2, 1, 10)$ . By a similar argument, we see that  $M$  does not exist for the remaining  $(x_1, x_2, x_3) \in S_3$ .

**Case 2.** If  $M$  is of the type  $[3^4.6 : 3.6.3.6]$ , then observe that two distinct hexagonal faces can share at most one vertex. Now, without loss of generality, let  $f_{seq}(0) = (3.6.3.6)$  and  $\text{lk}(0) = C_{10}(1, \mathbf{2}, \mathbf{3}, \mathbf{4}, 5, 6, \mathbf{7}, \mathbf{8}, \mathbf{9}, 10)$ . Then  $f_{seq}(1) = (3.6.3.6)$  or  $(3^4.6)$ . If  $f_{seq}(1) = (3.6.3.6)$ , then  $\text{lk}(1) = C_{10}(2, \mathbf{3}, \mathbf{4}, \mathbf{5}, 0, 10, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, x_4)$ . Note that  $x_1 = 11$  and then  $x_4 \in \{7, 8, 9\}$ . But for all these values of  $x_4$ , we see that two distinct hexagonal faces share two vertices. So  $f_{seq}(1) \neq (3.6.3.6)$ .

On the other hand, if  $f_{seq}(1) = (3^4.6)$ , then  $\text{lk}(1) = C_8(2, \mathbf{3}, \mathbf{4}, \mathbf{5}, 0, 10, 11, x_1)$ , where  $x_1 \in \{7, 8, 9\}$ . If  $x_1 = 7$ , then  $\text{lk}(7) = C_8(8, \mathbf{9}, \mathbf{10}, \mathbf{0}, 6, 11, 1, 2)$  or  $\text{lk}(7) = C_8(8, \mathbf{9}, \mathbf{10}, \mathbf{0}, 6, 2, 1, 11)$ , which implies  $\text{lk}(2) = C_8(3, \mathbf{4}, \mathbf{5}, \mathbf{0}, 1, 7, 8, x_2)$  or  $\text{lk}(2) = C_8(3, \mathbf{4}, \mathbf{5}, \mathbf{0}, 1, 7, 6, x_2)$  respectively. Observe that for both the cases of  $\text{lk}(2)$ ,  $x_2$  has no suitable value in  $V(M)$ . So  $x_1 \neq 7$ . Proceeding similarly, we see that  $x_1 \neq 8$  or  $9$ .

**Case 3.** If  $M$  is of the type  $[3^2.6^2 : 3.6.3.6]$ , then observe that two distinct hexagonal faces can share at most two vertices. Without loss of generality, let  $0$  be a critical vertex with  $f_{seq}(0) = (3^2.6^2)$  and  $\text{lk}(0) = C_{10}(1, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, 6, 7, \mathbf{8}, \mathbf{9}, \mathbf{10})$ . Then  $f_{seq}(6) = (3^6)$ . This implies  $\text{lk}(6) = C_6(5, 0, 7, x_1, x_2, x_3)$ . It is easy to see that  $(x_1, x_2, x_3) \in \{(2, 1, 10), (2, 3, 11), (3, 2, 11), (3, 4, 11), (4, 3, 11), (11, 8, 9), (11, 9, 8), (11, 9, 10), (11, 10, 9)\}$ . If  $(x_1, x_2, x_3) = (2, 1, 10)$ , i.e.,  $\text{lk}(6) = C_6(5, 0, 7, 2, 1, 10)$ , then  $\text{lk}(2) = C_{10}(3, \mathbf{4}, \mathbf{5}, \mathbf{0}, 1, 6, 7, \mathbf{8}, \mathbf{x}_4, \mathbf{x}_5)$ . Observe that  $x_4 = 11$  and  $x_5$  has no suitable value in  $V(M)$ . So for  $(x_1, x_2, x_3) = (2, 1, 10)$ , we do not get  $M$ . Proceeding similarly, we see that  $M$  does not exist for the remaining values of  $(x_1, x_2, x_3)$ .

**Case 4.** If  $M$  is of the type  $[3.4^2.6 : 3.6.3.6]$ , then observe that two distinct hexagonal faces can share at most one vertex. Let  $f_{seq}(0) = (3.6.3.6)$  and  $\text{lk}(0) = C_{10}(1, \mathbf{2}, \mathbf{3}, \mathbf{4}, 5, 6, \mathbf{7}, \mathbf{8}, \mathbf{9}, 10)$ . Then  $f_{seq}(1) = (3.6.3.6)$  or  $(3.4^2.6)$ . In the first case, we get  $\text{lk}(1) = C_{10}(2, \mathbf{3}, \mathbf{4}, \mathbf{5}, 0, 10, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, x_4)$ . Now following the fact that two distinct hexagonal faces can share at most one vertex, we do not get suitable values for  $x_1, x_2, x_3, x_4$  in  $V(M)$  so that  $\text{lk}(1)$  can be completed. On the other hand, if  $f_{seq}(1) = (3.4^2.6)$ , then  $\text{lk}(1) = C_9(0, \mathbf{5}, \mathbf{4}, \mathbf{3}, 2, \mathbf{x}_3, x_2, \mathbf{x}_1, 10)$ . It is easy to see that  $x_2 = 11$ , this gives  $x_1 = 9$  and  $x_3 \in \{6, 7, 8\}$ . If  $x_3 = 6$ , then  $\text{lk}(11) = C_9(x_4, \mathbf{x}_5, \mathbf{x}_6, \mathbf{x}_7, 6, \mathbf{2}, 1, \mathbf{10}, 9)$  or  $\text{lk}(11) = C_9(x_4, \mathbf{x}_5, \mathbf{x}_6, \mathbf{x}_7, 9, \mathbf{10}, 1, \mathbf{2}, 6)$ . Again by the fact that two distinct quadrangular faces can share at most 1 vertex, we see that in both the cases  $\text{lk}(11)$  can not be completed. Similarly for  $x_3 = 7$  and  $8$ , we see that  $\text{lk}(11)$  can not be completed.

**Case 5.** If  $M$  is of the type  $[3.4^2.6 : 3.4.6.4]$ , then observe that two distinct hexagonal faces can not share any vertex. Let  $f_{seq}(0) = (3.4^2.6)$  and  $\text{lk}(0) = C_9(1, \mathbf{2}, \mathbf{3}, \mathbf{4}, 5, \mathbf{6}, \mathbf{7}, \mathbf{8}, 9)$ . This implies  $\text{lk}(7) = C_9(x_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, 6, \mathbf{5}, 0, \mathbf{9}, 8)$  or  $\text{lk}(7) = C_9(x_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, 8, \mathbf{9}, 0, \mathbf{5}, 6)$ . Then by the fact that two distinct hexagonal faces are disjoint, successively, we see  $x_1 = 10$  and  $x_2 = 11$ . Now observe that

$x_3, x_4$  have no suitable values in  $V(M)$  so that  $\text{lk}(7)$  can be completed.

**Case 6.** If  $M$  is of the type  $[3^4.6 : 3^2.6^2]$ , then observe that two distinct hexagonal faces can share at most two adjacent vertices (an edge). Let  $f_{seq}(0) = (3^2.6^2)$  and  $\text{lk}(0) = C_{10}(1, \mathbf{2}, \mathbf{3}, \mathbf{4}, 5, 6, 7, \mathbf{8}, \mathbf{9}, \mathbf{10})$ . This implies  $\text{lk}(1) = C_{10}(0, \mathbf{5}, \mathbf{4}, \mathbf{3}, 2, 11, 10, \mathbf{9}, \mathbf{8}, \mathbf{7})$ . Then  $f_{seq}(6) = (3^2.6^2)$  or  $(3^4.6)$ . If  $f_{seq}(6) = (3^2.6^2)$ , then by the above observation, we see easily that  $\text{lk}(6)$  can not be completed. on the other hand, If  $f_{seq}(6) = (3^4.6)$ , then  $\text{lk}(6) = C_8(5, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, x_4, x_5, 7, 0)$  or  $\text{lk}(6) = C_8(7, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, x_4, x_5, 5, 0)$  or  $\text{lk}(6) = C_8(x_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, x_5, 7, 0, 5)$ . The first case of  $\text{lk}(6)$  implies  $x_1 = 4$ . Then there exist two distinct hexagonal faces at 5, which gives  $C_6(0, 4, x_2, x_3, x_4, 6) \subseteq \text{lk}(5)$ . This is not possible. Similarly, we see that the second case of  $\text{lk}(6)$  is also not possible. In the last case of  $\text{lk}(6)$ , it is easy to see that  $(x_1, x_2, x_3, x_4, x_5) \in \{(8, 9, 3, 2, 11), (8, 9, 4, 3, 11), (8, 9, 11, 2, 3), (8, 9, 11, 3, 4), (8, 9, 11, 4, 3), (9, 8, 3, 2, 11), (9, 8, 11, 2, 3), (9, 8, 11, 3, 4), (9, 8, 11, 4, 3), (11, 10, 9, 3, 4), (11, 10, 9, 4, 3)\}$ . If  $(x_1, x_2, x_3, x_4, x_5) = (8, 9, 3, 2, 11)$ , then  $\text{lk}(8) = C_{10}(9, \mathbf{10}, \mathbf{1}, \mathbf{0}, 7, 4, 6, \mathbf{11}, \mathbf{2}, \mathbf{3})$  and  $\text{lk}(9) = C_{10}(8, \mathbf{7}, \mathbf{0}, \mathbf{1}, 10, 4, 3, \mathbf{2}, \mathbf{11}, \mathbf{6})$ . This implies  $C_9(0, 1, 2, 11, 6, 8, 9, 4, 5) \subseteq \text{lk}(3)$ . This is not possible. Proceeding similarly for the remaining cases, we get no map. For the detailed computation, we refer to see [15]. Thus the proof.  $\square$

**Lemma 3.** *Let  $M$  be a 2-semi equivelar map of type  $[3^6 : 3^4.6]$  on  $\leq 12$  vertices. Then  $M$  is isomorphic to one of  $\mathcal{A}_1(K)$ ,  $\mathcal{A}_2(T)$  or  $\mathcal{A}_3(T)$  given in Section 3.*

*Proof.* Without loss of generality, let 0 be a vertex of face-sequence  $(3^4.6)$  and  $\text{lk}(0) = C_8(1, \mathbf{2}, \mathbf{3}, \mathbf{4}, 5, 6, 7, 8)$ . Since the vertices 1, 2, 3, 4, 5 appear in hexagonal face  $[0, 1, 2, 3, 4, 5]$ , the vertices have the face-sequence  $(3^4.6)$ . This gives  $\text{lk}(1) = C_8(2, \mathbf{3}, \mathbf{4}, \mathbf{5}, 0, 8, x_1, x_2)$ . It is easy to see that  $(x_1, x_2) \in \{(6, 7), (6, 9), (9, 6), (9, 10), (9, 7)\}$ . The case  $(6, 9) \cong (9, 7)$  by the map  $(0, 1)(2, 5)(3, 4)(6, 7, 9)$ . So, we need not consider the last case.

**Claim 2.**  $(x_1, x_2) = (9, 6)$  or  $(9, 10)$ .

If  $(x_1, x_2) = (6, 7)$ , then  $\text{lk}(1) = C_8(2, \mathbf{3}, \mathbf{4}, \mathbf{5}, 0, 8, 6, 7)$  and this gives  $f_{seq}(6)$  is  $(3^6)$  or  $(3^4.6)$ . By the fact that two distinct hexagonal faces are disjoint, we see that for the given number of vertices,  $\text{lk}(6)$  can not be completed if  $f_{seq}(6) = (3^4.6)$ . So,  $f_{seq}(6) = (3^6)$ . This gives  $\text{lk}(6) = C_6(0, 5, 9, 8, 1, 7)$ . Again by the same fact as above, we see  $f_{seq}(7) = (3^6)$  and then we get  $\text{lk}(7) = C_6(0, 6, 1, 2, x_3, 8)$ , where  $x_3 \in \{9, 10\}$ . If  $x_3 = 9$ , then  $C_5(0, 1, 6, 9, 7) \subseteq \text{lk}(8)$ , which can not be true. If  $x_3 = 10$ , then  $\text{lk}(7) = C_6(0, 6, 1, 2, 10, 8)$ . This implies  $\text{lk}(8) = C_6(0, 1, 6, 9, 10, 7)$ ,  $\text{lk}(2) = C_8(3, \mathbf{4}, \mathbf{5}, \mathbf{0}, 1, 7, 10, 11)$ . Now, considering the same fact as above, we see  $f_{seq}(10) = (3^6)$ . This gives  $\text{lk}(10) = C_6(9, 8, 7, 2, 11, x_4)$ . But observe that  $x_4$  has no value in  $V(M)$  so that  $\text{lk}(10)$  can be completed. So  $(x, y) \neq (6, 7)$ .

If  $(x_1, x_2) = (6, 9)$ , then  $\text{lk}(1) = C_8(2, \mathbf{3}, \mathbf{4}, \mathbf{5}, 0, 8, 6, 9)$ ,  $\text{lk}(6) = C_6(0, 5, 8, 1, 9, 7)$ ,  $\text{lk}(5) = C_8(0, \mathbf{1}, \mathbf{2}, \mathbf{3}, 4, 10, 8, 6)$ ,  $\text{lk}(8) = C_6(0, 1, 6, 5, 10, 7)$ . This gives  $f_{seq}(7) = (3^6)$ ,

which implies  $\text{lk}(7) = C_6(9, 6, 0, 8, 10, x_3)$ , where  $x_3 \in \{3, 4, 11\}$ . If  $x_3 = a$ , for  $a \in \{3, 11\}$ , we see that the  $f_{seq}(9) = (3^6)$ , otherwise the vertices 2 and 3 lie in two distinct hexagonal faces which is not allowed. So  $\text{lk}(9) = C_6(2, 1, 6, 7, a, x_4)$ . Now observe that  $x_4$  has no value in  $V(M)$  so that  $\text{lk}(9)$  can be completed. If  $x_4 = 4$ , then  $\text{lk}(7) = C_6(9, 6, 0, 8, 10, 4)$  and  $\text{lk}(4) = C_8(5, \mathbf{0}, \mathbf{1}, \mathbf{2}, 3, 9, 7, 10)$ . This implies  $C_6(0, 1, 2, 9, 4, 5) \subseteq \text{lk}(3)$ . This can not be true as  $f_{seq}(3) = (3^4.6)$ . This proves the claim.

**Case 1.**  $(x_1, x_2) = (9, 6)$ , i.e.,  $\text{lk}(1) = C_8(2, \mathbf{3}, \mathbf{4}, \mathbf{5}, 0, 8, 9, 6)$ . Then  $\text{lk}(6) = C_6(0, 5, 9, 1, 2, 7)$  and  $\text{lk}(2) = C_8(3, \mathbf{4}, \mathbf{5}, \mathbf{0}, 1, 6, 7, x_3)$ , where  $x_3 \in \{8, 10\}$ . If  $x_3 = 8$ , then  $\text{lk}(2) = C_8(3, \mathbf{4}, \mathbf{5}, \mathbf{0}, 1, 6, 7, 8)$ ,  $\text{lk}(8) = C_6(0, 1, 9, 3, 2, 7)$ ,  $\text{lk}(9) = C_6(1, 6, 5, 10, 3, 8)$ ,  $\text{lk}(3) = C_8(4, \mathbf{5}, \mathbf{0}, \mathbf{1}, 2, 8, 9, 10)$ ,  $\text{lk}(5) = C_8(0, \mathbf{1}, \mathbf{2}, \mathbf{3}, 4, 10, 9, 6)$ . This implies  $C_4(3, 4, 5, 9) \subseteq \text{lk}(10)$ . So  $x_3 = 10$ . Then  $\text{lk}(2) = C_8(3, \mathbf{4}, \mathbf{5}, \mathbf{0}, 1, 6, 7, 10)$  and  $\text{lk}(7) = C_6(8, 0, 6, 2, 10, x_4)$ , where  $x_4 \in \{4, 11\}$ .

In case  $x_4 = 11$ , considering  $\text{lk}(8) = C_6(9, 1, 0, 7, 11, x_5)$ , we get  $x_5 = 4$ . But then considering  $\text{lk}(9)$  and  $\text{lk}(3)$  successively, we see that  $\deg(5) > 5$ . So,  $x_4 = 4$ . Then  $\text{lk}(8) = C_6(9, 1, 0, 7, 4, 3)$ , completing successively, we get  $\text{lk}(4) = C_8(5, \mathbf{0}, \mathbf{1}, \mathbf{2}, 3, 8, 7, 10)$ ,  $\text{lk}(5) = C_8(0, \mathbf{1}, \mathbf{2}, \mathbf{3}, 4, 10, 9, 6)$ ,  $\text{lk}(9) = C_6(1, 6, 5, 10, 3, 8)$ ,  $\text{lk}(3) = C_8(4, \mathbf{5}, \mathbf{0}, \mathbf{1}, 2, 10, 9, 8)$ , and  $\text{lk}(10) = C_6(2, 3, 9, 5, 4, 7)$ . Then  $M \cong \mathcal{A}_1(K)$  by the identity map.

**Case 2.**  $(x_1, x_2) = (9, 10)$ , i.e.,  $\text{lk}(1) = C_8(2, \mathbf{3}, \mathbf{4}, \mathbf{5}, 0, 8, 9, 10)$ . This implies  $\text{lk}(2) = C_8(3, \mathbf{4}, \mathbf{5}, \mathbf{0}, 1, 10, x_3, x_4)$ . It is easy to see that  $(x_3, x_4) \in \{(6, 7), (6, 11), (7, 6), (7, 8), (7, 9), (7, 11), (8, 7), (8, 9), (11, 6), (11, 7), (11, 9)\}$ . Here, the cases  $(6, 7) \cong (7, 6)$  and  $(6, 11) \cong (11, 7)$  by the maps  $(0, 2)(3, 5)(6, 7)(8, 10)$  and  $(0, 2)(3, 5)(6, 7, 11)(8, 10)$  respectively. So, we need not to discuss the cases  $(x_3, x_4) = (6, 7)$  or  $(11, 7)$ .

**Claim 2.**  $(x_3, x_4) = (6, 11), (7, 6)$  or  $(7, 8)$ .

If  $(x_3, x_4) = (7, 9)$ , then  $\text{lk}(9) = C_6(1, 8, 7, 2, 3, 10)$ ,  $\text{lk}(7) = C_6(0, 6, 10, 2, 9, 8)$ ,  $\text{lk}(10) = C_6(1, 2, 7, 6, 3, 9)$ , and  $\text{lk}(3) = C_8(4, \mathbf{5}, \mathbf{0}, \mathbf{1}, 2, 9, 10, 6)$ . Now considering  $\text{lk}(6)$ , we see that the set  $\{3, 4, 5\}$  forms a triangular face, which is not true in  $\text{lk}(0)$ .

If  $(x_3, x_4) = (7, 11)$ , then  $\text{lk}(2) = C_8(3, \mathbf{4}, \mathbf{5}, \mathbf{0}, 1, 10, 7, 11)$ . This implies  $\text{lk}(7) = C_6(0, 6, 10, 2, 11, 8)$  or  $\text{lk}(7) = C_6(0, 6, 11, 2, 10, 8)$ . In the first case,  $\text{lk}(10) = C_6(1, 2, 7, 6, 4, 9)$ ,  $\text{lk}(4) = C_8(5, \mathbf{0}, \mathbf{1}, \mathbf{2}, 3, 6, 10, 9)$ . This implies  $\text{lk}(6) = C_6(0, 5, 3, 4, 10, 7)$ , which contradicts the fact that  $\{5, 6\}$  forms a non-edge in  $\text{lk}(0)$ . So  $\text{lk}(7) = C_6(0, 6, 11, 2, 10, 8)$ . Then  $\text{lk}(10) = C_6(1, 2, 7, 8, 4, 9)$ ,  $\text{lk}(8) = C_6(0, 1, 9, 4, 10, 7)$ , this gives  $C_4(1, 8, 4, 10) \subseteq \text{lk}(9)$ .

If  $(x_3, x_4) = (8, 7)$ , then  $\text{lk}(2) = C_8(3, \mathbf{4}, \mathbf{5}, \mathbf{0}, 1, 10, 8, 7)$ . This implies  $\text{lk}(8) = C_6(0, 1, 9, 10, 2, 7)$ , and we get triangular face  $[8, 9, 10]$ , which is not true if we consider  $\text{lk}(1)$ .

If  $(x_3, x_4) = (8, 9)$ , then  $\text{lk}(2) = C_8(3, \mathbf{4}, \mathbf{5}, \mathbf{0}, 1, 10, 8, 9)$ . This implies  $\text{lk}(8) = C_6(0, 1, 9, 2, 10, 7)$ , and  $\text{lk}(10) = C_6(7, 8, 2, 1, 9, x_5)$ , where  $x_5 \in \{4, 11\}$ . If  $x_5 = 4$ , then  $\text{lk}(9) = C_6(1, 8, 2, 3, 4, 10)$  and we get  $C_6(0, 1, 2, 9, 4, 5) \subseteq \text{lk}(3)$ . If  $x_5 = 11$ , then successively, we get  $\text{lk}(9) = C_6(1, 8, 2, 3, 11, 10)$ ,  $\text{lk}(7) = C_6(0, 6, 4, 11, 10, 8)$ . Observe that  $\text{lk}(4) = C_8(5, \mathbf{0}, \mathbf{1}, \mathbf{2}, 3, 6, 7, 11)$ , this implies  $\text{lk}(3) = C_8(4, \mathbf{5}, \mathbf{0}, \mathbf{1}, 2, 9, 11, 6)$ , and we see  $\deg(11) > 6$ .

If  $(x_3, x_4) = (11, 6)$ , then  $\text{lk}(2) = C_8(3, \mathbf{4}, \mathbf{5}, \mathbf{0}, 1, 10, 11, 6)$ . This implies  $\text{lk}(6) = C_6(0, 5, 11, 2, 3, 7)$ ,  $\text{lk}(3) = C_8(4, \mathbf{5}, \mathbf{0}, \mathbf{1}, 2, 6, 7, 9)$ , and  $\text{lk}(7) = C_6(8, 0, 6, 3, 9, x_5)$ , where  $x_5 \in \{4, 10\}$ . If  $x_5 = 4$ , considering  $\text{lk}(4)$  and  $\text{lk}(8)$ , we see  $\deg(9) > 6$ . If  $x_5 = 10$ , then considering  $\text{lk}(10)$ ,  $\text{lk}(8)$  and  $\text{lk}(9)$  successively, we see  $\deg(3) > 5$ .

If  $(x_3, x_4) = (11, 9)$ , then  $\text{lk}(2) = C_8(3, \mathbf{4}, \mathbf{5}, \mathbf{0}, 1, 10, 11, 9)$ . This gives  $\text{lk}(9) = C_6(1, 8, 11, 2, 3, 10)$ , and  $\text{lk}(10) = C_6(3, 9, 1, 2, 11, x_5)$ . Here  $x_5 \in \{4, 6, 7\}$ . If  $x_5 = 4$ , then  $C_7(0, 1, 2, 9, 10, 4, 5) \subseteq \text{lk}(3)$ . If  $x_5 = a \in \{6, 7\}$ , then considering  $\text{lk}(6)$ , we see that  $\deg(a) > 6$ . This proves the claim.

**Subcase 2.1.** If  $(x_3, x_4) = (6, 11)$ , then  $\text{lk}(6) = C_6(0, 5, 10, 2, 11, 7)$  or  $\text{lk}(6) = C_6(0, 5, 11, 2, 10, 7)$ . In the first case,  $\text{lk}(10)$  can not be completed. So,  $\text{lk}(6) = C_6(0, 5, 11, 2, 10, 7)$ . Then  $\text{lk}(10) = C_6(1, 2, 6, 7, 4, 9)$  and  $\text{lk}(7) = C_6(0, 6, 10, 4, 3, 8)$ . Completing successively, we get  $\text{lk}(4) = C_8(5, \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, 7, 10, 9)$ ,  $\text{lk}(5) = C_8(0, \mathbf{1}, \mathbf{2}, \mathbf{3}, 4, 9, 11, 6)$ ,  $\text{lk}(3) = C_8(4, \mathbf{5}, \mathbf{0}, \mathbf{1}, 2, 11, 8, 7)$ ,  $\text{lk}(8) = C_6(0, 1, 9, 11, 3, 7)$ ,  $\text{lk}(9) = C_6(1, 8, 11, 5, 4, 10)$ . Then  $M \cong \mathcal{A}_2(T)$  by the identity map.

**Subcase 2.2.** If  $(x_3, x_4) = (7, 6)$ , then  $\text{lk}(2) = C_8(3, \mathbf{4}, \mathbf{5}, \mathbf{0}, 1, 10, 7, 6)$  and  $\text{lk}(6) = C_6(3, 2, 7, 0, 5, x_5)$ , where  $x_5 \in \{9, 11\}$ . If  $x_5 = 11$ , then  $\text{lk}(3) = C_8(4, \mathbf{5}, \mathbf{0}, \mathbf{1}, 2, 6, 11, 9)$  and we see that  $\text{lk}(7)$  can not be completed. So  $x_5 = 9$ . Then  $\text{lk}(6) = C_6(3, 2, 7, 0, 5, 9)$ , this implies,  $\text{lk}(9) = C_6(1, 8, 5, 6, 3, 10)$  or  $\text{lk}(9) = C_6(1, 8, 3, 6, 5, 10)$ . In the first case of  $\text{lk}(9)$ , completing successively we get  $\text{lk}(5) = C_8(0, \mathbf{1}, \mathbf{2}, \mathbf{3}, 4, 8, 9, 6)$ ,  $\text{lk}(8) = C_6(0, 1, 9, 5, 4, 7)$ ,  $\text{lk}(7) = C_6(0, 6, 2, 10, 4, 8)$ ,  $\text{lk}(4) = C_8(5, \mathbf{0}, \mathbf{1}, \mathbf{2}, 3, 10, 7, 8)$ ,  $\text{lk}(3) = C_8(4, \mathbf{5}, \mathbf{0}, \mathbf{1}, 2, 6, 9, 10)$ , and  $\text{lk}(10) = C_6(1, 2, 7, 4, 3, 9)$ . Then  $M \cong \mathcal{C}_1(K)$  by the map  $(1, 5)(2, 4)(6, 8)$ . Also, when  $\text{lk}(9) = C_6(1, 8, 3, 6, 5, 10)$ , completing successively we get  $\text{lk}(5) = C_8(0, \mathbf{1}, \mathbf{2}, \mathbf{3}, 4, 10, 9, 6)$ ,  $\text{lk}(10) = C_6(1, 2, 7, 4, 5, 9)$ ,  $\text{lk}(7) = C_6(0, 6, 2, 10, 4, 8)$ ,  $\text{lk}(4) = C_8(5, \mathbf{0}, \mathbf{1}, \mathbf{2}, 3, 8, 7, 10)$ ,  $\text{lk}(3) = C_8(4, \mathbf{5}, \mathbf{0}, \mathbf{1}, 2, 6, 9, 8)$ . Then  $M \cong \mathcal{A}_3(T)$  by the identity map.

**Subcase 2.3.** If  $(x_3, x_4) = (7, 8)$ , then  $\text{lk}(8) = C_6(0, 1, 9, 3, 2, 7)$  and  $\text{lk}(3) = C_8(4, \mathbf{5}, \mathbf{0}, \mathbf{1}, 2, 8, 9, x_5)$ . Observe that  $x_5 = 6$ . Now completing successively, we get  $\text{lk}(3) = C_8(4, \mathbf{5}, \mathbf{0}, \mathbf{1}, 2, 8, 9, 6)$ ,  $\text{lk}(6) = C_6(0, 5, 9, 3, 4, 7)$ ,  $\text{lk}(4) = C_8(5, \mathbf{0}, \mathbf{1}, \mathbf{2}, 3, 6, 7, 10)$ ,  $\text{lk}(10) = C_6(1, 2, 7, 4, 5, 9)$ ,  $\text{lk}(5) = C_8(0, \mathbf{1}, \mathbf{2}, \mathbf{3}, 4, 10, 9, 6)$ ,  $\text{lk}(9) = C_6(1, 8, 3, 6, 5, 10)$ , and  $\text{lk}(7) = C_6(0, 6, 4, 10, 2, 8)$ . Then  $M \cong \mathcal{A}_1(K)$  by the map  $(0, 5, 4, 3, 2, 1)(6, 10, 8)(7, 9)$ . Thus the proof.  $\square$

**Lemma 4.** *Let  $M$  be a 2-semi equivelar map of type  $[3^3.4^2 : 3.4.6.4]$  on  $\leq 12$  vertices. Then  $M$  is isomorphic to  $\mathcal{B}_1(K)$  or  $\mathcal{B}_2(T)$  given in example Section 3.*

*Proof.* If  $M$  is of the type  $[3^3.4^2 : 3.4.6.4]$ . Then, observe that: (i) if  $i$  is a vertex with the face-sequence  $(3^3.4^2)$  or  $(3^2.4.3.4)$ , then  $\deg(i) = 5$  or 4 respectively (ii) two distinct hexagonal faces do not share any vertex, and (iii) the non-empty intersection of a hexagonal face and a quadrangular face is an edge. Without loss of generality, let  $f_{seq}(0) = (3.4.6.4)$  and  $\text{lk}(0) = C_9(1, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{7}, \mathbf{8}, \mathbf{9})$ .

**Claim 3.** The vertices 6, 7, 8, 9 can not have the face-sequence  $(3.4.6.4)$ .

Note that the cases,  $f_{seq}(6) = (3.4.6.4)$  and  $f_{seq}(7) = (3.4.6.4)$  are similar to the cases  $f_{seq}(9) = (3.4.6.4)$  and  $f_{seq}(8) = (3.4.6.4)$  respectively. So, first assume that  $f_{seq}(6) = (3.4.6.4)$ . Then  $\text{lk}(6) = C_9(5, \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, y_4, \mathbf{y}_5, y_6, 7, \mathbf{0})$  or  $\text{lk}(6) = C_9(7, \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, y_4, \mathbf{y}_5, y_6, 5, \mathbf{0})$ , where  $y_1, y_2, y_3, y_4, y_5, y_6 \in V(M)$ . In the first case of  $\text{lk}(6)$ , we see that 5 appears in two distinct hexagonal faces  $[0, 1, 2, 3, 4, 5]$  and  $[5, y_1, y_2, y_3, y_4]$ , which is not allowed. While for the later case of  $\text{lk}(6)$ , we see that  $y_1, y_2, y_3, y_4 \neq 8$ , as if  $y_1 = 8$ , then it contradicts the fact that  $[0, 7, 8]$  is a triangular face and if any  $x_j = 8$ , for  $j \in \{2, 3, 4\}$ , then it contradicts the fact that 78 is an edge. Now, observe that we have four vertices  $y_1, y_2, y_3, y_4$  in  $\text{lk}(6)$  which have possible choices from the set  $\{9, 10, 11\}$ . This is not possible. So  $f_{seq}(6) \neq (3.4.6.4)$ . Similarly, we see that  $f_{seq}(7) \neq (3.4.6.4)$ . Thus the claim.

From  $\text{lk}(0)$ , we get  $\text{lk}(1) = C_9(2, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{0}, \mathbf{8}, 9, x_1, \mathbf{x}_2)$ , where  $x_1, x_2 \in V(M)$ . It is easy to see that  $(x_1, x_2) \in \{(6, 7), (6, 10), (7, 6), (7, 10), (10, 6), (10, 7), (10, 11)\}$ .

**Claim 4.**  $(x_1, x_2) = (10, 11)$ .

By Claim 1, we know if there exist two quadrangular faces at 6, 7, 8 or 9, then these faces share an edge. It follows that  $(x_1, x_2) \notin \{(6, 10), (7, 10), (10, 6), (10, 7)\}$ . If  $(x_1, x_2) = (7, 6)$ , then  $\text{lk}(7) = C_7(6, \mathbf{2}, 1, 9, 8, \mathbf{0}, \mathbf{5})$ . This implies  $C_4(0, 1, 9, 7) \subseteq \text{lk}(8)$ . If  $(x_1, x_2) = (6, 7)$ , then  $\text{lk}(6) = C_7(7, \mathbf{2}, 1, 9, 10, 5, \mathbf{0})$ ,  $\text{lk}(7) = C_7(6, \mathbf{1}, 2, 11, 8, \mathbf{0}, \mathbf{5})$ , and  $\text{lk}(2) = C_9(1, \mathbf{0}, \mathbf{5}, \mathbf{4}, 3, \mathbf{10}, 11, 7, \mathbf{6})$ . This implies  $\text{lk}(10) = C_7(3, \mathbf{2}, 11, 9, 6, 5, \mathbf{x}_3)$  or  $\text{lk}(10) = C_7(11, \mathbf{2}, 3, 5, 6, 9, \mathbf{x}_3)$ . In the first case of  $\text{lk}(10)$ , we see that the faces  $[0, 1, 2, 3, 4, 5]$  and  $[3, 10, 5, x_3]$  share two non-adjacent vertices  $\{3, 5\}$ , which contradicts (iii) given above. In the later case of  $\text{lk}(10)$ , we see that the set  $\{3, 5\}$  forms an edge and non-edge both. Thus the claim.

Let  $(x_1, x_2) = (10, 11)$ , i.e.,  $\text{lk}(1) = C_9(2, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{0}, \mathbf{8}, 9, 10, \mathbf{11})$ . Then  $\text{lk}(2) = C_9(3, \mathbf{4}, \mathbf{5}, \mathbf{0}, 1, \mathbf{10}, 11, x_3, \mathbf{x}_4)$ . Considering the above facts, we observe that  $(x_3, x_4) \in \{(6, 7), (7, 6), (8, 9), (9, 8)\}$ .

**Claim 5.**  $(x_3, x_4) = (7, 6)$  or  $(8, 9)$ .

If  $(x_3, x_4) = (6, 7)$ , then  $\text{lk}(6) = C_7(7, \mathbf{0}, 5, 10, 11, 2, \mathbf{3})$ . This implies  $\text{lk}(7) = C_7(6, \mathbf{2}, 3, x_5, 8, \mathbf{0}, \mathbf{5})$ . Now observe that  $x_5$  has no value in  $V(M)$ . Similarly if  $(x_3, x_4) = (9, 8)$ , then  $\text{lk}(9) = C_7(8, \mathbf{0}, 1, 6, 11, 2, \mathbf{3})$ . This implies  $\text{lk}(8) = C_7(9, \mathbf{1}, 0, 7, x_5, 3, \mathbf{2})$ . Now again we see that  $x_5$  has no value in  $V(M)$ . This proves the claim.

**Case 1.** If  $(x_3, x_4) = (8, 9)$ , then completing successively, we get  $\text{lk}(8) = C_7(9, \mathbf{1}, 0, 7, 11, 2, \mathbf{3})$ ,  $\text{lk}(9) = C_7(8, \mathbf{0}, 1, 10, 6, 3, \mathbf{2})$ ,  $\text{lk}(6) = C_7(7, \mathbf{0}, 5, 10, 9, 3, \mathbf{4})$ ,  $\text{lk}(7) = C_7(6, \mathbf{3}, 4, 11, 8, \mathbf{0}, \mathbf{5})$ ,  $\text{lk}(10) = C_7(11, \mathbf{2}, 1, 9, 6, 5, \mathbf{4})$ ,  $\text{lk}(11) = C_7(10, \mathbf{1}, 2, 8, 7, 4, \mathbf{5})$ ,  $\text{lk}(3) = C_9(2, \mathbf{1}, \mathbf{0}, \mathbf{5}, 4, \mathbf{7}, 6, 9, \mathbf{8})$ ,  $\text{lk}(4) = C_9(3, \mathbf{2}, \mathbf{1}, \mathbf{0}, 5, \mathbf{10}, 11, 7, \mathbf{6})$ . Then  $M \cong \mathcal{B}_1(K)$  by the identity map.



**Case 2.** If  $(x_3, x_4) = (7, 6)$ , then  $\text{lk}(7) = C_7(6, \mathbf{3}, 2, 11, 8, 0, \mathbf{5})$ . Observe that  $(x_5, x_6) \in \{(9, 10), (10, 9)\}$ . If  $(x_5, x_6) = (9, 10)$ , then completing successively, we get  $\text{lk}(6) = C_7(7, \mathbf{0}, 5, 9, 10, 3, \mathbf{2})$ ,  $\text{lk}(9) = C_7(8, \mathbf{0}, 1, 10, 6, 5, \mathbf{4})$ ,  $\text{lk}(8) = C_7(9, \mathbf{1}, 0, 7, 11, 4, \mathbf{5})$ ,  $\text{lk}(10) = C_7(11, \mathbf{2}, 1, 9, 6, 3, \mathbf{4})$ ,  $\text{lk}(11) = C_7(10, \mathbf{1}, 2, 7, 8, 4, \mathbf{3})$ ,  $\text{lk}(3) = C_9(2, \mathbf{1}, \mathbf{0}, \mathbf{5}, 4, 11, 10, 6, \mathbf{7})$ ,  $\text{lk}(4) = C_9(3, \mathbf{2}, \mathbf{1}, \mathbf{0}, 5, 9, 8, 11, \mathbf{10})$ . Then  $M \cong \mathcal{B}_1(K)$  by the map  $(0, 2, 4)(1, 3, 5)(6, 10)(7, 11)$ . If  $(x_5, x_6) = (10, 9)$ , then completing successively, we get  $\text{lk}(6) = C_7(7, \mathbf{0}, 5, 10, 9, 3, \mathbf{2})$ ,  $\text{lk}(9) = C_7(8, \mathbf{0}, 1, 10, 6, 3, \mathbf{4})$ ,  $\text{lk}(8) = C_7(9, \mathbf{1}, 0, 7, 11, 4, \mathbf{3})$ ,  $\text{lk}(10) = C_7(11, \mathbf{2}, 1, 9, 6, 5, \mathbf{4})$ ,  $\text{lk}(11) = C_7(10, \mathbf{1}, 2, 7, 8, 4, \mathbf{5})$ ,  $\text{lk}(3) = C_9(2, \mathbf{1}, \mathbf{0}, \mathbf{5}, 4, 8, 9, 6, \mathbf{7})$ ,  $\text{lk}(4) = C_9(3, \mathbf{2}, \mathbf{1}, \mathbf{0}, 5, 10, 11, 8, \mathbf{9})$ . Then  $M \cong \mathcal{B}_2(T)$  by the identity map. Thus the proof.  $\square$

**Lemma 5.** *Let  $M$  be a 2-semi equivelar map of the type  $[3^2.4.3.4 : 3.4.6.4]$  on  $\leq 12$  vertices. Then  $M$  is isomorphic to  $\mathcal{C}_1(K)$  or  $\mathcal{C}_2(T)$  given in example Section 3.*

*Proof.* Without loss of generality, let  $f_{seq}(0) = (3.4.6.4)$  and  $\text{lk}(0) = C_9(1, \mathbf{2}, \mathbf{3}, \mathbf{4}, 5, \mathbf{6}, 7, 8, \mathbf{9})$ . Since, the vertices 1, 2, 3, 4, and 5 lie on the hexagonal face  $[0, 1, 2, 3, 4, 5]$ ,  $f_{seq}(1) = f_{seq}(2) = f_{seq}(3) = f_{seq}(4) = f_{seq}(5) = (3.4.6.4)$ . Then  $\text{lk}(1) = C_9(2, \mathbf{3}, \mathbf{4}, \mathbf{5}, 0, \mathbf{8}, 9, x_1, \mathbf{x}_2)$ , where observe that  $x_1 \in \{6, 7, 10\}$ . If  $x_1 = 6$ , then  $x_2 \in \{7, 10\}$ , but for  $x_2 = 7$  we see that the edge 67 appears in two distinct quadrangular faces  $[0, 5, 6, 7]$  and  $[1, 2, 7, 6]$ , which is not possible. If  $x_1 = 7$ , then  $x_2 = 6$  and we see again that two distinct quadrangular faces share more than one vertices. If  $x_1 = 10$ , then  $x_2 \in \{6, 7, 11\}$ . This gives  $(x_1, x_2) \in \{(6, 10), (10, 6), (10, 7), (10, 11)\}$ . Since  $(6, 10) \cong (10, 7)$  by the map  $(0, 1)(2, 5)(3, 4)(6, 7, 10)(8, 9)$ , we need not consider the case  $(6, 10)$ .

**Claim 6.**  $(x_1, x_2) = (10, 6)$  or  $(10, 7)$ .

For  $(x_1, x_2) = (10, 11)$ , we get  $\text{lk}(1) = C_9(2, \mathbf{3}, \mathbf{4}, \mathbf{5}, 0, \mathbf{8}, 9, 10, \mathbf{11})$ . This implies  $\text{lk}(2) = C_9(3, \mathbf{4}, \mathbf{5}, \mathbf{0}, 1, \mathbf{10}, 11, x_3, \mathbf{x}_4)$ , where  $(x_3, x_4) \in \{(6, 8), (6, 9), (7, 8), (8, 7)\}$ . If  $(x_3, x_4) = (6, 8)$ , then  $\text{lk}(6) = C_7(2, 11, 7, \mathbf{0}, 5, 8, \mathbf{3})$ , which implies  $\text{deg}(8) > 5$ . If  $(x_3, x_4) = (6, 9)$ , then  $\text{lk}(6) = C_7(2, 11, 7, \mathbf{0}, 5, 9, \mathbf{3})$  or  $\text{lk}(6) = C_7(2, 11, 5, \mathbf{0}, 7, 9, \mathbf{3})$ . But for both the cases, we see  $\text{deg}(9) > 5$ . If  $(x_3, x_4) = (8, 7)$ , then  $\text{lk}(7) = C_7(3, x_5, 6, \mathbf{5}, 0, 8, \mathbf{2})$ . Now observe that  $x_5$  has no value in  $V(M)$ . If  $(x_3, x_4) = (7, 8)$ , then  $\text{lk}(7) = C_7(2, 11, 6, \mathbf{5}, 0, 8, \mathbf{3})$  and  $\text{lk}(3) = C_9(4, \mathbf{5}, \mathbf{0}, \mathbf{1}, 2, \mathbf{7}, 8, x_5, \mathbf{x}_6)$ , where  $(x_5, x_6) \in \{(6, 11), (10, 9)\}$ . In case  $(x_5, x_6) = (6, 11)$  (or  $(10, 9)$ ), considering  $\text{lk}(8)$ , we see  $\text{deg}(6) > 5$  (resp.  $C_4(3, 4, 9, 8) \subseteq \text{lk}(10)$ ). This proves the claim.

**Case 1.**  $(x_1, x_2) = (10, 6)$ , i.e.,  $\text{lk}(1) = C_9(2, \mathbf{3}, \mathbf{4}, \mathbf{5}, 0, \mathbf{8}, 9, 10, \mathbf{6})$  then  $\text{lk}(2) = C_9(3, \mathbf{4}, \mathbf{5}, \mathbf{0}, 1, \mathbf{10}, 6, x_3, \mathbf{x}_4)$ , where  $(x_3, x_4) \in \{(7, 9), (8, 7), (11, 7), (11, 8), (11, 9)\}$ . If  $(x_3, x_4) = (7, 9)$  then considering  $\text{lk}(7)$  we get  $C_4(0, 1, 9, 7) \subseteq \text{lk}(8)$ . If  $(x_3, x_4) = (8, 7)$ , then considering  $\text{lk}(6)$ , we get three quadrangular faces  $[6, 5, 0, 7]$ ,  $[6, 2, 1, 10]$  and  $[6, 9, x_5, x_6]$  at 6, which is not allowed. If  $(x_3, x_4) = (11, 7)$  or  $(11, 9)$  then  $\text{lk}(6) = C_7(5, 11, 2, \mathbf{1}, 10, 7, \mathbf{0})$  or  $\text{lk}(6) = C_7(7, 11, 2, \mathbf{1}, 10, 5, \mathbf{0})$ . In the first case  $\text{deg}(7) > 5$

and in the second case we get triangular face  $[6, 7, 11]$  which is not possible, see  $\text{lk}(2)$ . Now we search for the remaining cases.

If  $(x_3, x_4) = (11, 8)$ , then  $\text{lk}(2) = C_9(3, \mathbf{4}, \mathbf{5}, \mathbf{0}, \mathbf{1}, \mathbf{10}, 6, 11, \mathbf{8})$  and  $\text{lk}(3) = C_9(4, \mathbf{5}, \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{11}, 8, x_5, \mathbf{x}_6)$ . Here,  $(x_5, x_6) \in \{(7, 9), (7, 10), (9, 7)\}$ . In case  $(x_5, x_6) = (7, 9)$ , considering  $\text{lk}(7)$ , we get  $\text{deg}(6) > 5$ . In case  $(x_5, x_6) = (9, 7)$ , considering  $\text{lk}(9)$ , we get  $\text{deg}(7) > 5$ . If  $(x_5, x_6) = (7, 10)$ , then  $\text{lk}(7) = C_7(0, 8, 3, \mathbf{4}, 10, 6, \mathbf{5})$ ,  $\text{lk}(6) = C_7(2, 11, 5, \mathbf{0}, 7, 10, \mathbf{1})$ . Now completing successively, we get  $\text{lk}(5) = C_9(0, \mathbf{1}, \mathbf{2}, \mathbf{3}, 4, \mathbf{9}, 11, 6, \mathbf{7})$ ,  $\text{lk}(11) = C_7(2, 6, 5, \mathbf{4}, 9, 8, \mathbf{3})$ ,  $\text{lk}(8) = C_7(0, 7, 3, \mathbf{2}, 11, 9, \mathbf{1})$ ,  $\text{lk}(4) = C_9(5, \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{7}, 10, 9, \mathbf{11})$ ,  $\text{lk}(9) = C_7(1, 10, 4, \mathbf{5}, 11, 8, \mathbf{0})$ , and  $\text{lk}(10) = C_7(1, 9, 4, \mathbf{3}, 7, 6, \mathbf{2})$ . Then  $M \cong C_2(T)$  by the identity map.

If  $(x_3, x_4) = (11, 9)$ , then  $\text{lk}(2) = C_9(3, \mathbf{4}, \mathbf{5}, \mathbf{0}, \mathbf{1}, \mathbf{10}, 6, 11, \mathbf{9})$  and  $\text{lk}(3) = C_9(4, \mathbf{5}, \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{11}, 9, x_5, \mathbf{x}_6)$ . Here,  $(x_5, x_6) \in \{(8, 10), (10, 7), (10, 8)\}$ . If  $(x_5, x_6) = (8, 10)$ , we get  $\text{lk}(8) = C_7(0, 7, 10, 4, \mathbf{3}, 9, \mathbf{1})$ , which implies  $\text{lk}(4)$  can not be completed. If  $(x_5, x_6) = (10, 8)$ , then considering  $\text{lk}(9)$  and  $\text{lk}(10)$ , we get  $\text{deg}(6) > 5$ . If  $(x_5, x_6) = (10, 7)$ , then  $\text{lk}(4) = C_9(5, \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{10}, 7, 8, \mathbf{11})$ ,  $\text{lk}(8) = C_7(0, 7, 4, \mathbf{5}, 11, 9, \mathbf{1})$ ,  $\text{lk}(7) = C_7(0, 8, 4, \mathbf{3}, 10, 6, \mathbf{5})$ ,  $\text{lk}(6) = C_7(2, 11, 5, \mathbf{0}, 7, 10, \mathbf{1})$ . Now completing successively, we get  $\text{lk}(11) = C_7(2, 6, 5, \mathbf{4}, 8, 9, \mathbf{3})$ ,  $\text{lk}(5) = C_9(0, \mathbf{1}, \mathbf{2}, \mathbf{3}, 4, \mathbf{8}, 11, 6, \mathbf{7})$ ,  $\text{lk}(9) = C_7(1, 10, 3, \mathbf{2}, 11, 8, \mathbf{0})$ . Then  $M \cong C_1(K)$  by the identity map.

**Case 2.**  $(x_1, x_2) = (10, 7)$ , i.e.,  $\text{lk}(1) = C_9(2, \mathbf{3}, \mathbf{4}, \mathbf{5}, 0, \mathbf{8}, 9, 10, \mathbf{7})$ . Then  $\text{lk}(7) = C_7(0, 8, 10, \mathbf{1}, 2, 6, \mathbf{5})$  or  $\text{lk}(7) = C_7(0, 8, 2, \mathbf{1}, 10, 6, \mathbf{5})$ . In the first case of  $\text{lk}(7)$ , we get  $\text{lk}(2) = C_9(3, \mathbf{4}, \mathbf{5}, \mathbf{0}, \mathbf{1}, \mathbf{10}, 7, 6, \mathbf{x}_3)$ , where  $x_3 \in \{9, 11\}$ . But for both the cases of  $x_3$ ,  $\text{lk}(6)$  can not be completed. While for  $\text{lk}(7) = C_7(0, 8, 2, \mathbf{1}, 10, 6, \mathbf{5})$ , we get  $\text{lk}(2) = C_9(3, \mathbf{4}, \mathbf{5}, \mathbf{0}, \mathbf{1}, \mathbf{10}, 7, 8, \mathbf{x}_3)$ , where  $x_3 \in \{6, 11\}$ . If  $x_3 = 6$ , considering  $\text{lk}(2)$  and  $\text{lk}(8)$ , we see  $\text{deg}(6) > 5$ . So, let  $x_3 = 11$ . Then  $\text{lk}(2) = C_9(3, \mathbf{4}, \mathbf{5}, \mathbf{0}, \mathbf{1}, \mathbf{10}, 7, 8, \mathbf{11})$ ,  $\text{lk}(8) = C_7(0, 7, 2, \mathbf{3}, 11, 9, \mathbf{1})$ . Note that  $\text{lk}(3) = C_9(4, \mathbf{5}, \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{8}, 11, 6, \mathbf{10})$ . Now completing successively, we get  $\text{lk}(6) = C_7(3, 11, 5, \mathbf{0}, 7, 10, \mathbf{4})$ ,  $\text{lk}(10) = C_7(1, 9, 4, \mathbf{3}, 6, 7, \mathbf{2})$ ,  $\text{lk}(9) = C_7(1, 10, 4, \mathbf{5}, 11, 8, \mathbf{0})$ ,  $\text{lk}(4) = C_9(5, \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{6}, 10, 9, \mathbf{11})$ ,  $\text{lk}(5) = C_9(0, \mathbf{1}, \mathbf{2}, \mathbf{3}, 4, \mathbf{9}, 11, 6, \mathbf{7})$ . Then  $M \cong C_1$  by the map  $(1, 5)(2, 4)(6, 9)(7, 8)(10, 11)$ . This proves the lemma.  $\square$

**Theorem A.** Let  $M$  be a 2-semi equivelar map of type  $[3^6 : 3^2.4.3.4]$  on  $\leq 12$ . Then  $M$  is isomorphic to  $\mathcal{D}_1(K)$  given in example Section 3.

*Proof.* Without loss of generality, let 0 be a critical vertex in  $M$  with the face-sequence  $(3^6)$  and  $\text{lk}(0) = C_6(1, 2, 3, 4, 5, 6)$ . Since, each vertex in the  $\text{lk}(0)$  has two triangular faces, without loss of generality, let  $f_{seq}(1) = (3^2.4.3.4)$ . This gives  $\text{lk}(1) = C_7(2, 0, 6, \mathbf{x}_4, x_3, x_2, \mathbf{x}_1)$ . It is easy to see that  $(x_1, x_2, x_3, x_4) \in \{(5, 4, 7, 8), (8, 7, 4, 3), (9, 7, 8, 10)\}$ .

**Case 1.**  $(x_1, x_2, x_3, x_4) = (5, 4, 7, 8)$ , i.e.,  $\text{lk}(1) = C_7(2, 0, 6, \mathbf{8}, 7, 4, \mathbf{5})$ . This implies  $\text{lk}(4) = C_7(3, 0, 5, \mathbf{2}, 1, 7, \mathbf{9})$  and  $\text{lk}(3) = C_7(2, 0, 4, \mathbf{7}, 9, x_5, \mathbf{x}_6)$ , where  $(x_5, x_6) \in \{(10, 8), (10, 11), (8, 10)\}$ . If  $(x_5, x_6) = (10, 8)$ , then  $\text{lk}(2) = C_7(1, 0, 3, \mathbf{10}, 8, 5, \mathbf{4})$ . Now considering  $\text{lk}(5)$  and  $\text{lk}(1)$ , we see that the set  $\{6, 8\}$  forms both an edge and a non-edge. If  $(x_5, x_6) = (10, 11)$ , then  $\text{lk}(2) = C_7(1, 0, 3, \mathbf{10}, 11, 5, \mathbf{4})$ ,  $\text{lk}(5) =$

$C_7(4, 0, 6, \mathbf{9}, 11, 2, \mathbf{1})$ ,  $\text{lk}(6) = C_7(1, 0, 5, \mathbf{11}, 9, 8, \mathbf{7})$ , and  $\text{lk}(7) = C_7(8, x_7, 9, \mathbf{3}, 4, \mathbf{1}, \mathbf{6})$ . Now observe that  $x$  has no value in  $V(M)$  so that  $\text{lk}(7)$  can be completed. If  $(x_5, x_6) = (8, 10)$ , then  $\text{lk}(2) = C_7(1, 0, 3, \mathbf{8}, 10, 5, \mathbf{4})$ ,  $\text{lk}(5) = C_7(6, 0, 4, \mathbf{1}, 2, 10, \mathbf{x}_7)$ . It is easy to see that  $x_7 = 9$ , completing successively, we get  $\text{lk}(5) = C_7(4, 0, 6, \mathbf{9}, 10, 2, \mathbf{1})$ ,  $\text{lk}(6) = C_7(1, 0, 5, \mathbf{10}, 9, 8, \mathbf{7})$ ,  $\text{lk}(9) = C_7(3, 8, 6, \mathbf{5}, 10, 7, \mathbf{4})$ ,  $\text{lk}(7) = C_7(8, 10, 9, \mathbf{3}, 4, \mathbf{1}, \mathbf{6})$ ,  $\text{lk}(10) = C_7(8, 7, 9, \mathbf{6}, 5, 2, \mathbf{3})$  and  $\text{lk}(8) = C_7(3, 9, 6, \mathbf{1}, 7, 10, \mathbf{2})$ . Then  $M \cong \mathcal{D}_1(K)$  by the identity map.

**Case 2.**  $(x_1, x_2, x_3, x_4) = (8, 7, 4, 3)$ , i.e.,  $\text{lk}(1) = C_7(2, 0, 6, \mathbf{3}, 4, 7, \mathbf{8})$ . Then  $\text{lk}(4) = C_7(3, 0, 5, \mathbf{9}, 7, 1, \mathbf{6})$  and  $\text{lk}(7) = C_7(8, 10, 9, \mathbf{5}, 4, 1, \mathbf{2})$ . This implies  $\text{lk}(2) = C_7(3, 0, 1, \mathbf{7}, 8, x_5, \mathbf{x}_6)$ , where,  $(x_5, x_6) \in \{(11, 9), (9, 10)\}$ . If  $(x_5, x_6) = (11, 9)$ , then  $\text{lk}(3) = C_7(2, 0, 4, \mathbf{1}, 6, 9, \mathbf{11})$ ,  $\text{lk}(6) = C_7(1, 0, 5, \mathbf{10}, 9, 3, \mathbf{4})$ , and  $\text{lk}(9) = C_7(10, x_7, 11, \mathbf{2}, 3, 6, \mathbf{5})$ , but observe that  $x_7$  has no value in  $V(M)$ . On the other hand, if  $(x_5, x_6) = (9, 10)$ , completing successively, we get  $\text{lk}(2) = C_7(1, 0, 3, \mathbf{10}, 9, 8, \mathbf{7})$ ,  $\text{lk}(3) = C_7(2, 0, 4, \mathbf{1}, 6, 10, \mathbf{9})$ ,  $\text{lk}(10) = C_7(8, 7, 9, \mathbf{2}, 3, 6, \mathbf{5})$ ,  $\text{lk}(5) = C_7(4, 0, 6, \mathbf{10}, 8, 9, \mathbf{7})$ ,  $\text{lk}(8) = C_7(2, 9, 5, \mathbf{6}, 10, 7, \mathbf{1})$ ,  $\text{lk}(9) = C_7(2, 8, 5, \mathbf{4}, 7, 10, \mathbf{3})$ . Then  $M \cong \mathcal{D}_1(K)$  by the map  $(2, 6)(3, 5)$ .

**Case 3.**  $(x_1, x_2, x_3, x_4) = (9, 7, 8, 10)$ , i.e.,  $\text{lk}(1) = C_7(2, 0, 6, \mathbf{10}, 8, 7, \mathbf{9})$ . This implies  $\text{lk}(2) = C_7(3, 0, 1, \mathbf{7}, 9, x_5, \mathbf{x}_6)$ , where  $(x_5, x_6) \in \{(10, 11), (11, 8), (11, 10), (5, 6)\}$ . If  $(x_5, x_6) = (10, 11)$ , then  $\text{lk}(10) = C_7(2, 9, 8, \mathbf{1}, 6, 11, \mathbf{3})$  or  $\text{lk}(10) = C_7(2, 9, 6, \mathbf{1}, 8, 11, \mathbf{3})$ . In the first case, we get  $\text{lk}(6) = C_7(1, 0, 5, \mathbf{7}, 11, 10, \mathbf{8})$  and  $\text{lk}(11) = C_7(3, x_7, 7, \mathbf{5}, 6, 10, \mathbf{2})$ , but observe that  $x_7$  has no value in  $V(M)$ . On the other hand when,  $\text{lk}(10) = C_7(2, 9, 6, \mathbf{1}, 8, 11, \mathbf{3})$ , then considering  $\text{lk}(10)$ ,  $\text{lk}(6)$ , and  $\text{lk}(10)$  successively, we see that  $\text{lk}(11)$  can not be completed. If  $(x_5, x_6) = (11, 8)$ , then  $\text{lk}(8) = C_7(1, 7, 3, \mathbf{2}, 11, 10, \mathbf{6})$  or  $\text{lk}(8) = C_7(1, 7, 11, \mathbf{2}, 3, 10, \mathbf{6})$ . Now, as in previous case, we see that link of all vertices can not be completed. If  $(x_5, x_6) = (11, 10)$ , then  $\text{lk}(2) = C_7(1, 0, 3, \mathbf{10}, 11, 9, \mathbf{7})$ . This implies  $\text{lk}(3) = C_7(4, 0, 2, \mathbf{11}, 10, x_7, \mathbf{x}_8)$ , where  $(x_7, x_8) \in \{(7, 8), (8, 7)\}$ . But for both the cases,  $\text{lk}(7)$  can not be completed. If  $(x_5, x_6) = (5, 6)$ , then  $\text{lk}(2) = C_7(1, 0, 3, \mathbf{6}, 5, 9, \mathbf{7})$ , and  $\text{lk}(5) = C_7(4, 0, 6, \mathbf{3}, 2, 9, \mathbf{x}_7)$ , where  $x_7 \in \{8, 10\}$ . If  $x_7 = 10$ , then considering  $\text{lk}(5)$  and  $\text{lk}(6)$  we see that  $\text{lk}(3)$  can not be completed. So,  $x_7 = 8$ . Completing successively, we get  $\text{lk}(5) = C_7(4, 0, 6, \mathbf{3}, 2, 9, \mathbf{8})$ ,  $\text{lk}(6) = C_7(1, 0, 5, \mathbf{2}, 3, 10, \mathbf{8})$ ,  $\text{lk}(3) = C_7(2, 0, 4, \mathbf{7}, 10, 5, \mathbf{6})$ ,  $\text{lk}(4) = C_7(3, 0, 5, \mathbf{9}, 8, 7, \mathbf{10})$ ,  $\text{lk}(7) = C_7(1, 8, 4, \mathbf{3}, 10, 9, \mathbf{2})$ ,  $\text{lk}(8) = C_7(1, 7, 4, \mathbf{5}, 9, 10, \mathbf{6})$ ,  $\text{lk}(9) = C_7(7, 10, 8, \mathbf{4}, 5, 2, \mathbf{1})$ . Then  $M \cong \mathcal{D}_1(K)$  by the map  $(1, 6, 5, 4, 3, 2)(7, 8, 9)$ . This proves the lemma.  $\square$

**Lemma 6.** *Let  $M$  be a 2-semi equivelar map of type  $[3^6 : 3^3.4^2]$  on  $\leq 12$  vertices. Then  $M$  is isomorphic to one of  $\mathcal{E}_1(K)$ ,  $\mathcal{E}_2(K)$ ,  $\mathcal{E}_3(T)$ ,  $\mathcal{E}_4(T)$ ,  $\mathcal{E}_5(K)$ ,  $\mathcal{E}_6(T)$ ,  $\mathcal{E}_7(K)$ ,  $\mathcal{E}_8(T)$ ,  $\mathcal{E}_9(K)$ ,  $\mathcal{E}_{10}(K)$ ,  $\mathcal{E}_{11}(T)$ ,  $\mathcal{E}_{12}(T)$  or  $\mathcal{E}_{13}(T)$  given in example Section 3.*

*Proof.* Without loss, let  $f_{seq}(0) = (3^3.4^2)$  and  $\text{lk}(0) = C_7(1, \mathbf{2}, 3, 4, 5, 6, \mathbf{7})$ . Then the vertices 1, 2, 3, 6 and 7 have the face-sequence  $(3^3.4^2)$ . Therefore,  $\text{lk}(1) = C_7(0, \mathbf{6}, 7, x_1, x_2, 2, \mathbf{3})$ . Now observe that  $(x_1, x_2) \in$

$\{(4, 5), (4, 8), (5, 4), (5, 8), (8, 4), (8, 5), (8, 9)\}$ . Here, the case  $(4, 8) \cong (8, 5)$  by the map  $(0, 1)(2, 3)(4, 5, 8)(6, 7)$ . So, we need not discuss the case  $(x_1, x_2) = (8, 5)$ .

**Case 1.**  $(x_1, x_2) = (4, 5)$ , i.e.,  $\text{lk}(1) = C_7(0, \mathbf{6}, 7, 4, 5, 2, \mathbf{3})$ . This implies  $\text{lk}(4) = C_6(0, 3, 8, 7, 1, 5)$  and  $\text{lk}(3) = C_7(2, \mathbf{1}, 0, 4, 8, x_3, \mathbf{x}_4)$ , where  $(x_3, x_4) \in \{(6, 7), (9, 10)\}$ . In case  $(x_3, x_4) = (6, 7)$ , completing successively, we get  $\text{lk}(3) = C_7(2, \mathbf{1}, 0, 4, 8, 6, \mathbf{7})$ ,  $\text{lk}(7) = C_7(\mathbf{6}, \mathbf{0}, 1, 4, 8, 2, \mathbf{3})$ ,  $\text{lk}(2) = C_7(3, \mathbf{0}, 1, 5, 8, 7, \mathbf{6})$ ,  $\text{lk}(5) = C_6(0, 4, 1, 2, 8, 6)$ ,  $\text{lk}(6) = C_7(7, \mathbf{1}, 0, 5, 8, 3, \mathbf{2})$ ,  $\text{lk}(8) = C_6(2, 5, 6, 3, 4, 7)$ . Then  $M \cong \mathcal{E}_1(K)$  by the identity map. If  $(x_3, x_4) = (9, 10)$ , then  $\text{lk}(2) = C_7(3, \mathbf{0}, 1, 5, x_3, 10, \mathbf{9})$ . Observe that  $x_3 = 11$ . Now completing successively, we get  $\text{lk}(5) = C_6(0, 4, 1, 2, 11, 6)$ ,  $\text{lk}(6) = C_7(7, \mathbf{1}, 0, 5, 11, 9, \mathbf{10})$ ,  $\text{lk}(9) = C_7(10, \mathbf{2}, 3, 8, 11, 6, \mathbf{7})$ ,  $\text{lk}(8) = C_6(3, 4, 7, 10, 11, 9)$ ,  $\text{lk}(10) = C_7(9, \mathbf{3}, 2, 11, 8, 7, \mathbf{6})$ ,  $\text{lk}(7) = C_7(6, \mathbf{0}, 1, 4, 8, 10, \mathbf{9})$ . Then  $M \cong \mathcal{E}_2(K)$  by the identity map.

**Case 2.**  $(x_1, x_2) = (4, 8)$ . Then  $\text{lk}(4) = C_6(0, 3, 7, 1, 8, 5)$  or  $\text{lk}(4) = C_6(0, 3, 8, 1, 7, 5)$ . If  $\text{lk}(4) = C_6(0, 3, 7, 1, 8, 5)$ , then  $\text{lk}(3) = C_7(2, \mathbf{1}, 0, 4, 7, 9, \mathbf{10})$ ,  $\text{lk}(2) = C_7(3, \mathbf{0}, 1, 8, x_3, 10, \mathbf{9})$ . Observe that the possible value of  $x_3 = 5$ , but then considering  $\text{lk}(5)$ , we get  $C_4(1, 2, 5, 4)$ . So  $\text{lk}(4) = C_6(0, 3, 8, 1, 7, 5)$ . Then  $\text{lk}(3) = C_7(2, \mathbf{1}, 0, 4, 8, x_3, \mathbf{x}_4)$ , where  $(x_3, x_4) \in \{(6, 7), (9, 10)\}$ . If  $(x_3, x_4) = (6, 7)$ , completing successively, we get  $\text{lk}(3) = C_7(2, \mathbf{1}, 0, 4, 8, 6, \mathbf{7})$ ,  $\text{lk}(6) = C_7(7, \mathbf{1}, 0, 5, 8, 3, \mathbf{2})$ ,  $\text{lk}(7) = C_7(\mathbf{6}, \mathbf{0}, 1, 4, 5, 2, \mathbf{3})$ ,  $\text{lk}(2) = C_7(3, \mathbf{0}, 1, 8, 5, 7, \mathbf{6})$ ,  $\text{lk}(5) = C_6(0, 4, 7, 2, 8, 6)$ , and  $\text{lk}(8) = C_6(1, 2, 5, 6, 3, 4)$ . Then  $M \cong \mathcal{E}_3(T)$  by the identity map. If  $(x_3, x_4) = (9, 10)$ , then  $\text{lk}(2) = C_7(3, \mathbf{0}, 1, 8, 11, 10, \mathbf{9})$ , completing successively, we get  $\text{lk}(8) = C_6(1, 2, 11, 9, 3, 4)$ ,  $\text{lk}(9) = C_7(10, \mathbf{2}, 3, 8, 11, 6, \mathbf{7})$ ,  $\text{lk}(6) = C_7(7, \mathbf{1}, 0, 5, 11, 9, \mathbf{10})$ ,  $\text{lk}(7) = C_7(\mathbf{6}, \mathbf{0}, 1, 4, 5, 10, \mathbf{9})$ ,  $\text{lk}(5) = C_6(0, 4, 7, 10, 11, 6)$ ,  $\text{lk}(10) = C_7(9, \mathbf{3}, 2, 11, 5, 7, \mathbf{6})$ . Then  $M \cong \mathcal{E}_4(T)$  by the identity map.

**Case 3.**  $(x_1, x_2) = (5, 4)$ . Then  $\text{lk}(5) = C_6(0, 4, 1, 7, 8, 6)$ . This implies  $\text{lk}(4) = C_6(0, 3, 8, 2, 1, 5)$  or  $\text{lk}(4) = C_6(0, 3, 9, 2, 1, 5)$ . In the first case, when  $\text{lk}(4) = C_6(0, 3, 8, 2, 1, 5)$ , then  $\text{lk}(8) = C_6(2, 4, 3, 7, 5, 6)$  or  $\text{lk}(8) = C_6(2, 4, 3, 6, 5, 7)$ . In case  $\text{lk}(8) = C_6(2, 4, 3, 7, 5, 6)$ , completing successively, we get  $\text{lk}(7) = C_7(\mathbf{6}, \mathbf{0}, 1, 5, 8, 3, \mathbf{2})$ ,  $\text{lk}(6) = C_7(7, \mathbf{1}, 0, 5, 8, 2, \mathbf{3})$ ,  $\text{lk}(2) = C_7(3, \mathbf{0}, 1, 4, 8, 6, \mathbf{7})$ , and  $\text{lk}(3) = C_7(2, \mathbf{1}, 0, 4, 8, 7, \mathbf{6})$ . Then  $M \cong \mathcal{E}_5(K)$  by the identity map. While, for  $\text{lk}(8) = C_6(2, 4, 3, 6, 5, 7)$ , completing successively, we get  $\text{lk}(7) = C_7(\mathbf{6}, \mathbf{0}, 1, 5, 8, 2, \mathbf{3})$ ,  $\text{lk}(6) = C_7(7, \mathbf{1}, 0, 5, 8, 3, \mathbf{2})$ ,  $\text{lk}(2) = C_7(3, \mathbf{0}, 1, 4, 8, 7, \mathbf{6})$ . Then  $M \cong \mathcal{E}_6(T)$  by the identity map.

On the other hand, when  $\text{lk}(4) = C_6(0, 3, 9, 2, 1, 5)$ , then  $\text{lk}(2) = C_7(3, \mathbf{0}, 1, 4, 9, x_3, \mathbf{x}_4)$ , where we see easily that  $(x_3, x_4) = (10, 11)$ . Then  $\text{lk}(3) = C_7(2, \mathbf{1}, 0, 4, 9, 11, \mathbf{10})$ ,  $\text{lk}(9) = C_6(2, 4, 3, 11, 8, 10)$ , and  $\text{lk}(10) = C_7(11, \mathbf{3}, 2, 9, 8, x_5, \mathbf{x}_6)$ , where  $(x_5, x_6) \in \{(6, 7), (7, 6)\}$ . If  $(x_5, x_6) = (6, 7)$ , completing successively, we get  $\text{lk}(6) = C_7(7, \mathbf{1}, 0, 5, 8, 10, \mathbf{11})$ ,  $\text{lk}(7) = C_7(\mathbf{6}, \mathbf{0}, 1, 5, 8, 11, \mathbf{10})$ ,  $\text{lk}(8) = C_6(5, 6, 10, 9, 11, 7)$ . Then  $M \cong \mathcal{E}_7(K)$  by the identity map. Also, if  $(x_5, x_6) = (7, 6)$ , completing successively, we get  $\text{lk}(7) = C_7(\mathbf{6}, \mathbf{0}, 1, 5, 8, 10, \mathbf{11})$ ,  $\text{lk}(6) = C_7(7, \mathbf{1}, 0, 5, 8, 11, \mathbf{10})$ ,  $\text{lk}(8) = C_6(5, 6, 11, 9, 10, 7)$ . Then  $M \cong \mathcal{E}_8(T)$  by the identity map.

**Case 4.**  $(x_1, x_2) = (5, 8)$ . Then  $\text{lk}(5) = C_6(0, 4, 7, 1, 8, 6)$ . This implies  $\text{lk}(6) =$

$C_7(7, \mathbf{1}, 0, 5, 8, x_3, \mathbf{x}_4)$ , where  $(x_3, x_4) \in \{(2, 3), (3, 2), (9, 10)\}$ . If  $(x_3, x_4) = (2, 3)$ , then  $\text{lk}(7) = C_7(6, \mathbf{0}, 1, 5, 4, \mathbf{3}, \mathbf{2})$  and we get  $C_4(0, 3, 7, 5) \subseteq \text{lk}(4)$ . If  $(x_3, x_4) = (3, 2)$ , then  $\text{lk}(7) = C_7(6, \mathbf{0}, 1, 5, 4, \mathbf{2}, \mathbf{3})$  and  $\text{lk}(4) = C_6(0, 3, 8, 2, 7, 5)$ . Completing successively, we get  $\text{lk}(8) = C_6(1, 2, 4, 3, 6, 5)$ ,  $\text{lk}(6) = C_7(7, \mathbf{1}, 0, 5, 8, \mathbf{3}, \mathbf{2})$ ,  $\text{lk}(7) = C_7(6, \mathbf{0}, 1, 5, 4, \mathbf{2}, \mathbf{3})$ . Then  $M \cong \mathcal{E}_1(K)$  by the map  $(0, 3)(1, 2)(5, 8)$ .

If  $(x_3, x_4) = (9, 10)$ , then  $\text{lk}(7) = C_7(6, \mathbf{0}, 1, 5, 4, 10, \mathbf{9})$  and  $\text{lk}(4) = C_6(0, 3, 11, 10, 7, 5)$ . Now completing successively, we get  $\text{lk}(10) = C_7(9, \mathbf{3}, 2, 11, 4, 7, \mathbf{6})$ ,  $\text{lk}(3) = C_7(2, \mathbf{1}, 0, 4, 11, 9, \mathbf{10})$ ,  $\text{lk}(9) = C_7(10, \mathbf{2}, 3, 11, 8, 6, \mathbf{7})$ ,  $\text{lk}(2) = C_7(3, \mathbf{0}, 1, 8, 11, 10, \mathbf{9})$ ,  $\text{lk}(8) = C_6(1, 2, 11, 9, 6, 5)$ ,  $\text{lk}(11) = C_6(2, 8, 9, 3, 4, 10)$ . Then  $M \cong \mathcal{E}_9(K)$  by the identity map.

**Case 5.**  $(x_1, x_2) = (8, 4)$ . Then  $\text{lk}(4) = C_6(0, 3, 8, 1, 2, 5)$ . This implies  $\text{lk}(2) = C_7(3, \mathbf{0}, 1, 4, 5, x_3, \mathbf{x}_4)$ , where we see easily that  $(x_3, x_4) \in \{(7, 6), (9, 10)\}$ . If  $(x_3, x_4) = (7, 6)$ , then completing successively, we get  $\text{lk}(3) = C_7(2, \mathbf{1}, 0, 4, 8, 6, \mathbf{7})$ ,  $\text{lk}(6) = C_7(7, \mathbf{1}, 0, 5, 8, \mathbf{3}, \mathbf{2})$ ,  $\text{lk}(7) = C_7(6, \mathbf{0}, 1, 8, 5, 2, \mathbf{3})$ ,  $\text{lk}(5) = C_6(0, 4, 2, 7, 8, 6)$ . Then  $M \cong \mathcal{E}_1(K)$  by the map  $(0, 6)(1, 7)(4, 8)$ .

If  $(x_3, x_4) = (9, 10)$ , then  $\text{lk}(3) = C_7(2, \mathbf{1}, 0, 4, 8, 10, \mathbf{9})$  and  $\text{lk}(8) = C_6(1, 4, 3, 10, 11, 7)$ . Now completing successively, we get  $\text{lk}(7) = C_7(6, \mathbf{0}, 1, 8, 11, 9, \mathbf{10})$ ,  $\text{lk}(9) = C_7(10, \mathbf{3}, 2, 5, 11, 7, \mathbf{6})$ ,  $\text{lk}(10) = C_7(9, \mathbf{2}, 3, 8, 11, 6, \mathbf{7})$ ,  $\text{lk}(5) = C_6(0, 4, 2, 9, 11, 6)$ ,  $\text{lk}(6) = C_7(7, \mathbf{1}, 0, 5, 11, 10, \mathbf{9})$ . Then  $M \cong \mathcal{E}_9(K)$  by the map  $(0, 9, 1, 10)(4, 11, 5, 8)$ .

**Case 6.**  $(x_1, x_2) = (8, 9)$ . Then  $\text{lk}(1) = C_7(0, \mathbf{3}, 2, 9, 8, 7, \mathbf{6})$ . This implies  $\text{lk}(2) = C_7(3, \mathbf{0}, 1, 9, x_3, x_4, \mathbf{x}_5)$ . Then it is easy to see that  $(x_3, x_4, x_5) \in \{(4, 5, 8), (4, 5, 10), (4, 6, 7), (4, 7, 6), (4, 10, 5), (4, 10, 8), (4, 10, 11), (5, 6, 7), (5, 7, 6), (5, 10, 8), (5, 10, 11), (6, 5, 8), (6, 5, 10), (7, 8, 5), (7, 8, 10), (10, 6, 7), (10, 7, 6), (10, 11, 5), (10, 11, 8)\}$ .

**Claim 1.**  $(x_3, x_4, x_5) = (4, 10, 11), (5, 10, 11)$  or  $(10, 7, 6)$ .

If  $(x_3, x_4, x_5) = (4, 5, 8)$  (or  $(4, 5, 10)$ ), then  $\text{lk}(5) = C_7(8, \mathbf{3}, 2, 4, 0, 6, \mathbf{7})$  (resp.  $\text{lk}(5) = C_7(10, \mathbf{3}, 2, 4, 0, 6, \mathbf{7})$ ), which implies  $C_5(0, 1, 7, 8, 5) \subseteq \text{lk}(6)$  (resp.  $C_5(0, 1, 7, 10, 5) \subseteq \text{lk}(6)$ ). If  $(x_3, x_4, x_5) = (4, 6, 7)$ , then considering  $\text{lk}(6)$ , we get  $C_3(0, 4, 5) \subseteq \text{lk}(5)$ . If  $(x_3, x_4, x_5) = (4, 7, 6)$ , considering  $\text{lk}(7)$ , we get  $\text{deg}(4) > 6$ . If  $(x_3, x_4, x_5) = (4, 10, 5)$ , then considering  $\text{lk}(5)$ , we see that  $\text{lk}(4)$  can not be completed. If  $(x_3, x_4, x_5) = (4, 10, 8)$ , then  $\text{lk}(4) = C_6(0, 3, 10, 2, 9, 5)$  or  $\text{lk}(4) = C_6(0, 3, 9, 2, 10, 5)$ , but for both the cases,  $\text{lk}(9)$  can not be completed. By a similar computation we see easily that  $M$  does not exist for  $(x_3, x_4, x_5) \in \{(5, 6, 7), (5, 7, 6), (5, 10, 8), (7, 8, 10)\}$ .

If  $(x_3, x_4, x_5) = (6, 5, 8)$ , then  $\text{lk}(6) = C_7(7, \mathbf{1}, 0, 5, 2, 9, \mathbf{x}_6)$ , where  $x_6 \in \{4, 10\}$ . If  $x_6 = 10$ , then  $\text{lk}(9) = C_7(10, \mathbf{5}, 8, 1, 2, 6, \mathbf{x}_7)$ , and we see that  $\text{lk}(7)$  can not be completed. If  $x_6 = 4$ , then completing successively, we get  $\text{lk}(9) = C_7(4, \mathbf{5}, 8, 1, 2, 6, \mathbf{7})$ ,  $\text{lk}(4) = C_7(9, \mathbf{6}, 7, 3, 0, 5, \mathbf{8})$ ,  $\text{lk}(7) = C_7(6, \mathbf{0}, 1, 8, 3, 4, \mathbf{9})$ ,  $\text{lk}(3) = C_7(2, \mathbf{1}, 0, 4, 7, 8, \mathbf{5})$ ,  $\text{lk}(5) = C_7(8, \mathbf{3}, 2, 6, 0, 4, \mathbf{9})$ ,  $\text{lk}(8) = C_7(5, \mathbf{2}, 3, 7, 1, 9, \mathbf{4})$ . But, this gives a semi-equivelar map.

If  $(x_3, x_4, x_5) = (6, 5, 10)$ , then  $\text{lk}(6) = C_7(7, \mathbf{1}, 0, 5, 2, 9, \mathbf{x}_6)$ , where  $x_6 \in \{4, 11\}$ . If  $x_6 = 11$ , then  $\text{lk}(9)$  can not be completed. If  $x_6 = 4$ , as in previous case, we get a semi-equivelar map.

If  $(x_3, x_4, x_5) = (7, 8, 5)$ , then  $\text{lk}(7) = C_7(6, \mathbf{0}, 1, 8, 2, 9, \mathbf{x}_6)$ , where  $x_6 = 4$ . Now,

completing as in previous case, we get a semi-equivelar map.

If  $(x_3, x_4, x_5) = (7, 8, 10)$ , then  $\text{lk}(8) = C_7(10, \mathbf{3}, 2, 7, 1, 9, \mathbf{x}_6)$ , where  $x_6 \in \{5, 11\}$ . If  $x_6 = 5$ , then  $\text{lk}(9) = C_7(5, \mathbf{6}, 7, 2, 1, 8, \mathbf{10})$  and we get  $C_5(0, 1, 7, 9, 5) \subseteq \text{lk}(6)$ . If  $x_6 = 11$ , then considering  $\text{lk}(9)$ ,  $\text{lk}(7)$ ,  $\text{lk}(6)$ , and  $\text{lk}(10)$  successively, we get  $C_5(6, 7, 9, 8, 10) \subseteq \text{lk}(5)$ . This proves the claim.

**Subcase 6.1.**  $(x_3, x_4, x_5) = (4, 10, 11)$ . Completing successively, we get  $\text{lk}(2) = C_7(3, \mathbf{0}, 1, 9, 4, 10, \mathbf{11})$ ,  $\text{lk}(4) = C_6(0, 3, 9, 2, 10, 5)$ ,  $\text{lk}(3) = C_7(2, \mathbf{1}, 0, 4, 9, 11, \mathbf{10})$ ,  $\text{lk}(9) = C_6(1, 2, 4, 3, 11, 8)$ ,  $\text{lk}(10) = C_7(11, \mathbf{3}, 2, 4, 5, 7, \mathbf{6})$ ,  $\text{lk}(11) = C_7(10, \mathbf{2}, 3, 9, 8, 6, \mathbf{7})$ ,  $\text{lk}(6) = C_7(7, \mathbf{1}, 0, 5, 8, 11, \mathbf{10})$ ,  $\text{lk}(5) = C_6(0, 4, 10, 7, 8, 6)$ ,  $\text{lk}(8) = C_6(1, 7, 5, 6, 11, 9)$ . Then  $M \cong \mathcal{E}_2(K)$  by the map  $(0, 6, 9, 4, 5, 11, 3)(1, 7, 10, 2)$ .

**Subcase 6.2.**  $(x_3, x_4, x_5) = (5, 10, 11)$ . Then  $\text{lk}(5) = C_6(0, 4, 9, 2, 10, 6)$  or  $\text{lk}(5) = C_6(0, 4, 10, 2, 9, 6)$ . In the first case, considering  $\text{lk}(9)$ , we see that  $\text{lk}(4)$  can not be completed. So,  $\text{lk}(5) = C_6(0, 4, 10, 2, 9, 6)$ . Completing successively, we get  $\text{lk}(9) = C_6(1, 2, 5, 6, 11, 8)$ ,  $\text{lk}(6) = C_7(7, \mathbf{1}, 0, 5, 9, 11, \mathbf{10})$ ,  $\text{lk}(10) = C_7(11, \mathbf{3}, 2, 5, 4, 7, \mathbf{6})$ ,  $\text{lk}(11) = C_7(10, \mathbf{2}, 3, 8, 9, 6, \mathbf{7})$ ,  $\text{lk}(3) = C_7(2, \mathbf{1}, 0, 4, 8, 11, \mathbf{10})$ ,  $\text{lk}(4) = C_6(0, 3, 8, 7, 10, 5)$ . Then  $M \cong \mathcal{E}_{11}(T)$  by the identity map.

**Subcase 6.3.**  $(x_3, x_4, x_5) = (10, 7, 6)$ . Then  $\text{lk}(2) = C_7(3, \mathbf{0}, 1, 9, 10, 7, \mathbf{6})$ ,  $\text{lk}(7) = C_7(6, \mathbf{0}, 1, 8, 10, 2, \mathbf{3})$ ,  $\text{lk}(6) = C_7(7, \mathbf{1}, 0, 5, 11, 3, \mathbf{2})$ ,  $\text{lk}(3) = C_7(2, \mathbf{1}, 0, 4, 11, 6, \mathbf{7})$ . This implies  $\text{lk}(10) = C_6(8, 7, 2, 9, x_6, x_7)$  or  $\text{lk}(10) = C_7(x_7, \mathbf{x}_8, 8, 7, 2, 9, \mathbf{x}_6)$ .

**Subcase 6.3.1.** If  $\text{lk}(10) = C_6(8, 7, 2, 9, x_6, x_7)$ , then  $(x_6, x_7) \in \{(4, 5), (5, 4), (11, 4), (11, 5)\}$ . If  $(x_6, x_7) = (4, 5)$ , completing successively, we get  $\text{lk}(4) = C_6(0, 3, 11, 9, 10, 5)$ ,  $\text{lk}(5) = C_6(0, 4, 10, 8, 11, 6)$ ,  $\text{lk}(8) = C_6(1, 7, 10, 5, 11, 9)$ ,  $\text{lk}(9) = C_6(1, 2, 10, 4, 11, 8)$ , and  $\text{lk}(11) = C_6(3, 4, 9, 8, 5, 6)$ . Then  $M \cong \mathcal{E}_{10}(K)$  by the identity map.

If  $(x_6, x_7) = (5, 4)$ , completing successively, we get  $\text{lk}(4) = C_6(0, 3, 11, 8, 10, 5)$ ,  $\text{lk}(5) = C_6(0, 4, 10, 9, 11, 6)$ ,  $\text{lk}(9) = C_6(1, 2, 10, 5, 11, 8)$ ,  $\text{lk}(8) = C_6(1, 7, 10, 4, 11, 9)$ , and  $\text{lk}(11) = C_6(3, 4, 8, 9, 5, 6)$ . Then  $M \cong \mathcal{E}_{12}(T)$  by the identity map.

If  $(x_6, x_7) = (11, 4)$ , completing successively, we get  $\text{lk}(4) = C_6(0, 3, 11, 10, 8, 5)$ ,  $\text{lk}(8) = C_6(1, 7, 10, 4, 5, 9)$ ,  $\text{lk}(5) = C_6(0, 4, 8, 9, 11, 6)$ ,  $\text{lk}(9) = C_6(1, 2, 10, 11, 5, 8)$ , and  $\text{lk}(11) = C_6(3, 4, 10, 9, 5, 6)$ . Then  $M \cong \mathcal{E}_{10}(K)$  by the map  $(0, 6)(1, 7)(4, 11)(9, 10)$ .

If  $(x_6, x_7) = (11, 5)$ , completing successively, we get  $\text{lk}(5) = C_6(0, 4, 8, 10, 11, 6)$ ,  $\text{lk}(8) = C_6(1, 7, 10, 5, 4, 9)$ ,  $\text{lk}(4) = C_6(0, 3, 11, 9, 8, 5)$ ,  $\text{lk}(9) = C_6(1, 2, 10, 11, 4, 8)$ , and  $\text{lk}(11) = C_6(3, 4, 9, 10, 5, 6)$ . Then  $M \cong \mathcal{E}_{13}(T)$  by the identity map.

**Subcase 6.3.2.** If  $\text{lk}(10) = C_7(x_7, \mathbf{x}_8, 8, 7, 2, 9, \mathbf{x}_6)$ , then it is easy to see that  $(x_6, x_7, x_8) \in \{(5, 4, 11), (11, 4, 5), (4, 5, 11), (11, 5, 4), (4, 11, 5), (5, 11, 4)\}$ . But for all these cases of  $(x_6, x_7, x_8)$  we see at least one quadrangular face incident at each vertex  $i$ , for  $0 \leq i \leq 11$ , which means there is no vertex in  $M$  with face-sequence  $(3^6)$ . Thus for these cases,  $M$  does not exist. This proves the lemma.  $\square$

**Theorem B.** Let  $M$  be a 2-semi equivelar map of type  $[3^3.4^2 : 4^4]$  on  $\leq 12$  vertices. Then  $M$  is isomorphic to one of  $\mathcal{F}_1(K)$ ,  $\mathcal{F}_2(T)$ ,  $\mathcal{F}_3(T)$ ,  $\mathcal{F}_4(K)$ ,  $\mathcal{F}_5(T)$ ,  $\mathcal{F}_6(T)$ ,  $\mathcal{F}_7(T)$ ,  $\mathcal{F}_8(K)$  or  $\mathcal{F}_9(T)$  given in example Section 3.

**Proof:** Assume that  $f_{seq}(0) = (4^4)$  and  $\text{lk}(0) = C_8(1, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{7}, \mathbf{8})$  in  $M$ . Then we have two cases, either  $f_{seq}(1) = (3^3.4^2)$  or  $f_{seq}(1) = (4^4)$ .

**Case 1:**  $f_{seq}(1) = (3^3.4^2)$ . Then  $\text{lk}(1) = C_7(0, \mathbf{7}, \mathbf{8}, x_1, x_2, \mathbf{2}, \mathbf{3})$ . It is easy to see that  $(x_1, x_2) \in \{(4, 5), (4, 6), (4, 9), (5, 6), (5, 4), (6, 4), (6, 5), (6, 9), (9, 4), (9, 6), (9, 10)\}$ . Here  $(4, 6) \cong (6, 4)$  by the map  $(2, 8)(3, 7)(4, 6)$ . So, we need not consider the case  $(6, 4)$ .

**Subcase 1.1.**  $(x_1, x_2) = (4, 5)$ . Then  $\text{lk}(1) = C_7(0, \mathbf{7}, \mathbf{8}, \mathbf{4}, \mathbf{5}, \mathbf{2}, \mathbf{3})$ ,  $\text{lk}(5) = C_7(0, \mathbf{3}, \mathbf{4}, \mathbf{1}, \mathbf{2}, \mathbf{6}, \mathbf{7})$ . This implies  $\text{lk}(2) = C_7(3, \mathbf{0}, \mathbf{1}, \mathbf{5}, \mathbf{6}, x_3, \mathbf{x}_4)$ , where  $(x_3, x_4) \in \{(9, 4), (9, 10)\}$ . If  $(x_3, x_4) = (9, 4)$ , then considering  $\text{lk}(4)$ , we get  $C_6(0, \mathbf{1}, \mathbf{2}, \mathbf{9}, \mathbf{4}, \mathbf{5}) \subseteq \text{lk}(3)$ . If  $(x_3, x_4) = (9, 10)$ , then it is easy to see that  $\text{lk}(3) = C_8(0, \mathbf{1}, \mathbf{2}, \mathbf{9}, \mathbf{10}, \mathbf{11}, \mathbf{4}, \mathbf{5})$ ,  $\text{lk}(4) = C_7(3, \mathbf{0}, \mathbf{5}, \mathbf{1}, \mathbf{8}, \mathbf{11}, \mathbf{10})$  and  $\text{lk}(8) = C_7(7, \mathbf{0}, \mathbf{1}, \mathbf{4}, \mathbf{11}, x_5, \mathbf{x}_6)$ , where  $(x_5, x_6) = (9, 10)$ . Completing successively, we get  $\text{lk}(8) = C_7(7, \mathbf{0}, \mathbf{1}, \mathbf{4}, \mathbf{11}, \mathbf{9}, \mathbf{10})$ ,  $\text{lk}(9) = C_7(10, \mathbf{3}, \mathbf{2}, \mathbf{6}, \mathbf{11}, \mathbf{8}, \mathbf{7})$ ,  $\text{lk}(6) = C_7(7, \mathbf{0}, \mathbf{5}, \mathbf{2}, \mathbf{9}, \mathbf{11}, \mathbf{10})$ ,  $\text{lk}(11) = C_7(10, \mathbf{3}, \mathbf{4}, \mathbf{8}, \mathbf{9}, \mathbf{6}, \mathbf{7})$ ,  $\text{lk}(7) = C_8(0, \mathbf{1}, \mathbf{8}, \mathbf{9}, \mathbf{10}, \mathbf{11}, \mathbf{6}, \mathbf{5})$ . Then  $M \cong \mathcal{F}_1(K)$  by the identity map.

**Subcase 1.2.**  $(x_1, x_2) = (4, 6)$ . Then  $\text{lk}(1) = C_7(0, \mathbf{7}, \mathbf{8}, \mathbf{4}, \mathbf{6}, \mathbf{2}, \mathbf{3})$ . This implies  $\text{lk}(4) = C_7(3, \mathbf{0}, \mathbf{5}, \mathbf{8}, \mathbf{1}, \mathbf{6}, \mathbf{x}_3)$ , where  $x_3 = 7$ . Completing successively, we get  $\text{lk}(6) = C_7(7, \mathbf{0}, \mathbf{5}, \mathbf{2}, \mathbf{1}, \mathbf{4}, \mathbf{3})$ ,  $\text{lk}(2) = C_7(3, \mathbf{0}, \mathbf{1}, \mathbf{6}, \mathbf{5}, \mathbf{8}, \mathbf{7})$ ,  $\text{lk}(8) = C_7(7, \mathbf{0}, \mathbf{1}, \mathbf{4}, \mathbf{5}, \mathbf{2}, \mathbf{3})$ ,  $\text{lk}(7) = C_8(0, \mathbf{1}, \mathbf{8}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{6}, \mathbf{5})$ ,  $\text{lk}(5) = C_7(0, \mathbf{3}, \mathbf{4}, \mathbf{8}, \mathbf{2}, \mathbf{6}, \mathbf{7})$ ,  $\text{lk}(3) = C_8(0, \mathbf{1}, \mathbf{2}, \mathbf{8}, \mathbf{7}, \mathbf{6}, \mathbf{4}, \mathbf{5})$ . Then  $M \cong \mathcal{F}_2(T)$  by the identity map.

**Subcase 1.3.**  $(x_1, x_2) = (4, 9)$ . Then  $\text{lk}(4) = C_7(3, \mathbf{0}, \mathbf{5}, \mathbf{9}, \mathbf{1}, \mathbf{8}, \mathbf{x}_3)$  or  $\text{lk}(4) = C_7(5, \mathbf{0}, \mathbf{3}, \mathbf{9}, \mathbf{1}, \mathbf{8}, \mathbf{x}_3)$  or  $\text{lk}(4) = C_7(5, \mathbf{0}, \mathbf{3}, \mathbf{8}, \mathbf{1}, \mathbf{9}, \mathbf{x}_3)$ , or  $\text{lk}(4) = C_7(3, \mathbf{0}, \mathbf{5}, \mathbf{8}, \mathbf{1}, \mathbf{9}, \mathbf{x}_3)$ , where  $x_3 \in V(M)$ . If  $\text{lk}(4) = C_7(3, \mathbf{0}, \mathbf{5}, \mathbf{9}, \mathbf{1}, \mathbf{8}, \mathbf{x}_3)$  (or  $\text{lk}(4) = C_7(5, \mathbf{0}, \mathbf{3}, \mathbf{9}, \mathbf{1}, \mathbf{8}, \mathbf{x}_3)$ ), then  $x_3 = 7$  and we get  $C_5(0, \mathbf{1}, \mathbf{4}, \mathbf{3}, \mathbf{7}) \subseteq \text{lk}(8)$  (resp.  $C_5(0, \mathbf{1}, \mathbf{4}, \mathbf{5}, \mathbf{7}) \subseteq \text{lk}(8)$ ). If  $\text{lk}(4) = C_7(5, \mathbf{0}, \mathbf{3}, \mathbf{8}, \mathbf{1}, \mathbf{9}, \mathbf{x}_3)$ , then  $x_3 = 10$ . This implies  $\text{lk}(5) = C_8(0, \mathbf{3}, \mathbf{4}, \mathbf{9}, \mathbf{10}, \mathbf{11}, \mathbf{6}, \mathbf{7})$ , but then  $\text{lk}(9)$  can not be completed. If  $\text{lk}(4) = C_7(3, \mathbf{0}, \mathbf{5}, \mathbf{8}, \mathbf{1}, \mathbf{9}, \mathbf{x}_3)$ , then  $\text{lk}(5) = C_7(0, \mathbf{3}, \mathbf{4}, \mathbf{8}, \mathbf{10}, \mathbf{6}, \mathbf{7})$  and  $\text{lk}(8) = C_7(7, \mathbf{0}, \mathbf{1}, \mathbf{4}, \mathbf{5}, \mathbf{10}, \mathbf{x}_4)$ , where  $x_4 \in \{9, 11\}$ . In case  $x_4 = 9$ ,  $\text{lk}(7)$  can not be completed. So,  $x_4 = 11$ . Completing successively, we get  $\text{lk}(8) = C_7(7, \mathbf{0}, \mathbf{1}, \mathbf{4}, \mathbf{5}, \mathbf{10}, \mathbf{11})$ ,  $\text{lk}(7) = C_8(0, \mathbf{1}, \mathbf{8}, \mathbf{10}, \mathbf{11}, \mathbf{9}, \mathbf{6}, \mathbf{5})$ ,  $\text{lk}(6) = C_7(7, \mathbf{0}, \mathbf{5}, \mathbf{10}, \mathbf{2}, \mathbf{9}, \mathbf{11})$ ,  $\text{lk}(2) = C_7(3, \mathbf{0}, \mathbf{1}, \mathbf{9}, \mathbf{6}, \mathbf{10}, \mathbf{11})$ ,  $\text{lk}(3) = C_8(0, \mathbf{1}, \mathbf{2}, \mathbf{10}, \mathbf{11}, \mathbf{9}, \mathbf{4}, \mathbf{5})$ ,  $\text{lk}(4) = C_7(3, \mathbf{0}, \mathbf{5}, \mathbf{8}, \mathbf{1}, \mathbf{9}, \mathbf{11})$ ,  $\text{lk}(9) = C_7(11, \mathbf{3}, \mathbf{4}, \mathbf{1}, \mathbf{2}, \mathbf{6}, \mathbf{7})$ ,  $\text{lk}(10) = C_7(11, \mathbf{3}, \mathbf{2}, \mathbf{6}, \mathbf{5}, \mathbf{8}, \mathbf{7})$ . Then  $M \cong \mathcal{F}_3(T)$  by the identity map.

**Subcase 1.4.**  $(x_1, x_2) = (5, 6)$ , i.e.,  $\text{lk}(1) = C_7(0, \mathbf{3}, \mathbf{2}, \mathbf{6}, \mathbf{5}, \mathbf{8}, \mathbf{7})$ . Then  $\text{lk}(5) = C_7(0, \mathbf{3}, \mathbf{4}, \mathbf{8}, \mathbf{1}, \mathbf{6}, \mathbf{7})$  and  $\text{lk}(6) = C_7(7, \mathbf{0}, \mathbf{5}, \mathbf{1}, \mathbf{2}, x_3, \mathbf{x}_4)$ . Here, we see that  $(x_3, x_4) \in \{(4, 3), (9, 10)\}$ . If  $(x_3, x_4) = (4, 3)$ , completing successively, we get  $\text{lk}(4) = C_7(3, \mathbf{0}, \mathbf{5}, \mathbf{8}, \mathbf{2}, \mathbf{6}, \mathbf{7})$ ,  $\text{lk}(8) = C_7(7, \mathbf{0}, \mathbf{1}, \mathbf{5}, \mathbf{4}, \mathbf{2}, \mathbf{3})$ ,  $\text{lk}(2) = C_7(3, \mathbf{0}, \mathbf{1}, \mathbf{6}, \mathbf{4}, \mathbf{8}, \mathbf{7})$ ,  $\text{lk}(3) = C_8(0, \mathbf{1}, \mathbf{2}, \mathbf{8}, \mathbf{7}, \mathbf{6}, \mathbf{4}, \mathbf{5})$ . Then  $M \cong \mathcal{F}_4(K)$  by the identity map.

On the other hand, if  $(x_3, x_4) = (9, 10)$ , completing successively, we get  $\text{lk}(2) = C_7(3, \mathbf{0}, \mathbf{1}, \mathbf{6}, \mathbf{9}, \mathbf{11}, \mathbf{10})$ ,  $\text{lk}(3) = C_8(0, \mathbf{1}, \mathbf{2}, \mathbf{11}, \mathbf{10}, \mathbf{9}, \mathbf{4}, \mathbf{5})$ ,  $\text{lk}(9) = C_7(10, \mathbf{3}, \mathbf{4}, \mathbf{11}, \mathbf{2}, \mathbf{6}, \mathbf{7})$ ,  $\text{lk}(4) = C_7(3, \mathbf{0}, \mathbf{5}, \mathbf{8}, \mathbf{11}, \mathbf{9}, \mathbf{10})$ ,  $\text{lk}(7) = C_8(0, \mathbf{1}, \mathbf{8}, \mathbf{11}, \mathbf{10}, \mathbf{9}, \mathbf{6}, \mathbf{5})$ ,  $\text{lk}(8) = C_7(7, \mathbf{0}, \mathbf{1}, \mathbf{5}, \mathbf{4}, \mathbf{11}, \mathbf{10})$ ,  $\text{lk}(11) = C_7(10, \mathbf{3}, \mathbf{2}, \mathbf{9}, \mathbf{4}, \mathbf{8}, \mathbf{7})$ . Then  $M \cong \mathcal{F}_1(K)$  by the map  $(2, 8)(3, 7)(4, 6)(9, 11)$ .

**Subcase 1.5.**  $(x_1, x_2) = (5, 4)$ , i.e.,  $\text{lk}(1) = C_7(0, \mathbf{3}, \mathbf{2}, \mathbf{4}, \mathbf{5}, \mathbf{8}, \mathbf{7})$ . Then

$\text{lk}(5) = C_7(0, \mathbf{3}, 4, 1, 8, 6, \mathbf{7})$  and  $\text{lk}(8) = C_7(7, \mathbf{0}, 1, 5, 6, x_3, \mathbf{x}_4)$ , where  $(x_3, x_4) \in \{(2, 3), (9, 10)\}$ . If  $(x_3, x_4) = (2, 3)$ , completing successively, we get  $\text{lk}(8) = C_7(7, \mathbf{0}, 1, 5, 6, 2, \mathbf{3})$ ,  $\text{lk}(2) = C_7(3, \mathbf{0}, 1, 4, 6, 8, \mathbf{7})$ ,  $\text{lk}(4) = C_7(3, \mathbf{0}, 5, 1, 2, 6, \mathbf{7})$ ,  $\text{lk}(6) = C_7(7, \mathbf{0}, 5, 8, 2, 4, \mathbf{3})$ ,  $\text{lk}(7) = C_8(0, \mathbf{1}, 8, \mathbf{2}, 3, \mathbf{4}, 6, \mathbf{5})$ ,  $\text{lk}(3) = C_8(0, \mathbf{1}, 2, \mathbf{8}, 7, \mathbf{6}, \mathbf{4}, \mathbf{5})$ . Then  $M \cong \mathcal{F}_5(T)$  by the identity map.

On the other hand, if  $(x_3, x_4) = (9, 10)$ , then  $\text{lk}(8) = C_7(7, \mathbf{0}, 1, 5, 6, 9, \mathbf{10})$ . This implies  $\text{lk}(6) = C_7(7, \mathbf{0}, 5, 8, 9, x_5, \mathbf{x}_6)$ , where  $(x_5, x_6) \in \{(2, 3), (11, 10)\}$ . If  $(x_5, x_6) = (2, 3)$ , then  $\text{lk}(6) = C_7(7, \mathbf{0}, 5, 8, 9, 2, \mathbf{3})$ ,  $\text{lk}(2) = C_7(3, \mathbf{0}, 1, 4, 9, 6, \mathbf{7})$ ,  $\text{lk}(9) = C_7(10, \mathbf{3}, 4, 2, 6, 8, \mathbf{7})$ , which implies that  $\text{lk}(3)$  can not be completed. So  $(x_5, x_6) = (11, 10)$ . Completing successively, we get  $\text{lk}(6) = C_7(7, \mathbf{0}, 5, 8, 9, 11, \mathbf{10})$ ,  $\text{lk}(7) = C_8(0, \mathbf{1}, 8, \mathbf{9}, 10, \mathbf{11}, 6, \mathbf{5})$ ,  $\text{lk}(9) = C_7(10, \mathbf{3}, 2, 11, 6, 8, \mathbf{7})$ ,  $\text{lk}(2) = C_7(3, \mathbf{0}, 1, 4, 11, 9, \mathbf{10})$ ,  $\text{lk}(3) = C_8(0, \mathbf{1}, 2, \mathbf{9}, 10, \mathbf{11}, \mathbf{4}, \mathbf{5})$ ,  $\text{lk}(4) = C_7(3, \mathbf{0}, 5, 1, 2, 11, \mathbf{10})$ ,  $\text{lk}(11) = C_7(10, \mathbf{3}, 4, 2, 9, 6, \mathbf{7})$ . Then  $M \cong \mathcal{F}_6(T)$  by the identity map.

**Subcase 1.6.**  $(x_1, x_2) = (6, 5)$ , i.e.,  $\text{lk}(1) = C_7(0, \mathbf{3}, 2, 5, 6, 8, \mathbf{7})$ . Then  $\text{lk}(5) = C_7(0, \mathbf{3}, 4, 2, 1, 6, \mathbf{7})$ . This implies  $\text{lk}(6) = C_7(7, \mathbf{0}, 5, 1, 8, x_3, \mathbf{x}_4)$ , where  $(x_3, x_4) \in \{(2, 3), (4, 3), (9, 10)\}$ . If  $(x_3, x_4) = (2, 3)$ , then considering  $\text{lk}(6)$  and  $\text{lk}(2)$ , we see that  $\text{lk}(8)$  can not be completed. If  $(x_3, x_4) = (4, 3)$ , then completing successively we get  $\text{lk}(4) = C_7(3, \mathbf{0}, 5, 2, 8, 6, \mathbf{7})$ ,  $\text{lk}(2) = C_7(3, \mathbf{0}, 1, 5, 4, 8, \mathbf{7})$ ,  $\text{lk}(8) = C_7(7, \mathbf{0}, 1, 6, 4, 2, \mathbf{3})$ ,  $\text{lk}(3) = C_8(2, \mathbf{1}, 0, \mathbf{5}, 4, \mathbf{6}, 7, \mathbf{8})$ . Then  $M \cong \mathcal{F}_5(T)$  by the map  $(0, 3)(1, 2)(4, 5)$ .

On the other hand, if  $(x_3, x_4) = (9, 10)$ , then  $\text{lk}(6) = C_7(7, \mathbf{0}, 5, 1, 8, 9, \mathbf{10})$ ,  $\text{lk}(7) = C_8(0, \mathbf{1}, 8, \mathbf{11}, 10, \mathbf{9}, 6, \mathbf{5})$ ,  $\text{lk}(8) = C_7(7, \mathbf{0}, 1, 6, 9, 11, \mathbf{10})$ . This implies  $\text{lk}(9) = C_7(10, \mathbf{7}, 6, 8, 11, x_5, \mathbf{x}_6)$ , where  $(x_5, x_6) \in \{(3, 2), (4, 3)\}$ . In case  $(x_5, x_6) = (3, 2)$ , considering  $\text{lk}(9)$  and  $\text{lk}(2)$ , we see that  $\text{lk}(10)$  can not be completed. If  $(x_5, x_6) = (4, 3)$ , completing successively, we get  $\text{lk}(9) = C_7(10, \mathbf{3}, 4, 11, 8, 6, \mathbf{7})$ ,  $\text{lk}(4) = C_7(3, \mathbf{0}, 5, 2, 11, 9, \mathbf{10})$ ,  $\text{lk}(3) = C_8(0, \mathbf{1}, 2, \mathbf{11}, 10, \mathbf{9}, 4, \mathbf{5})$ ,  $\text{lk}(2) = C_7(3, \mathbf{0}, 1, 5, 4, 11, \mathbf{10})$ ,  $\text{lk}(11) = C_7(10, \mathbf{3}, 2, 4, 9, 8, \mathbf{7})$ . Then  $M \cong \mathcal{F}_6(T)$  by the map  $(0, 10)(1, 9, 5, 11)$ .

**Subcase 1.7.**  $(x_1, x_2) = (6, 9)$ , i.e.,  $\text{lk}(1) = C_7(0, \mathbf{3}, 2, 9, 6, 8, \mathbf{7})$ . Then  $\text{lk}(6) = C_7(7, \mathbf{0}, 5, 8, 1, 9, \mathbf{x}_3)$ , where  $x_3 \in \{4, 10\}$ . If  $x_3 = 4$ , then considering  $\text{lk}(5)$  and  $\text{lk}(8)$ , we see that  $\text{lk}(7)$  can not be completed. If  $x_3 = 10$ , then  $\text{lk}(7) = C_8(0, \mathbf{1}, 8, \mathbf{11}, 10, \mathbf{9}, 6, \mathbf{5})$ ,  $\text{lk}(8) = C_7(7, \mathbf{0}, 1, 6, 5, 11, \mathbf{10})$ ,  $\text{lk}(5) = C_7(0, \mathbf{3}, 4, 11, 8, 6, \mathbf{7})$ . This implies  $\text{lk}(11) = C_7(10, \mathbf{3}, 2, 4, 5, 8, \mathbf{7})$ , completing successively, we get  $\text{lk}(2) = C_7(3, \mathbf{0}, 1, 9, 4, 11, \mathbf{10})$ ,  $\text{lk}(3) = C_8(0, \mathbf{1}, 2, \mathbf{11}, 10, \mathbf{9}, \mathbf{4}, \mathbf{5})$ ,  $\text{lk}(4) = C_7(3, \mathbf{0}, 5, 11, 2, 9, \mathbf{10})$ ,  $\text{lk}(9) = C_7(10, \mathbf{3}, 4, 2, 1, 6, \mathbf{7})$ . Then  $M \cong \mathcal{F}_1(K)$  by the map  $(0, 7, 10)(1, 8, 9, 4, 5, 6, 11, 2)$ .

**Subcase 1.8.**  $(x_1, x_2) = (9, 4)$ . Then  $\text{lk}(4) = C_7(3, \mathbf{0}, 5, 2, 1, 9, \mathbf{x}_3)$ , where  $x_3 \in \{6, 10\}$ . The case  $x_3 = 6$  implies  $\text{lk}(3) = C_8(0, \mathbf{1}, 2, \mathbf{x}_4, 6, \mathbf{9}, 4, \mathbf{5})$ . Observe that  $x_4 \in \{7, 10\}$ . If  $x_4 = 7$ , then considering  $\text{lk}(3)$ , we see that  $\text{lk}(2)$  can not be completed. While if  $x_4 = 10$ , considering  $\text{lk}(2)$  and  $\text{lk}(5)$ , we see that  $\text{lk}(6)$  can not be completed. On the other hand, if  $x_3 = 10$ , then  $\text{lk}(3) = C_8(0, \mathbf{1}, 2, \mathbf{x}_4, 10, \mathbf{9}, \mathbf{4}, \mathbf{5})$ . Observe that  $x_4 \in \{6, 11\}$ . As above, we see that  $x_4 \neq 6$ . So  $x_4 = 11$ . Now completing successively, we get  $\text{lk}(3) = C_8(0, \mathbf{1}, 2, \mathbf{11}, 10, \mathbf{9}, \mathbf{4}, \mathbf{5})$ ,  $\text{lk}(6) = C_7(7, \mathbf{0}, 5, 11, 8, 9, \mathbf{10})$ ,  $\text{lk}(9) = C_7(10, \mathbf{3}, 4, 1, 8, 6, \mathbf{7})$ ,  $\text{lk}(8) = C_7(7, \mathbf{0}, 1, 9, 6, 11, \mathbf{10})$ ,  $\text{lk}(7) = C_8(0, \mathbf{1}, 8, \mathbf{11}, 10, \mathbf{9}, 6, \mathbf{5})$ ,  $\text{lk}(11) = C_7(10, \mathbf{3}, 2, 5, 6, 8, \mathbf{7})$ . Then  $M \cong \mathcal{F}_1(K)$  by



the map  $(0, 3)(1, 2)(4, 5)(6, 11)(7, 10)$ .

**Subcase 1.9.**  $(x_1, x_2) = (9, 6)$ , i.e.,  $\text{lk}(1) = C_7(0, \mathbf{3}, 2, 6, 9, 8, \mathbf{7})$ . Then  $\text{lk}(6) = C_7(5, \mathbf{0}, 7, 2, 1, 9, \mathbf{x_3})$  or  $\text{lk}(6) = C_7(7, \mathbf{0}, 5, 2, 1, 9, \mathbf{x_3})$ , for  $x_3 \in V(M)$ . In the first case, we see  $x_3 \in \{4, 10\}$ . But, a small computation shows that for both of these values,  $M$  does not exist. For the later case of  $\text{lk}(6)$ , we get  $x_3 \in \{4, 10\}$ . If  $x_3 = 4$ , then considering  $\text{lk}(6)$  and  $\text{lk}(9)$ , we see  $\text{lk}(4)$  can not be completed. If  $x_3 = 10$ , then  $\text{lk}(6) = C_7(7, \mathbf{0}, 5, 2, 1, 9, \mathbf{10})$  and  $\text{lk}(7) = C_8(8, \mathbf{1}, 0, \mathbf{5}, 6, \mathbf{9}, 10, \mathbf{x_6})$ , where  $x_6 \in \{4, 11\}$ . If  $x_6 = 4$ , considering  $\text{lk}(7)$  and  $\text{lk}(5)$ , we see that  $\text{lk}(4)$  can not be completed. If  $x_6 = 11$ , completing successively, we get  $\text{lk}(8) = C_7(7, \mathbf{0}, 1, 9, 4, 11, \mathbf{10})$ ,  $\text{lk}(9) = C_7(10, \mathbf{3}, 4, 8, 1, 6, \mathbf{7})$ ,  $\text{lk}(4) = C_7(3, \mathbf{0}, 5, 11, 8, 9, \mathbf{10})$ ,  $\text{lk}(3) = C_8(0, \mathbf{1}, 2, \mathbf{11}, 10, \mathbf{9}, 4, \mathbf{5})$ ,  $\text{lk}(11) = C_7(10, \mathbf{3}, 2, 5, 4, 8, \mathbf{7})$ ,  $\text{lk}(2) = C_7(3, \mathbf{0}, 1, 6, 5, 11, \mathbf{10})$ . Then  $M \cong \mathcal{F}_3(T)$  by the map  $(0, 3)(1, 2)(4, 5)(6, 9)(7, 11, 8, 10)$ .

**Subcase 1.10.**  $(x_1, x_2) = (9, 10)$ , i.e.,  $\text{lk}(1) = C_7(0, \mathbf{3}, 2, 10, 9, 8, \mathbf{7})$ . Then  $\text{lk}(2) = C_7(3, \mathbf{0}, 1, 10, x_3, x_4, \mathbf{x_5})$ . Observe that,  $(x_3, x_4, x_5) \in \{(4, 8, 7), (6, 11, 9), (8, 9, 6), (8, 9, 11), (11, 6, 7), (11, 8, 7)\}$ . A small computation shows that  $M$  does not exist for these values of  $(x_3, x_4, x_5)$ , except  $(11, 8, 7)$ . If  $(x_3, x_4, x_5) = (11, 8, 7)$ , then  $\text{lk}(2) = C_7(3, \mathbf{0}, 1, 10, 11, 8, \mathbf{7})$ ,  $\text{lk}(8) = C_7(7, \mathbf{0}, 1, 9, 11, 2, \mathbf{3})$ ,  $\text{lk}(3) = C_8(0, \mathbf{1}, 2, \mathbf{8}, 7, \mathbf{6}, 4, \mathbf{5})$  and  $\text{lk}(7) = C_8(0, \mathbf{1}, 8, \mathbf{2}, 3, 4, 6, \mathbf{5})$ . This implies  $\text{lk}(9) = C_7(x_7, \mathbf{x_8}, 11, 8, 1, 10, \mathbf{x_6})$ . It is easy to see that  $(x_6, x_7, x_8) \in \{(4, 5, 6), (4, 6, 5), (5, 4, 6), (5, 6, 4), (6, 4, 5), (6, 5, 4)\}$ . Note that  $(5, 4, 6) \cong (4, 6, 5)$  by the map  $(0, 7, 3)(1, 8, 2)(4, 5, 6)(9, 11, 10)$ ,  $(5, 6, 4) \cong (4, 5, 6)$  by the map  $(0, 7)(1, 8)(5, 6)(10, 11)$ , and  $(6, 5, 4) \cong (4, 6, 5)$  by the map  $(0, 7)(1, 8)(5, 6)(10, 11)$ . So, we search for  $(x_6, x_7, x_8) \in \{(4, 5, 6), (4, 6, 5), (6, 4, 5)\}$ .

**Subcase 1.10.1.** If  $(x_6, x_7, x_8) = (4, 5, 6)$ , then completing successively, we get  $\text{lk}(9) = C_7(5, \mathbf{4}, 10, 1, 8, 11, \mathbf{6})$ ,  $\text{lk}(5) = C_8(0, \mathbf{3}, 4, \mathbf{10}, 9, \mathbf{11}, 6, \mathbf{7})$ ,  $\text{lk}(4) = C_8(3, \mathbf{0}, 5, \mathbf{9}, 10, \mathbf{11}, 6, \mathbf{7})$ ,  $\text{lk}(6) = C_8(4, \mathbf{3}, 7, \mathbf{0}, 5, \mathbf{9}, 11, \mathbf{10})$ ,  $\text{lk}(10) = C_7(4, \mathbf{5}, 9, 1, 2, 11, \mathbf{6})$ ,  $\text{lk}(11) = C_7(6, \mathbf{4}, 10, 2, 8, 9, \mathbf{5})$ . Then  $M \cong \mathcal{F}_7(T)$ .

**Subcase 1.10.2.** If  $(x_6, x_7, x_8) = (4, 6, 5)$ , then completing successively, we get  $\text{lk}(9) = C_7(6, \mathbf{4}, 10, 1, 8, 11, \mathbf{5})$ ,  $\text{lk}(6) = C_8(4, \mathbf{3}, 7, \mathbf{0}, 5, \mathbf{11}, 9, \mathbf{10})$ ,  $\text{lk}(4) = C_8(3, \mathbf{0}, 5, \mathbf{11}, 10, \mathbf{9}, 6, \mathbf{7})$ ,  $\text{lk}(10) = C_7(4, \mathbf{5}, 11, 2, 1, 9, \mathbf{6})$ ,  $\text{lk}(11) = C_7(5, \mathbf{4}, 10, 2, 8, 9, \mathbf{6})$ ,  $\text{lk}(5) = C_8(0, \mathbf{3}, 4, \mathbf{10}, 11, \mathbf{9}, 6, \mathbf{7})$ . Then  $M \cong \mathcal{F}_8(K)$ .

**Subcase 1.10.3.** If  $(x_6, x_7, x_8) = (6, 4, 5)$ , then completing successively, we get  $\text{lk}(9) = C_7(4, \mathbf{6}, 10, 1, 8, 11, \mathbf{5})$ ,  $\text{lk}(4) = C_8(3, \mathbf{0}, 5, \mathbf{11}, 9, \mathbf{10}, 6, \mathbf{7})$ ,  $\text{lk}(5) = C_8(0, \mathbf{3}, 4, \mathbf{9}, 11, \mathbf{10}, 6, \mathbf{7})$ ,  $\text{lk}(11) = C_7(5, \mathbf{4}, 9, 8, 2, 10, \mathbf{6})$ ,  $\text{lk}(10) = C_7(6, \mathbf{4}, 9, 1, 2, 11, \mathbf{5})$ . Then  $M \cong \mathcal{F}_9(T)$ .

**Case 2.** If  $f_{seq}(1) = (4^4)$ , then  $\text{lk}(1) = C_8(2, \mathbf{3}, 0, \mathbf{7}, 8, \mathbf{x_1}, x_2, \mathbf{x_3})$ . We see that,  $(x_1, x_2, x_3) \in \{(4, 5, 6), (4, 6, 5), (4, 9, 6), (4, 9, 10), (5, 4, 6), (5, 4, 9), (6, 4, 5), (6, 5, 4), (6, 9, 4), (6, 9, 10), (9, 4, 5), (9, 6, 5), (9, 10, 4), (9, 10, 6), (9, 10, 11)\}$ . Now, doing computation for these cases, we get no other map. The detailed enumeration of this is given in [Case 2 of Lemma B in [15]]. To save space here, hence, we refer to [15].  $\square$

**Proof of Theorem 2.** Let  $M_1$  and  $M_2$  be 2-semi-equivelar maps with the vertex sets  $V(M_1)$  and  $V(M_2)$  respectively. Then,  $M_1 \not\cong M_2$  if: (i) one is on the torus and other is on the Klein bottle or (ii) their types are distinct or (iii)  $|V(M_1)| \neq |V(M_2)|$ , where

$|V(M_i)|$  denotes the cardinality of  $V(M_i)$ . Now, the proof follows from Theorem 1, Lemma 1 and Lemmas 2-B.  $\square$

The above classification is given below in tabular form.

**Table 1:** 2-semi-equivelar maps on the torus and Klein bottle on  $|V| \leq 12$

S. No.	Map Type	$ V(M) $	—Maps—	On Torus	On Klein bottle
1	$[3^6 : 3^4.6]$	11	2	$\mathcal{A}_3(T)$	$\mathcal{A}_1(K)$
		12	1	$\mathcal{A}_2(T)$	
2	$[3^3.4^2 : 3.4.6.4]$	12	2	$\mathcal{B}_2(T)$	$\mathcal{B}_1(K)$
3	$[3^2.4.3.4 : 3.4.6.4]$	12	2	$\mathcal{C}_2(T)$	$\mathcal{C}_1(K)$
4	$[3^6 : 3^2.4.3.4]$	11	1		$\mathcal{D}_1(K)$
5	$[3^6 : 3^3.4^2]$	9	4	$\mathcal{E}_3(T), \mathcal{E}_6(T)$	$\mathcal{E}_1(K), \mathcal{E}_5(K)$
		12	9	$\mathcal{E}_4(T), \mathcal{E}_8(T), \mathcal{E}_{11}(T), \mathcal{E}_{12}(T), \mathcal{E}_{13}(T)$	$\mathcal{E}_2(K), \mathcal{E}_7(K), \mathcal{E}_9(K), \mathcal{E}_{10}(K)$
6	$[3^3.4^2 : 4^4]$	9	3	$\mathcal{F}_3(T), \mathcal{F}_5(T)$	$\mathcal{F}_4(K)$
		12	6	$\mathcal{F}_3(T), \mathcal{F}_6(T), \mathcal{F}_7(T), \mathcal{F}_9(T)$	$\mathcal{F}_1(K), \mathcal{F}_8(K)$

## 5. Conclusion

The 2-semi equivelar maps are generalization of Johnson solids, as are 1-semi equivelar maps of Platonic solids and Archimedean solids. In this article, 2-semi-equivelar maps with curvature 0 have been studied for the surfaces of Euler characteristic 0. It has been obtained that there are exactly 16 types 2-semi-equivelar maps on these surfaces. Further, enumerating the maps for these types on at most 12 vertices, we have obtained 31 2-semi-equivelar maps. Out of which, 18 are on the torus and remaining 13 are on the Klein bottle. In [4], the authors have shown that there are exactly eleven types,  $[3^6]$ ,  $[3^4.6]$ ,  $[3^3.4^2]$ ,  $[3^2.4.3.4]$ ,  $[3.4.6.4]$ ,  $[3.6.3.6]$ ,  $[3.12^2]$ ,  $[4^4]$ ,  $[4.6.12]$ ,  $[4.8^2]$  and  $[6^3]$  semi equivelar maps are possible on the torus. Thus a natural questions occurs here: are there exist finitely many types of doubly semi-equivelar maps on the torus? If yes, then what are the remaining types?

**Conflict of interest.** The authors declare that they have no conflict of interest.

**Data Availability.** Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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