Research Article



Some results on the complete sigraphs with exactly three non-negative eigenvalues

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Abstract: Let (K_n, H^-) be a complete sigraph of order *n* whose negative edges induce a subgraph *H*. In this paper, we characterize (K_n, H^-) with exactly 3 nonnegative eigenvalues, where *H* is a non-spanning two-cyclic subgraph of K_n .

Keywords: sigraph, complete graph, two-cyclic graph, non-negative eigenvalues.

AMS Subject classification: 05C22, 05C50, 15A18

1. Introduction

Let G be a simple graph. As usual, V(G) and E(G) denote the set of vertices and the set of edges of G, respectively. If $V(G) = \{v_1, \ldots, v_n\}$, then n = |V(G)| is called the order of G. The set of all neighbors of v_i in G is denoted by $N(v_i)$. A pendant vertex is a vertex of degree one. The girth of G, denoted by gr(G), is the order of the shortest cycle contained in G. A graph H is a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph H with $|V(H)| \neq |V(G)|$ is said to be a non-spanning subgraph (briefly, ns-subgraph) of G. Also, a subgraph H of G is induced if E(H)contains all edges of G that have both ends in V(H). Let K_n , P_n and C_n denote the complete graph, the path and the cycle of order n, respectively. A two-cyclic graph is a connected graph with exactly two cycles.

A pair $\Gamma = (G, \sigma)$ is said to be a signed graph (called also sigraph), where $\sigma : E(G) \rightarrow \{-,+\}$ is a function defined on E(G). The graph G is called the underlying graph of

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Γ, and σ is called the *signature*. We use (K_n, H^-) to denote a complete sigraph of order *n* whose negative edges induce a subgraph *H*. If *H* is a disjoint union of two graphs H_1 and H_2 , then we denote (K_n, H^-) by $(K_n, H_1^- \cup H_2^-)$. Let $A(G) = (a_{ij})$ be the adjacency matrix of *G*. The *adjacency matrix* of a sigraph $\Gamma = (G, \sigma)$ is a matrix $A(\Gamma) = (a_{ij}^{\sigma})$, where $a_{ij}^{\sigma} = \sigma(v_i v_j) a_{ij}$. The nullity of a graph *G*, denoted by n(G), is the nullity of A(G). By $\varphi(A)$, we denote the characteristic polynomial of a square matrix *A*. If Γ is a sigraph, then we use $\varphi(\Gamma, \lambda)$ instead of $\varphi(A(\Gamma))$. The spectrum of $A(\Gamma)$ is referred to as the spectrum of Γ . The class of all sigraphs having exactly $r \geq 1$ non-negative eigenvalues (including their multiplicities) is denoted by $\mathcal{L}(r)$. Let $\lambda_1 > \cdots > \lambda_s$ be the distinct eigenvalues of a sigraph Γ with the corresponding multiplicities $m_{\Gamma}(\lambda_1), \ldots, m_{\Gamma}(\lambda_s)$. The spectrum of Γ is denoted by

Spec
$$\Gamma = \begin{pmatrix} \lambda_1 & \dots & \lambda_s \\ m_{\Gamma}(\lambda_1) & \dots & m_{\Gamma}(\lambda_s) \end{pmatrix}$$
.

For some recent results on the spectra of sigraphs see [3, 5–7, 14, 15].

Let $\Gamma_1 = (G, \sigma)$ be a sigraph and $S \subset V(\Gamma_1)$. If Γ_2 is the sigraph obtained from Γ_1 by reversing the signs of all edges between S and $V(\Gamma_1) \setminus S$, then two graphs Γ_1 and Γ_2 are called *switching equivalent*, and denoted by $\Gamma_1 \sim \Gamma_2$. If two sigraphs Γ_1 and Γ_2 are switching equivalent, then they are cospectral, see [17].

Characterizing graphs with a few non-negative eigenvalues has received a great deal of attention in literature. In [11–13], the authors characterized all graphs with exactly one or two non-negative eigenvalues. The authors in [9] determined all of the sigraphs (K_n, σ) belonging to $\mathcal{L}(1)$ or $\mathcal{L}(2)$. Also, in [9, 10], they provided a characterization of $(K_n, H^-) \in \mathcal{L}(3)$, where H is either a non-spanning tree or a unicyclic ns-subgraph of K_n . In this paper, we characterize $(K_n, H^-) \in \mathcal{L}(3)$, where H is a two-cyclic ns-subgraph of K_n . After our Theorem 4, the next natural step toward the complete structural characterization of complete sigraph in $\mathcal{L}(3)$ is to detect all (K_n, H^-) in that set with H being a θ -graph. We plan to attack this problem in a future paper.

2. Preliminaries

To prove the main theorem, we need the following results.

Theorem 1. (Interlacing Theorem [8, Theorem 1.3.11]) Let Γ be a sigraph with n vertices and eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$, and let Γ' be an induced subgraph of Γ of order m. If $\lambda'_1 \geq \cdots \geq \lambda'_m$ are the eigenvalues of Γ' , then

$$\lambda_{n-m+i} \le \lambda_i' \le \lambda_i \quad (i = 1, \dots, m).$$

Theorem 2. [1, Corollary 1] Let $\Gamma = (K_n, H^-)$ be a complete sigraph and |V(H)| = t < n. Then

$$\varphi(\Gamma,\lambda) = (\lambda+1)^{n-t-1}\varphi\left(\begin{bmatrix} A(K_t,H^-) & (n-t)J_{t\times 1} \\ J_{1\times t} & n-t-1 \end{bmatrix}\right),$$

and so $m_{\Gamma}(-1) \geq n-t-1$.

Theorem 3. [2, Theorem 3] Let $\Gamma = (K_n, H^-)$ be a complete sigraph and |V(H)| = t < n. Then $m_{\Gamma}(-1) = n - t - 1 + n(H)$.

Remark 1. Let H be a connected graph and consider the following equivalence relation on the vertex set V(H): two vertices $v_i, v_j \in V(H)$ are related if and only if $N(v_i) = N(v_j)$. The corresponding quotient graph C(H) is called the canonical graph of H. Let $n_+(H)$ and $n_-(H)$ denote the numbers of positive and negative eigenvalues of H, respectively. By [16, Proposition 1], we know that $n_+(H) = n_+(C(H))$ and $n_-(H) = n_-(C(H))$. Thus n(H) - n(C(H)) = |V(H)| - |V(C(H))|. If $\Gamma = (K_n, H^-)$ and |V(H)| < n, then by Theorem 3, we conclude that

$$m_{\Gamma}(-1) = n - 1 + n(C(H)) - |V(C(H))|.$$

3. Main result

Let H be a two-cyclic ns-subgraph of K_n . In this section, we characterize $(K_n, H^-) \in \mathcal{L}(3)$. First, we have the next lemma.

Lemma 1. Let H be a two-cyclic ns-subgraph of K_n , and let C_g be a cycle of H. If $(K_n, H^-) \in \mathcal{L}(3)$, then $g \in \{3, 4\}$.

Proof. If $g \ge 5$, then (K_n, H^-) contains $(K_7, P_4^- \cup K_2^-)$ as an induced subgraph. By a computer search, one can see that

Spec
$$(K_7, P_4^- \cup K_2^-) = \begin{pmatrix} 4.01 & 2.24 & 1 & 0.09 & -1.58 & -2.24 & -3.52 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$
.

Note that the values in the spectrum are approximate. So $(K_7, P_4^- \cup K_2^-) \in \mathcal{L}(4)$ and hence by Theorem 1, we deduce that $(K_n, H^-) \in \mathcal{L}(r)$ for some $r \geq 4$, a contradiction.

Let $q \ge 1$ be an integer. Let H(q) be the graph with q + 7 vertices obtained by two quadrangles sharing a vertex u_1 , by attaching q pendant vertices to u_1 . Note that $C(H(q)) \cong T$, see Figure 1.



Figure 1. The two-cyclic graph H(q) and its canonical graph T.

Now, we prove the main result of the paper.

Theorem 4. Let (K_n, σ) be a complete sigraph and $(K_n, \sigma) \sim (K_n, H^-)$, where H is a two-cyclic ns-subgraph of K_n . Then $(K_n, H^-) \in \mathcal{L}(3)$ if and only if one of the next assertions holds:

- 1. $H \cong Q_1$ for n = 7 or $H \cong Q_2$ for n > 7 or $H \cong Q_3$ for n > 8, see Fig. 2.
- 2. $H \cong H(1)$ for $9 \le n \le 12$ or $H \cong H(2)$ for n = 10.



Figure 2. The two-cyclic graphs Q_1 , Q_2 and Q_3 .

Proof. First we consider the following cases:

1. Let $H \cong Q_1$, depicted in Figure 2, and n = 7. By a computer search, we find the spectrum of (K_7, Q_1^-) as follows:

Spec
$$(K_7, Q_1^-) = \begin{pmatrix} 3.86 & 2.33 & 1 & -0.02 & -1 & -2.54 & -3.63 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$
.

Hence $(K_7, Q_1^-) \in \mathcal{L}(3)$.

Now, let $H \cong Q_2$, shown in Figure 2, and $\Gamma = (K_n, Q_2^-)$, where n > 7. Since $n(Q_2) = 3$, by Theorem 3, we have $m_{\Gamma}(-1) = n - 5$. The following are the eigenvalues of (K_8, Q_2^-) :

Spec
$$(K_8, Q_2^-) = \begin{pmatrix} 4.46 & 3 & 1.83 & -1 & -2.46 & -3.83 \\ 1 & 1 & 1 & 3 & 1 & 1 \end{pmatrix}$$
.

So (K_8, Q_2^-) has three positive eigenvalues and two negative eigenvalues smaller than -1. The sigraph $\Gamma = (K_n, Q_2^-)$ contains (K_8, Q_2^-) as an induced subgraph, for each $n \ge 8$. By Theorem 1, we conclude that $\Gamma = (K_n, Q_2^-) \in \mathcal{L}(3)$, for each n > 7.

Next, suppose that $H \cong Q_3$ (shown in Figure 2) and $\Gamma = (K_n, Q_3^-)$, where n > 8. By Theorem 2, we find that

$$\varphi(\Gamma,\lambda) = (\lambda+1)^{n-9}\varphi\left(\begin{bmatrix} A(K_8,Q_3^-) & (n-8)J_{8\times 1} \\ J_{1\times 8} & n-9 \end{bmatrix}\right) = (\lambda+1)^{n-7}g(\lambda),$$

where $g(\lambda) = \lambda^7 + (7-n)\lambda^6 + (21-6n)\lambda^5 + (21n-133)\lambda^4 + (124n-829)\lambda^3 + (805-119n)\lambda^2 + (3751-502n)\lambda + 217-29n$. It is easy to check that $g(-1) \neq 0$ and also g(0) = 217 - 29n < 0, for each n > 8. On the other hand, we have Spec (K_9, Q_3^-) as follows:

$$\begin{pmatrix} 4.46 & 3.69 & 2.56 & -0.06 & -1 & -1.56 & -2.46 & -4.63 \\ 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 \end{pmatrix}.$$

Thus (K_9, Q_3^-) contains three positive eigenvalues and three negative eigenvalues smaller than -1. Since (K_9, Q_3^-) is an induced subgraph of $\Gamma = (K_n, Q_3^-)$, by Theorem 1, we deduce that $\Gamma \in \mathcal{L}(3)$ or $\Gamma \in \mathcal{L}(4)$. If $\lambda_1, \ldots, \lambda_7$ are the roots of $g(\lambda)$, then $g(0) = -\prod_{i=1}^7 \lambda_i$. Now, g(0) < 0 yields that $\Gamma = (K_n, Q_3^-) \in \mathcal{L}(3)$, for each n > 8.

2. Let $H \cong H(q)$ and $\Gamma = (K_n, H(q)^-)$, where m = n - (q+7) > 0. We have $C(H(q)) \cong T$ (cf. Figure 1) and n(T) = 0. By Remark 1, we find that

 $m_{\Gamma}(-1) = n - 1 + n(T) - |V(T)| = n - 7.$

The spectrum of $(K_9, H(1)^-)$ is as follows:

$$\begin{pmatrix} 5.62 & 3.14 & 1.83 & -0.22 & -1 & -1.95 & -2.58 & -3.83 \\ 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 \end{pmatrix}.$$

Hence $(K_9, H(1)^-)$ has 3 positive eigenvalues and 3 negative eigenvalues smaller than -1. Since Γ has $(K_9, H(1)^-)$ as an induced subgraph, by Theorem 1, $\Gamma \in \mathcal{L}(3)$ or $\Gamma \in \mathcal{L}(4)$. Now, we compute $\varphi(A(\Gamma))$. Suppose that V(H(q)) is partitioned into the parts $X_1 = \{u_1\}, X_2 = \{u_2, v_2\}, X_3 = \{u_3, v_3\}, X_4 = \{u_4\},$ and $X_5 = \{u_5\}$, see Fig. 1. Let X_6 be the set of pendant vertices of H(q) and $X_7 = V(K_n) \setminus V(H(q))$. Note that $|X_6| = q$ and $|X_7| = m = n - q - 7$. If B is the quotient matrix of $A(\Gamma)$ related to the equitable partition $\Box = \{X_1, \ldots, X_7\}$ of $V(\Gamma)$, then

$$B = \begin{bmatrix} 0 & -2 & -2 & 1 & 1 & -q & m \\ -1 & 1 & 2 & -1 & 1 & q & m \\ -1 & 2 & 1 & 1 & -1 & q & m \\ 1 & -2 & 2 & 0 & 1 & q & m \\ 1 & 2 & -2 & 1 & 0 & q & m \\ -1 & 2 & 2 & 1 & 1 & q - 1 & m \\ 1 & 2 & 2 & 1 & 1 & q & m -1 \end{bmatrix}$$

If $h(\lambda) = \varphi(B)$, then $h(\lambda) = \lambda^7 + (7-n)\lambda^6 + (21-6n)\lambda^5 + (17m+9q+4mq-6)\lambda^4 + (108m+76q+16mq+87)\lambda^3 + (q-15m-40mq+44)\lambda^2 + (-262m-166q-112mq-99)\lambda + 196mq - 105q - 161m - 70$. By [4, Lemma 2.3.1], $h(\lambda)$ divides $\varphi(A(\Gamma))$. A direct check shows that if h(-1) = 0, then mq = 0, a contradiction. Hence, $\varphi(\Gamma, \lambda) = (\lambda + 1)^{n-7}h(\lambda)$. Since h(0) = (196q - 161)m - 105q - 70, so if $m < \frac{105q+70}{196q-161}$, then $\Gamma = (K_n, H(q)^-) \in \mathcal{L}(3)$. Otherwise, $\Gamma = (K_n, H(q)^-) \in \mathcal{L}(4)$. The function $f(q) := \frac{105q+70}{196q-161}$ is strictly decreasing for $q \ge 1$. Moreover, f(1) = 5 and f(3) < 1 < f(2) < 2. This means that only H(1) and H(2) can possibly satisfy the conditions m < f(q) and m = n - (q+7) > 0. Hence, $(K_n, H(1)^-) \in \mathcal{L}(3)$ for $9 \le n \le 12$, and $(K_n, H(2)^-) \in \mathcal{L}(3)$ for n = 10.



Figure 3. The two-cyclic graphs G_1, G_2, G_3 and G_4

Conversely, assume that $\Gamma = (K_n, H^-) \in \mathcal{L}(3)$, where H is a two-cyclic ns-subgraph of K_n . Since two sigraphs $(K_5, C_3^- \cup K_2^-)$ and $(K_7, P_4^- \cup K_2^-)$ belong to the class $\mathcal{L}(4)$, so they cannot appear as induced subgraphs of Γ . By Lemma 1, H has no cycle of length greater than 4. First, suppose that gr(H) = 3. It is not difficult to verify that $H \cong Q_1$ or one of the graphs G_1, G_2, G_3 (shown in Figure 3) is an induced subgraph of H, for otherwise the sigraphs $(K_5, C_3^- \cup K_2^-)$ or $(K_7, P_4^- \cup K_2^-)$ will appear as induced subgraphs of $\Gamma = (K_n, H^-)$. A direct check shows that the graphs $(K_8, Q_1^-), (K_6, G_1^-), (K_8, G_2^-)$, and (K_8, G_3^-) belong to the class $\mathcal{L}(4)$. Thus $H \cong Q_1$ and n = 7. Next, assume that $\operatorname{gr}(H) = 4$. Again, to avoid $(K_7, P_4^- \cup K_2^-)$ as an induced subgraph, one can deduce that $H \cong Q_2$ or $H \cong Q_3$ or $H \cong H(q)$ (for some positive integer q) or the two-cyclic graph G_4 (shown in Figure 3) is an induced subgraph of H. It is easy to check that $(K_9, G_4^-) \in \mathcal{L}(4)$. As we saw above, if $H \cong H(q)$, then q = 1 and $9 \le n \le 12$ or q = 2 and n = 10. Also, the sigraphs $\Gamma = (K_n, Q_2^-)$, for each n > 7, and $\Gamma = (K_n, Q_3^-)$, for each n > 8, belong to the class $\mathcal{L}(3).$

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References

- S. Akbari, S. Dalvandi, F. Heydari, and M. Maghasedi, On the eigenvalues of signed complete graphs, Linear Multilinear Algebra 67 (2019), no. 3, 433–441. https://doi.org/10.1080/03081087.2017.1403548.
- [2] _____, On the multiplicity of -1 and 1 in signed complete graphs, Util. Math. **116** (2020), 21–32.
- F. Belardo, M. Brunetti, M. Cavaleri, and A. Donno, Constructing cospectral signed graphs, Linear Multilinear Algebra 69 (2021), no. 14, 2717–2732. https://doi.org/10.1080/03081087.2019.1694483.

- [4] A.E. Brouwer and W.H. Haemers, Spectra of Graphs, Springer Science & Business Media, 2011.
- M. Brunetti and A. Ciampella, Signed bicyclic graphs with minimal index, Commun. Comb. Optim. 8 (2023), no. 1, 207–241. https://doi.org/10.22049/cco.2022.27346.1241.
- [6] M. Brunetti and Z. Stanić, Ordering signed graphs with large index, Ars Math. Contemp. 22 (2022), no. 4, #P4.05 https://doi.org/10.26493/1855-3974.2714.9b3.
- [7] _____, Unbalanced signed graphs with extremal spectral radius or index, Comput. Appl. Math. 41 (2022), no. 3, Article number: 118 https://doi.org/10.1007/s40314-022-01814-5.
- [8] D. Cvetković, P. Rowlinson, and S. Simić, An Introduction to the Theory of Graph Spectra, Cambridge University Press, London, 2009.
- [9] S. Dalvandi, F. Heydari, and M. Maghasedi, Signed complete graphs with exactly m non-negative eigenvalues, Bull. Malays. Math. Sci. Soc. 45 (2022), no. 5, 2107– 2122.

https://doi.org/10.1007/s40840-022-01331-y.

[10] _____, A characterization of (K_n, U^-) in the class L(3), Ric. Mat. (2024), In press

https://doi.org/10.1007/s11587-023-00844-3.

- [11] M.R. Oboudi, Characterization of graphs with exactly two non-negative eigenvalues, Ars Math. Contemp. 12 (2016), no. 2, 271–286.
- M. Petrović, Graphs with a small number of nonnegative eigenvalues, Graphs Combin. 15 (1999), no. 2, 221–232. https://doi.org/10.1007/s003730050042.
- [13] J.H. Smith, Symmetry and multiple eigenvalues of graphs, Glas. Mat. Ser. III 12 (1977), no. 1, 3–8.
- [14] M. Souri, F. Heydari, and M. Maghasedi, Maximizing the largest eigenvalues of signed unicyclic graphs, Discrete Math. Algorithms Appl. 12 (2020), no. 2, Article ID: 2050016.

https://doi.org/10.1142/S1793830920500160.

- [15] Z. Stanić, Some relations between the skew spectrum of an oriented graph and the spectrum of certain closely associated signed graphs, Rev. de la Union Mat. Argentina 63 (2022), no. 1, 41–50.
 - https://doi.org/10.33044/revuma.1914.
- [16] A. Torgašev, Graphs with exactly two negative eigenvalues, Math. Nachr. 122 (1985), no. 1, 135–140. https://doi.org/10.1002/mana.19851220113.
- [17] T. Zaslavsky, Matrices in the theory of signed simple graphs, Advances in Discrete Mathematics and Applications, Ramanujan Mathematical Society Lecture Notes Series 13, Ramanujan Mathematical Society, Mysore, 2010, pp. 207–229.