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Research Article

Total coalitions of cubic graphs of order at most 10

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Abstract: A total coalition in a graph $G = (V, E)$ consists of two disjoint sets of vertices V_1 and V_2, neither of which is a total dominating set but whose union $V_1 \cup V_2$, is a total dominating set. A total coalition partition in a graph G of order $n = |V|$ is a vertex partition $\tau = \{V_1, V_2, \dots, V_k\}$ such that every set $V_i \in \tau$ is not a total dominating set but forms a total coalition with another set $V_j \in \tau$ which is not a total dominating set. The total coalition number $TC(G)$ equals the maximum order k of a total coalition partition of G. In this paper, we determine the total coalition number of all cubic graphs of order $n \leq 10$.

Keywords: coalition, total coalition, cubic graphs, Petersen graph.

AMS Subject classification: 05C69

1. Introduction

Domination in graphs is one of the most studied areas in graph theory. The explosive growth of this field since 1998 has continued, and today several papers have been published on domination in graphs. Given a graph G, recall that a dominating set S of a graph G is a subset D of V such that every vertex in $V - D$ is adjacent to at least one member of D. The minimum cardinality of all dominating sets of G is called the domination number of G and is denoted by $\gamma(G)$. A set $S \subseteq V$ is a total dominating set of a graph G with no isolated vertex, if every vertex in V has at least one neighbour in S. The cardinality of a minimum total dominating set in G is called the *total domination number* of G and is denoted by $\gamma_t(G)$. Total domination in graphs was introduced in 1980 by Cockayne, Dawes, and Hedetniemi [6]. Domination and its variations have been extensively studied in the literature and surveyed in [15–17].

A domatic partition (or total domatic partition) is a partition of the vertex set into dominating sets (or total dominating sets). Formally, the domatic number (or total domatic number) $d(G)$ (or $d_t(G)$) equals the maximum order k of a vertex partition, called a domatic partition (total domatic partition), $\pi = \{V_1, V_2, \ldots, V_k\}$ such that every set V_i is a dominating set (or total dominating set) in G. The domatic number of a graph was introduced by Cockayne and Hedetniemi [7] and the total domatic number was introduced by Cockayne, Dawes and Hedetniemi in [6]. For more details on the domatic number and total domatic number refer to e.g., [19–21].

In 2020, a new concept called coalitions in graphs was introduced by Hedetniemi et. al [11]. A *coalition* in a graph $G = (V, E)$ consists of two disjoint sets V_1 and V_2 of vertices, such that neither V_1 nor V_2 is a dominating set, but the union $V_1 \cup V_2$ is a dominating set of G. A *coalition partition* in a graph G of order $n = |V|$ is a vertex partition $\mathcal{P} = \{V_1, V_2, \ldots, V_k\}$ such that every set V_i either is a dominating set consisting of a single vertex of degree $n-1$, or is not a dominating set but forms a coalition with another set V_j which is not a dominating set. The *coalition number* $C(G)$ of a graph G equals the maximum order of a coalition partition of G.

Unless otherwise stated, in what follows let G be an isolate-free graph. Let $U_1 \subset V$ and $U_2 \subset V$ denote two (disjoint) subsets of V.

A *total coalition* consists of two disjoint sets U_1 and U_2, neither of which is a total dominating set but the union $U_1 \cup U_2$ is a total dominating set. A *total coalition partition* is a vertex partition $\tau = \{U_1, U_2, \ldots, U_k\}$ no set of which is a total dominating set but every set U_i forms a total coalition with at least one other set U_j. For simplicity, we will call a total coalition partition a *tc-partition*. The *total coalition number* $TC(G)$ equals the maximum order of a total coalition partition of G.

Total coalitions in graphs were first studied in 2023 by Alikhani, Bakhshesh and Golmohammadi [1]. For some recent papers on coalitions in graphs see [2, 3, 5, 10, 12–14]. While many different types of dominating sets have been investigated for cubic graphs [8, 9, 18], recently, Alikhani, Golmohammadi and Konstantinova studied the coalition numbers of cubic graphs of order at most 10 [3]. In this paper, we investigate the total coalition numbers of cubic graphs of order at most 10.

This paper is organized as follows. In the next section, several known results about total coalitions are listed. In Section 3, we determine the total coalition numbers of all cubic graphs of order at most 10.

2. Preliminaries

In this section, we recall three important results which will be the key ingredients for our proofs.

Theorem 1. [1] *If G is an isolate-free graph with no full vertex and minimum degree $\delta(G)$, then $TC(G) \geq \delta(G) + 1$.*

Theorem 2. [5] *For any isolate-free graph G with maximum degree $\Delta(G)$, $TC(G) \leq \frac{(\Delta(G)+2)^2}{4}$.*

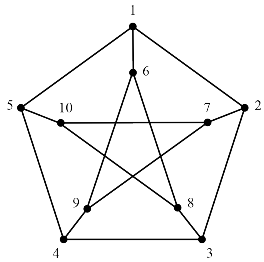

Figure 1. **Petersen graph P.**

We next present a key result, which gives us the number of total coalitions involving any set in a tc-partition of G.

Theorem 3. [1] *Let G be a graph with maximum degree $\Delta(G)$, and let π be a $TC(G)$-partition. If $X \in \pi$, then X is in at most $\Delta(G)$ total coalitions.*

Now, we present an observation with regards the Petersen graph (See Figure 1) which will be useful for our proofs.

Observation 4. (i) For the Petersen graph P, $\gamma_t(P) = 4$.

(ii) For the Petersen graph there are precisely 10 minimum total dominating sets, each one consists of the closed neighborhood of one of the 10 vertices.

(iii) Any two minimum total dominating sets of the Petersen graph have either one or two vertices in common.

(iv) Any two total dominating sets of the Petersen graph of order 5 have at most three vertices in common.

(v) Any total coalition in the Petersen graph consisting of two sets of cardinality 2 consists of consists of a pair of adjacent vertices and a pair of non-adjacent vertices.

3. Main results

In this section we determine the total coalition number of cubic graphs of order at most 10. Trivially, there is only one cubic graph of order 4, namely the complete graph K_4. It is clear that the singleton partition of K_4 is a tc-partition of order 4, and thus, $TC(K_4) = 4$. We consider cubic graphs of order 6 in the next subsection.

3.1. Cubic graphs of order 6

There are exactly two cubic graphs of order 6, which are denoted by G_1, G_2 in Figure 2 (see [3]).

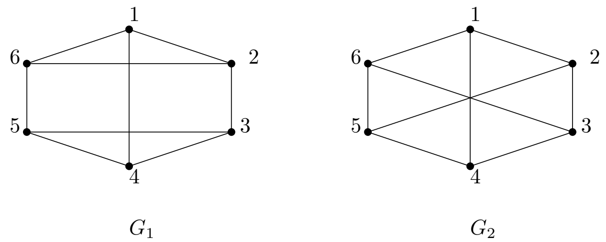

Figure 2. **Cubic graphs of order 6.**

Proposition 1. *The total coalition number of the cubic graphs of order* 6 *is* 6.

Proof. Using Theorems 1 and 2, we have $4 \leq TC(G) \leq 6$. We first compute $TC(G)$ for the graph G_1. We establish a partition of order 6. Let $\pi = \big\{V_1 = \{1,4\}, V_2 = \{2,3\}, V_3 = \{5,6\}\big\}$ be a total domatic partition of G_1, where $d_t(G_1) = 3$. Note that if we partition a minimal total dominating set into two non-empty sets, we obtain two non-total dominating sets that together form a total coalition. As a result, we can divide each non-singleton set $V_1 = \{1,4\}$, $V_2 = \{2,3\}$ and $V_3 = \{5,6\}$ into two sets such as $V_{1,1} = \{1\}$, $V_{1,2} = \{4\}$, $V_{2,1} = \{2\}$, $V_{2,2} = \{3\}$, $V_{3,1} = \{5\}$, and $V_{3,2} = \{6\}$. Each of $V_{1,1}$, $V_{2,1}$ and $V_{3,1}$ forms a total coalition with $V_{1,2}$, $V_{2,2}$ and $V_{3,2}$, respectively. Thus, $TC(G_1) \geq 6$. Moreover, as seen previously, $TC(G_1) \leq 6$, and so we have $TC(G_1) = 6$. Therefore, we can form a maximum *tc*-partition of G_1 of order 6 as follows: $\tau = \big\{V_{1,1} = \{1\}, V_{1,2} = \{4\}, V_{2,1} = \{2\}, V_{2,2} = \{3\}, V_{3,1} = \{5\}, V_{3,2} = \{6\}\big\}$. An identical argument shows that $TC(G_2) = 6$. □

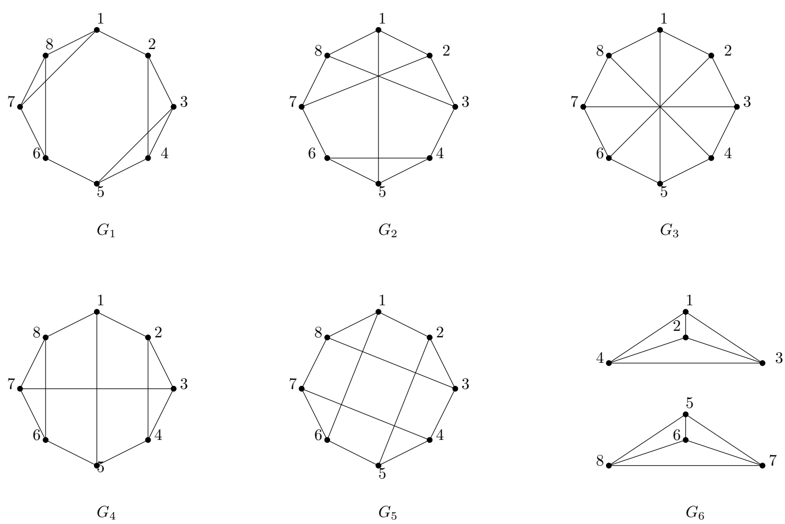

Figure 3. **Cubic graphs of order 8.**

3.2. Cubic graphs of order 8

In this subsection, we determine the total coalition numbers of cubic graphs of order 8. There are exactly 6 cubic graphs of order 8 which are denoted by $G_1, G_2, \ldots, G_6$ in Figure 3 (see [3]).

Theorem 5. *For the cubic graphs $G_1, G_2, \ldots, G_6$ of order 8 (Figure 3) we have:*

$$TC(G_i) = \begin{cases} 4 & i = 1, 5, 6 \\ 5 & i = 2, 4 \\ 6 & i = 3. \end{cases}$$

Proof. To prove Theorem 5, we partition the proof into three parts as follows.

(i) By Theorems 1 and 2, we have $4 \leq TC(G) \leq 6$. We show that there is a tc-partition of order 6 for the graph G_3. Assume that $\pi = \{V_1 = \{1, 2, 8\}, V_2 = \{3, 4, 5, 6, 7\}\}$ be a total domatic partition of the graph G_3, where $d_t(G_3) = 2$. Since any partition of a minimal total dominating set into two non-empty sets creates two non-total dominating sets whose union produces a total coalition, so the minimal total dominating set $V_1 = \{1, 2, 8\}$ can be divided into two sets such as $V_{1,1} = \{1, 2\}, V_{1,2} = \{8\}$, which together create a total coalition. Now we construct a partition τ, starting with the sets $V_{1,1} = \{1, 2\}$ and $V_{1,2} = \{8\}$. To obtain the other sets of this partition τ, let $V' = \{4, 5, 6\} \subset V_2$ be a minimal total dominating set contained in V_2. So, we shall partition it into two non-total dominating sets $V'_1 = \{5, 6\}$ and $V'_2 = \{4\}$, add these two sets to τ. The set $V'' = \{3, 7\}$ remains which is not a total dominating set, else there are at least 3 disjoint total dominating sets in G_3, a contradiction, because $d_t(G_3) = 2$. We divide the set V'' into two sets such as $V''_1 = \{3\}$, $V''_2 = \{7\}$. But $V''_1 = \{3\}$ forms a total coalition with $V_{1,1} = \{1, 2\}$ and $V''_2 = \{7\}$ forms a total coalition with $V'_1 = \{5, 6\}$. So, we can add $V''_1 = \{3\}$ and $V''_2 = \{7\}$ to τ. Hence, we have $TC(G_3) \geq 6$. As before, we know that $TC(G_3) \leq 6$. Thus, $TC(G_3) = 6$. Therefore, we can create a maximal tc-partition of G_3 of order 6 as follows.

$$\tau = \{V_{1,1} = \{1, 2\}, V_{1,2} = \{8\}, V'_1 = \{5, 6\}, V'_2 = \{4\}, V''_1 = \{3\}, V''_2 = \{7\}\}.$$

(ii) We next compute $TC(G)$ for the graph G_4. We first show there is no tc-partition of order 6 for G_4. For this purpose, assume that $\tau_1 = \{V_1, V_2, \ldots, V_6\}$ is a tc-partition of G_4. We consider two cases as follows.

Case 1. Suppose that there are 5 singleton sets in the partition τ_1. Then exactly one set of τ_1 must contain three vertices. Without loss of generality, let $|V_1| = 3$ and $|V_j| = 1$ for $2 \leq j \leq 6$. Suppose D is a total dominating set in the graph G_4. Since $|D| \geq 3$, no two singleton sets form a total coalition. It holds that each of V_j for $2 \leq j \leq 6$ must from a total coalition with V_1, contradicting Theorem 3, since by Theorem 3 V_1 is in at most $\Delta(G_4) = 3$ total coalitions. Therefore, we cannot establish a tc-partition of order 6 and so $TC(G_4) < 6$.

Case 2. Assume that there are 4 singleton sets in the partition and it has exactly two sets of τ_1 must contain two vertices. Without loss of generality, let $|V_1| = |V_2| = 2$ and $|V_j| = 1$ for $3 \leq j \leq 6$. Suppose D is a total dominating set in the graph G_4. Since $|D| \geq 3$, no two singleton sets form a total coalition. Now every set V_j for $3 \leq j \leq 6$ must produce a total coalition with V_1 or V_2. It can be seen that there are only two minimal total dominating sets of order 3, namely $U = \{1, 2, 8\}$ and $W = \{4, 5, 6\}$. So, each of V_1 and V_2 can be in a total coalition with only one singleton set. Therefore, neither of the remaining two singleton sets can form a total coalition with V_1 or V_2. This is a contradiction. It follows that the graph G_4 has no tc-partition of order 6. Thus, $TC(G_4) < 6$.

Now, we shall construct a tc-partition of order 5 for the graph G_4. Let $\pi' = \big\{V_1 = \{1, 2, 8\}, V_2 = \{3, 4, 5, 6, 7\}\big\}$ be a total domatic partition of G_4, where $d_t(G_4) = 2$. It is worth mentioning that by splitting a minimal total dominating set into two non-empty sets, we obtain two non-total dominating sets, which combine to form a total coalition. Thus, we can partition the minimal total dominating set $V_1 = \{1, 2, 8\}$ into two sets $V_{1,1} = \{1, 2\}$ and $V_{1,2} = \{8\}$, which form a total coalition. Now we create a partition τ' of sets and put the sets $V_{1,1} = \{1, 2\}$ and $V_{1,2} = \{8\}$ in this partition. Since $V_2' = \{4, 5, 6\} \subset V_2$ is a minimal total dominating, so to obtain the other sets of partition τ' we can partition it into two non-total dominating sets $V_{2,1}' = \{4, 5\}$ and $V_{2,2}' = \{6\}$, and add these two sets to τ'. The set $V'' = \{3, 7\}$ remains which is not a total dominating set, else there are at least 3 disjoint total dominating sets in G_4, a contradiction, because $d_t(G_4) = 2$. The set V'' produces a total coalition with the set $V_{2,1}' = \{4, 5\}$, so we can add V'' to $\mathcal{T}'$. So, we observe that $TC(G_4) \geq 5$. Moreover, as before, we have $TC(G_4) \leq 5$. Then, $TC(G_4) = 5$. It follows that we have a maximal tc-partition of G_4 of order 5 as follows.

$$\tau' = \big\{V_{1,1} = \{1, 2\}, V_{1,2} = \{8\}, V_{2,1}' = \{4, 5\}, V_{2,2}' = \{6\}, V'' = \{3, 7\}\big\}.$$

Now, we determine $TC(G)$ for the graph G_2. It can be seen that G_2 has six minimum total dominating sets such as {1,2,5}, {1,5,8}, {2,3,4}, {2,6,7}, {3,4,8} and {6,7,8}. Now, we may assume that the following three pairs of vertices are the only pairs of vertices that can form a total coalition with two singleton sets: (i) $V_1 = \{1, 5\}$ can appear with each of $V_4 = \{2\}$ and $V_5 = \{8\}$; (ii) $V_2 = \{3, 4\}$ can appear with each of $V_4 = \{2\}$ and $V_5 = \{8\}$, and finally (iii) $V_3 = \{6, 7\}$ can appear with each of $V_4 = \{2\}$ and $V_5 = \{8\}$. From this it follows that G_2 does not have a tc-partition of order 6. The only two possible sizes of a tc-partition of order 6 are 3, 1, 1, 1, 1, 1, which is not possible because of Theorem 3, and 2, 2, 1, 1, 1, 1, which is not possible because no set of four singleton sets exists which can combine with either of two sets of size two.

However, $\tau'' = \big\{V_1 = \{1, 5\}, V_2 = \{3, 4\}, V_3 = \{6, 7\}, V_4 = \{2\}, V_5 = \{8\}\big\}$ is a tc-partition of G_2 of order 5; thus, $TC(G_2) = 5$.

(iii) We next determine $TC(G)$ for the graph G_1. From our previous discussions, we can show that there is no tc-partition of order 6 for G_1. We next show there is no tc-partition of order 5 for G_1. Assume that G_1 has a tc-partition $\tau_2 = \{V_1, V_2, \ldots, V_5\}$. We consider the following three cases.

Case 1. Assume that there are 4 singleton sets in τ_2 and exactly one set of τ_2 contains four vertices. Without loss of generality, suppose that $|V_1| = 4$ and $|V_j| = 1$ for $2 \leq j \leq 5$. Let D be a total dominating set in the graph G_1. Since $|D| \geq 4$, no two singleton sets form a total coalition. It follows that each of V_j for $2 \leq j \leq 5$ must be in a total coalition with V_1. This contradicts Theorem 3. Then, there is no tc-partition with order 5. Hence, $TC(G_1) < 5$.

Case 2. Suppose that there are 2 singleton sets in the partition and then exactly three sets of τ_2 must contain two vertices. Without loss of generality, we may assume that $|V_1| = |V_2| = |V_3| = 2$ and $|V_j| = 1$ for $4 \leq j \leq 5$. Let D be a total dominating set in the graph G_1. Since $|D| \geq 4$, no two singleton sets produce a total coalition. Moreover, neither V_4 nor V_5 produce a total coalition with V_1, V_2, or V_3, this is a contradiction. Thus, we cannot create a tc-partition of order 5 in this case. Then, $TC(G_1) < 5$.

Case 3. Suppose that the tc-partition τ_2 contains 3 singleton sets, one set with two vertices and one set with three vertices. Without loss of generality, we may assume that $|V_1| = 3, |V_2| = 2$ and $|V_j| = 1$ for $3 \leq j \leq 5$. Let D be a total dominating set in the graph G_1. Since $|D| \geq 4$, no two singleton sets produce a total coalition. Moreover, no singleton set can produce a total coalition V_2. It holds therefore that each of V_i for $2 \leq i \leq 5$ must from a total coalition with V_1, contradicting Theorem 3. Consequently, there is no tc-partition of G_1 of order 5. Thus, $TC(G_1) < 5$.

Based on all cases, we conclude that $TC(G_1) \leq 4$. Furthermore, from our previous discussions, it is straightforward to verify that $TC(G_1) \geq 4$. Hence, $TC(G_1) = 4$. Now, we establish a tc-partition of order 4 such as the following. Note that V_1 and V_4 produce total coalitions with each of V_2 and V_3.

$\tau''' = \{V_1 = \{1, 7\}, V_2 = \{2, 4\}, V_3 = \{3, 5\}, V_4 = \{6, 8\}\}$.

The argument used above for G_1 can also be applied to prove that $TC(G_5) = TC(G_6) = 4$.

□

3.3. Cubic graphs of order 10

In this subsection, we consider the cubic graphs of order 10 and study their total coalition numbers. There are exactly 21 cubic graphs of order 10, denoted by $G_1, G_2, \ldots, G_{21}$ in Figure 4 (see [3, 4]). In particular, the graph G_{17} is isomorphic to the Petersen graph P.

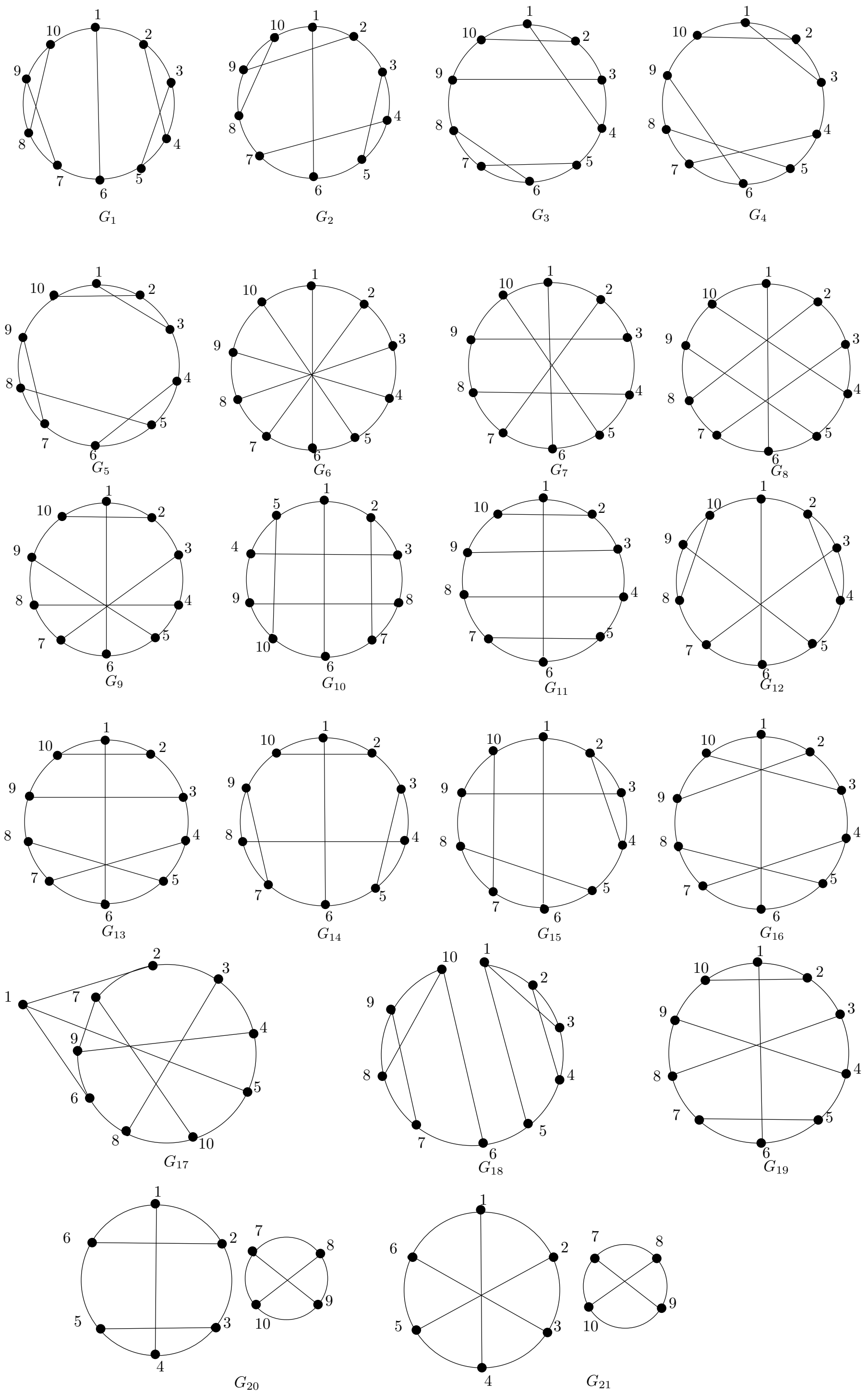

Figure 4. **Cubic graphs of order** 10.

Theorem 6. *For the cubic graphs $G_1, G_2, \ldots, G_{21}$ of order* 10 *(Figure 4) we have:*

$$TC(G_i) = \begin{cases} 4 & i = 12, 17 \\ 5 & i = 1, 8, 9, 14, 19, 20 \\ 6 & \text{otherwise.} \end{cases}$$

Proof. In order to prove Theorem 6, we shall divide the proof into three parts as follows.

(i) To show that $TC(G_i) = 4$ for $i = 12, 17$, let us pick the graph G_{17}. Now, we first need to prove that $TC(G_{17}) < 6$. Since the graph G_{17} and the Petersen graph P are isomorphic, we shall expose that none of the following cases are possible for the Petersen graph. Assume that P has a tc-partition $\tau_3 = \{V_1, V_2, \ldots, V_6\}$. Now, we consider the following cases.

Case 1. We may assume that there are five singleton sets and exactly one set of τ_3 must consist of five vertices. Without loss of generality, let $|V_1| = 5$ and $|V_j| = 1$ for $2 \leq j \leq 6$. Since P has no total dominating set with less than four vertices, so no two singleton sets produce a total coalition. It follows that each of V_j for $2 \leq j \leq 6$ must from a total coalition with V_1, contradicting Theorem 3, since by Theorem 3 V_1 is in at most $\Delta(P) = 3$ total coalitions. So, we cannot create a tc-partition of order 6. Hence, $TC(P) < 6$.

Case 2. We suppose $|V_1| = 4$, $|V_2| = 2$ and $|V_j| = 1$ for $3 \leq j \leq 6$. Since $\gamma_t(P) = 4$, so no two singleton sets can from a total coalition. Moreover, no singleton set can be in a total coalition with V_2. It holds that each of V_i for $2 \leq i \leq 6$ must be in a total coalition with V_1. This contradicts Theorem 3. Then, there is no tc-partition of order 6. So, $TC(P) < 6$.

Case 3. Let $|V_1| = |V_2| = 3$ and $|V_j| = 1$ for $3 \leq j \leq 6$. Since each of singleton sets must form a total coalition either of the two sets of cardinality 3, so it can be seen that at least two minimum total dominating sets have three vertices in common, contradicting Part (iii) of Observation 4. That means there is no tc-partition of order 6. Then, $TC(P) < 6$.

Case 4. Assume that $|V_1| = 3$, $|V_2| = |V_3| = 2$ and $|V_j| = 1$ for $4 \leq j \leq 6$. As before, we know that no two singleton sets can produce a total coalition. Furthermore, no singleton set forms a total coalition with V_2 and V_3. Now, we consider the following subcases.

Subcase 4.1. Let V_2 and V_3 do not form a total coalition. Therefore, each of V_i for $2 \leq i \leq 6$ must form a total coalition with V_1, contradicting Theorem 3.

Subcase 4.2.. Suppose that V_2 and V_3 form a total coalition. Since each of singleton sets must form a total coalition with V_1, so it holds that three minimum total dominating sets have three vertices in common, contradicting Part (iii) of Observation 4.

Based on both subcases, we conclude that P has no tc-partition of order 6. Hence, $TC(P) < 6$.

Case 5. Let $|V_j| = 2$ for $1 \leq j \leq 4$ and $|V_5| = |V_6| = 1$. By Part (i) of Observation 4, we have $\gamma_t(P) = 4$. So, no union of any two sets has cardinality 4. Consequently, there is no tc-partition of order 6. Then, $TC(P) < 6$.

Now, we demonstrate that the Petersen graph has no tc-partition of order 5. To achieve this aim, we show that none of the following cases are possible. Let P have a tc-partition $\tau_4 = \{V_1, V_2, \ldots, V_5\}$. Now, we consider the following subcases.

Subcase 5.1. Let $|V_1| = 6$ and $|V_j| = 1$ for $2 \leq j \leq 5$. Suppose D is a total dominating set in the graph P. Since $|D| \geq 4$, so no two singleton sets form a total coalition. It holds that each of V_j for $2 \leq j \leq 5$ must from a total coalition with V_1, contradicting Theorem 3, since by Theorem 3 V_1 is in at most $\Delta(P) = 3$ total coalitions. Thus, we cannot establish a tc-partition of order 5. Then, $TC(P) < 5$.

Subcase 5.2. Assume that $|V_1| = 5$, $|V_2| = 2$ and $|V_j| = 1$ for $3 \leq j \leq 5$. As seen early, no two singleton sets can produce a total coalition. Furthermore, no singleton set forms a total coalition with V_2. Thus, every set V_i for $2 \leq i \leq 5$ must be in a total coalition with V_1. This contradicts Theorem 3. It follows that P has no tc-partition of order 5. Hence, $TC(P) < 5$.

Subcase 5.3. Let $|V_1| = 4$, $|V_2| = 3$ and $|V_j| = 1$ for $3 \leq j \leq 5$. From our previous discussions, each of singleton sets must form a total coalition with V_1 or V_2. This would give rise to two minimum total dominating sets having three vertices in common, contradicting Part (iii) of Observation 4 or there would be two total dominating sets of order 5 having four vertices in common, contradicting Part (iv) of Observation 4. So, we observe that there is no tc-partition of order 5. Then, $TC(P) < 5$.

Subcase 5.4. Let $|V_1| = 4$, $|V_2| = |V_3| = 2$ and $|V_j| = 1$ for $4 \leq j \leq 5$. By Part (i) of Observation 4, we have $\gamma_t(P) = 4$. Thus, no union of any two singleton sets has cardinality 4. Also, no singleton set can produce a total coalition with V_2 and V_3. Now, we consider the following situations.

5.4.1. We may assume that V_2 and V_3 do not form a total coalition. Therefore, each of V_i for $2 \leq i \leq 5$ must form a total coalition with V_1, contradicting Theorem 3.

5.4.2. Let V_2 and V_3 form a total coalition. Since each of V_j for $4 \leq j \leq 5$ must form a total coalition with V_1, so it follows that two total dominating sets of order 5 have four vertices in common, contradicting Part (iv) of Observation 4.

Based on both subcases, we conclude that there is no tc-partition of order 5. So, $TC(P) < 5$.

Subcase 5.5. Suppose that $|V_1| = |V_2| = 3$, $|V_3| = 2$ and $|V_j| = 1$ for $4 \leq j \leq 5$. As before, each of V_i for $3 \leq i \leq 5$ must be in a total coalition with V_1 or V_2. We may assume that there are two subcases as follows.

5.5.1. Without less of generality, let V_1 and V_4 form a total coalition and V_1 and V_5 form a different total coalition, but this contradicts Part (iii) of Observation 4, since there would be two different minimum total dominating sets have three vertices in common.

5.5.2. V_1 and V_4 form a total coalition and V_2 and V_5 form a total coalition, but this would create two minimum dominating sets having no vertices in common, again contradicting Part (iii) of Observation 4.

Based on both subcases, there is no tc-partition of order 5. Hence, $TC(P) < 5$.

Subcase 5.6. Assume that $|V_1| = 3$, $|V_2| = 1$ and $|V_j| = 2$ for $3 \leq j \leq 5$. By Part (iii) of Observation 4, no two doubleton sets can form a total coalition, else we would have two minimum total dominating sets having no vertices in common, a contradiction. Thus, each of V_j for $3 \leq j \leq 5$ must form a total coalition with V_1. Moreover, from previous our discussions, V_2 must from total coalition with V_1. Therefore, each of V_i for $2 \leq i \leq 5$ must be in a total coalition with V_1. This contradicts Theorem 3. So, we observe that P has no tc-partition of order 5. Then, $TC(P) < 5$.

Subcase 5.7. Let $|V_j| = 2$ for $1 \leq j \leq 5$. In this case each set having two vertices must form a minimum total dominating set with another set having two vertices. We consider two subcases as follows.

5.7.1. Without less of generality, let each of the pairs $V_1 \cup V_2$ and $V_3 \cup V_4$ be total coalition partners but this is a contradiction since any two minimum total dominating sets must have a non-empty intersection.

5.7.2. Each of V_2, V_3, V_4, and V_5 forms a total coalition with V_1. Thus, from Part (v) of Observation 4, the pairs in V_2, V_3, V_4, and V_5 must either all be adjacent pairs or all be non-adjacent pairs. If V_1 consists of an adjacent pair, then it can only be in two minimum total dominating sets with non-adjacent pairs, which contradicts the fact that each non-adjacent pair in V_2, V_3, V_4, and V_5 must be in a minimum total dominating set with the adjacent pair in V_1. This is not possible. Conversely, if V_1 consists of a non-adjacent pair, then if this pair is in two minimum total dominating pairs with adjacent pairs, then the graph would have a 4-cycle, but a 4-cycle does not exist in the Petersen graph.

Based on both subcases, the Petresen graph P has no tc-partition of order 5. Then, $TC(P) < 5$.

To complete part (i), since the Petersen graph P has neither a tc-partition of order 5, nor a tc-partition of order 6, so $TC(P) = 4$. It holds that $TC(G_{17}) = 4$. Now, we construct a tc-partition of order 4 for the graph G_{17} as follows. Note that V_1 forms a total coalition with each of V_2 and V_3, and also V_3 forms a total coalition with V_4.

$$\tau = \big\{V_1 = \{1, 2, 3\}, V_2 = \{4, 5\}, V_3 = \{6, 7, 8\}, V_4 = \{9, 10\}\big\}.$$

Using the same approach, we can show that $TC(G_{12}) = 4$.

(ii) An identical argument to the one used in part (i) shows that $TC(G_i) < 6$, where $i \in \{1, 8, 9, 14, 19, 20\}$. Now, it suffices to determine the total coalition number for any graph belonging to $\{G_1, G_8, G_9, G_{14}, G_{19}, G_{20}\}$. Without loss of generality, therefore, we determine $TC(G_1)$.

We shall create a tc-partition of G_1 of order 5 and show that this partition is a maximum tc-partition. We first create a total domatic partition. We consider the total domatic partition $\pi = \big\{V_1 = \{1, 2, 6, 7\}, V_2 = \{3, 4, 5, 8, 9, 10\}\big\}$ of G_1, since $\gamma_t(G_1) = 4$ and therefore $d_t(G_1) = 2$. Regarding any division of a minimal total dominating set into two non-empty sets creates two non-total dominating sets whose union produces a total coalition, then it is possible to divide the minimal total dominating set $V_1 = \{1, 2, 6, 7\}$ into two sets $V_{1,1} = \{1, 6\}$ and $V_{1,2} = \{2, 7\}$, which together form a total coalition. Now we create a tc-partition τ containing the sets $V_{1,1} = \{1, 6\}$ and $V_{1,2} = \{2, 7\}$. To obtain the other sets of partition τ, say $V_2' = \{3, 5, 8, 10\} \subset V_2$ be a minimal total dominating set contained in V_2. We can partition this set into two non-total dominating sets $V_{2,1}' = \{3, 5\}$ and $V_{2,2}' = \{8, 10\}$ and add these two sets to τ. The set $V'' = \{4, 9\}$ remains which is not a total dominating set, else there are at least 3 disjoint total dominating sets in G_1, a contradiction, because $d_t(G_1) = 2$. The set V'' forms a total coalition with the set $V_{1,2} = \{2, 7\}$, so we can add V'' to τ. Then, we observe that $TC(G_1) \geq 5$. Furthermore, as before, we have $TC(G_1) \leq 5$. Hence, $TC(G_1) = 5$. And finally we can establish a maximal tc-partition of G of order 5 as follows.

$$\tau = \big\{V_{1,1} = \{1, 6\}, V_{1,2} = \{2, 7\}, V_{2,1}' = \{3, 5\}, V_{2,2}' = \{8, 10\}, V'' = \{4, 9\}\big\}.$$

Using the argument used to obtain the total coalition number of the graph G_1, we can show that the total coalition numbers of the other graphs belonging to this set of cubic graphs is 5.

(iii) To complete the proof, we proceed to show that $TC(G_i) = 6$ where $i \in \{2, 3, 4, 5, 6, 7, 10, 11, 13, 15, 16, 18, 21\}$. Using Theorems 1 and 2, we have $4 \leq TC(G) \leq 6$. We begin by constructing a tc-partition of G_6 of order 6. Let $\pi' = \big\{V_1 = \{1, 2, 3, 4\}, V_2 = \{5, 6, 7, 8, 9, 10\}\big\}$ be a total domatic partition of G_6, where $d_t(G_6) = 2$. As before, we know that by dividing a minimal total dominating set with more than one element into two non-empty sets, we obtain two non-total dominating sets that together form a total coalition. Thus, we can partition the minimal total dominating set $V_1 = \{1, 2, 3, 4\}$ into two sets $V_{1,1} = \{1, 2, 3\}$ and $V_{1,2} = \{4\}$. Now we construct a partition τ' and put the sets $V_{1,1} = \{1, 2, 3\}$ and $V_{1,2} = \{4\}$ in this partition. For the other sets of

τ', we may consider the minimal total dominating set $V_2' = \{6,7,8,9\} \subset V_2$. Thus, we can partition it into two non-total dominating sets $V_{2,1}' = \{6,7,8\}$ and $V_{2,2}' = \{9\}$, and add these two sets to τ'. The set $V'' = \{5,10\}$ remains, which is not a total dominating set, else there are at least 3 disjoint total dominating sets in G_6, a contradiction, because $d_t(G_6) = 2$. If we divide the set V'' into two parts such as $V_1'' = \{5\}$ and $V_2'' = \{10\}$, it can be seen that $V_1'' = \{5\}$ can form a total coalition with $V_{2,1}' = \{6,7,8\}$, and also $V_2'' = \{10\}$ can form a total coalition with $V_{1,1} = \{1,2,3\}$. Hence, we have $TC(G_6) \geq 6$. Moreover, as before, we have $TC(G_6) \leq 6$. Then, $TC(G_6) = 6$. Consequently, we have a maximal tc-partition of G of order 6 as follows.

$\tau' = \big\{V_{1,1} = \{1,2,3\}, V_{1,2} = \{4\}, V_{2,1}' = \{6,7,8\}, V_{2,2}' = \{9\}, V_1'' = \{5\}, V_2'' = \{10\}\big\}$.

Note that the total coalition number for other graphs in this section can be obtained using the same approach. Consequently, we have $TC(G_i) = 6$ where $i \in \{2,3,4,5,7,10,11,13,15,16,18,21\}$. □

4. Concluding Remarks and Open Problems

In this paper, we have determined the total coalition number of cubic graphs of order at most 10. We present the following open questions.

1. Characterize all connected cubic graphs G with $TC(G) = C(G)$.
2. Let $GP(n,k)$ be a generalized Petersen graph. Compute the total coalition number of $GP(n,k)$.
3. There are 85 connected cubic graphs of order 12. Compute the total coalition number of connected cubic graphs of order 12.

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