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Research Article

Reconfiguring minimum independent dominating sets in graphs

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Abstract: The independent domination number $i(G)$ of a graph G is the minimum cardinality of a maximal independent set of G, also called an $i(G)$-set. The i-graph of G, denoted $\mathscr{I}(G)$, is the graph whose vertices correspond to the $i(G)$-sets, and where two $i(G)$-sets are adjacent if and only if they differ by two adjacent vertices. We show that not all graphs are i-graph realizable, that is, given a target graph H, there does not necessarily exist a seed graph G such that $H \cong \mathscr{I}(G)$. Examples of such graphs include $K_4 - e$ and $K_{2,3}$. We build a series of tools to show that known i-graphs can be used to construct new i-graphs and apply these results to build other classes of i-graphs, such as block graphs, hypercubes, forests, cacti, and unicyclic graphs.

Keywords: independent domination number, graph reconfiguration, i-graph

AMS Subject classification: 05C69

1. Introduction

The i-graph H of a graph G is an example of a "reconfiguration graph". It has as its vertex set the minimum independent dominating sets of G, and two vertices of H are adjacent whenever the symmetric difference of their corresponding sets consists of two vertices that are adjacent in G. We consider the following realizability question: for which graphs H does there exist a graph G such that H is the i-graph of G? Following definitions and general discussions in the remainder of this section, we begin our investigation into i-graph realizability in Section 2 by composing a series of

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observations and technical lemmas concerning the adjacency of vertices in an i-graph and the structure of their associated i-sets in the seed graph. In Section 3, we present the three smallest graphs which are not i-graphs, and in Section 4, we show that several common graph classes, like trees and cycles, are i-graphs. We conclude by examining, in Section 5, how new i-graphs can be constructed from known ones.

1.1. Reconfiguration

In general, a reconfiguration problem asks whether it is possible to transform a given source solution to a given problem into a target solution through a series of incremental transformations (called reconfiguration steps) under some specified rule, such that each intermediate step is also a solution. The resulting chain of the source solution, intermediate solutions, and target solution is a reconfiguration sequence.
In graph theory, reconfiguration problems are often concerned with solutions that are vertex/edge subsets or labellings of a graph. In particular, when the solution is a vertex (or edge) subset, the reconfiguration problem can be viewed as a token manipulation problem, where a solution subset is represented by placing a token at each vertex or edge of the subset. The reconfiguration step for vertex subsets can be of one of three variants (edge subsets are handled analogously):

- ▷ **Token Slide (TS) Model**: A single token is slid along an edge between adjacent vertices.
- ▷ **Token Jump (TJ) Model**: A single token jumps from one vertex to another (without the vertices necessarily being adjacent).
- ▷ **Token Addition/Removal (TAR) Model:** A single token can either be added to a vertex or be removed from a vertex.

To represent the many possible solutions in a reconfiguration problem, each solution can be represented as a vertex of a new graph, referred to as a *reconfiguration graph*, where adjacency between vertices follows one of the three token adjacency models, producing the *slide graph*, the *jump graph*, or the *TAR graph*, respectively. See [16, 20]. More formally, given a graph G, the *slide graph* of G under some specified reconfiguration rule is the graph H such that each vertex of H represents a solution of some problem on G, and two vertices u and v of H are adjacent if and only if the solution in G corresponding to u can be transformed into the solution corresponding to v by sliding a single token along an edge of G.

1.2. γ-Graphs

We use the standard notation of $\gamma(G)$ for the cardinality of a minimum dominating set of a graph G. The *private neighbourhood* of a vertex v with respect to a vertex set S is the set $\mathrm{pn}(v, S) = N[v] - N[S - \{v\}]$; therefore, a dominating set S is minimal dominating if, for each $u \in S$, $\mathrm{pn}(u, S)$ is nonempty. The *external private neighbourhood* of v with respect to S is the set $\mathrm{epn}(v, S) = \mathrm{pn}(v, S) - \{v\}$. The *independent domination number* $i(G)$ of G is the minimum cardinality of a maximal independent set of G, or, equivalently, the minimum cardinality of an independent domination set of G. An independent dominating set of G of cardinality $i(G)$ is also called an *i-set* of G, or an *$i(G)$-set*.

In general, we follow the notation of [7]. In particular, the disjoint union of two graphs G and H is denoted $G \cup H$, whereas the join of G and H, denoted $G \vee H$, is the graph obtained from $G \cup H$ by joining every vertex of G with every vertex of H. For other domination principles and terminology, see [14, 15].

First defined by Fricke, Hedetniemi, Hedetniemi, and Hutson [10] in 2011, the *γ-graph of a graph* G is the graph $G(\gamma) = (V(G(\gamma)), E(G(\gamma)))$, where each vertex $v \in V(G(\gamma))$ corresponds to a γ-set S_v of G. The vertices u and v in $G(\gamma)$ are adjacent if and only if there exist vertices u' and v' in G such that $u'v' \in E(G)$ and $S_v = (S_u - u') \cup \{v'\}$; this is a token-slide model of adjacency.

An initial question of Fricke et al. [10] was to determine exactly which graphs are γ-graphs; they showed that every tree is the γ-graph of some graph and conjectured that every graph is the γ-graph of some graph. Later that year, Connelly, Hutson, and Hedetniemi [8] proved this conjecture to be true. For additional results on γ-graphs, see [3, 8–10]. Mynhardt and Teshima [19] investigated slide model reconfiguration graphs with respect to other domination parameters.

Subramanian and Sridharan [23] independently defined a different *γ-graph of a graph* G, denoted $\gamma \cdot G$. The vertex set of $\gamma \cdot G$ is the same as that of $G(\gamma)$; however, for $u, w \in V(\gamma \cdot G)$ with associated γ-sets S_u and S_w in G, u and w are adjacent in $\gamma \cdot G$ if and only if there exist some $v_u \in S_u$ and $v_w \in S_w$ such that $S_w = (S_u - \{v_u\}) \cup \{v_w\}$. This version of the γ-graph was dubbed the "single vertex replacement adjacency model" by Edwards [9], and is sometimes referred to as the "jump γ-graph" as it follows the TJ-Model for token reconfiguration. Further results concerning $\gamma \cdot G$ can be found in [17, 21, 22]. Notably, if G is a tree or a unicyclic graph, then there exists a graph H such that $\gamma \cdot H = G$ [22]. Conversely, if G is the (jump) γ-graph of some graph H, then G does not contain any induced $K_{2,3}$, $P_3 \vee K_2$, or $(K_1 \cup K_2) \vee 2K_1$ [17].

Using a token addition/removal model, Haas and Seyffarth [11] define the *k-dominating graph* $D_k(G)$ of G as the graph with vertices corresponding to the k-dominating sets of G (i.e., the dominating sets of cardinality at most k). Two vertices in the k-dominating graph are adjacent if and only if the symmetric difference of their associated k-dominating sets contains exactly one element. Additional results can be found in [1, 2, 12, 13, 24], and a survey on reconfiguration of colourings and dominating sets of graphs in [18].

1.3. i-Graphs

The *i-graph* of a graph G, denoted $\mathscr{I}(G) = (V(\mathscr{I}(G)), E(\mathscr{I}(G)))$, is the graph with vertices representing the minimum independent dominating sets of G (that is, the *i-sets* of G). As in the case of γ-graphs as defined in [10], adjacency in $\mathscr{I}(G)$ follows a slide model where $u, v \in V(\mathscr{I}(G))$, corresponding to the $i(G)$-sets S_u and S_v,

respectively, are adjacent in $\mathscr{I}(G)$ if and only if there exists $xy \in E(G)$ such that $S_u = (S_v - x) \cup \{y\}$. We say H *is an i-graph*, or is *i-graph realizable*, if there exists some graph G such that $\mathscr{I}(G) \cong H$. Moreover, we refer to G as the *seed graph* of the i-graph H. Going forward, we mildly abuse notation to denote both the i-set X of G and its corresponding vertex in H as X, so that $X \subseteq V(G)$ and $X \in V(H)$.
Imagine that there is a token on each vertex of an i-set S of G. Then S is adjacent, in $\mathscr{I}(G)$, to an $i(G)$-set S' if and only if a single token can be slid along an edge of G to transform S into S'. Notice that the *token jump model* of reconfiguration for independent domination is identical to the token-slide model. The reason for this is as follows. Suppose a token "jumps" to reconfigure an i-set S_1 into an i-set S_2 by "jumping" from $v \in S_1$ to $w \in S_2$. Since S_2 is independent, w is not adjacent to any vertex in S_2. But $S_2 - \{w\} = S_1 - \{v\}$, hence w is also not adjacent to any vertex in $S_1 - \{v\}$. Since S_1 is dominating, w is adjacent to v. Therefore, the "jump" action is the same as the "slide" action. A token is said to be *frozen* (in any reconfiguration model) if there are no available vertices to which it can slide/jump.
In acknowledgment of the slide-action in i-graphs, given i-sets $X = \{x_1, x_2, \ldots, x_k\}$ and $Y = \{y_1, x_2, \ldots, x_k\}$ of G with $x_1y_1 \in E(G)$, we denote the adjacency of X and Y in $\mathscr{I}(G)$ as $X \overset{x_1y_1}{\sim} Y$, where we imagine transforming the i-set X into Y by sliding the token at x_1 along an edge to y_1. When discussing several graphs, we use the notation $X \overset{x_1y_1}{\sim}_G Y$ to specify that the relationship is on G. More generally, we use $x \sim y$ to denote the adjacency of vertices x and y (and $x \not\sim y$ to denote non-adjacency); this is used in the context of both the seed graph and the target graph.

Although every graph is the γ-graph of some graph, there is no such tidy theorem for i-graphs; as we show in Section 3, not every graph is an i-graph, and determining which classes of graphs are (or are not) i-graphs has proven to be an interesting challenge.

2. Observations

To begin, we propose several observations about the structure of i-sets within given i-graphs which we then use to construct a series of useful lemmas.

Observation 1. Let G be a graph and $H = \mathscr{I}(G)$. A vertex $X \in V(H)$ has $\deg_H(X) \geq 1$ if and only if for some $v \in X \subseteq V(G)$, there exists $u \in \operatorname{epn}(v, X)$ such that u dominates $\operatorname{pn}(v, X)$.

From a token-sliding perspective, Observation 1 shows that a token on an i-set vertex v is frozen if and only if $\operatorname{epn}(v) = \varnothing$ or $G[\operatorname{epn}(v, X)]$ has no dominating vertex.
For some path $X_1, X_2, \ldots, X_k$ in H, only one vertex of the i-set is changed at each step, and so X_1 and X_k differ on at most k vertices. This yields the following observation.

Observation 2. Let G be a graph and $H = \mathscr{I}(G)$. Then for any i-sets X and Y of G, the distance $d_H(X, Y) \geq |X - Y|$.

Lemma 1. *Let G be a graph with $H = \mathscr{I}(G)$. Suppose XY and YZ are edges in H with $X \overset{xy_1}{\sim} Y$ and $Y \overset{y_2z}{\sim} Z$, with $X \neq Z$. Then XZ is an edge of H if and only if $y_1 = y_2$.*

Proof. Let $X = \{x, v_2, v_3, \ldots, v_k\}$ and $Y = \{y_1, v_2, v_3, \ldots, v_k\}$ so that $X \overset{xy_1}{\sim} Y$. To begin, suppose $y_1 = y_2$. Then $Y \overset{y_1z}{\sim} Z$ and $Z = \{z, v_2, v_3, \ldots, v_k\}$, hence $|X - Z| = 1$. Since X is dominating, z is adjacent to a vertex in $\{x, v_2, v_3, \ldots, v_k\}$; moreover, since Z is independent, z is not adjacent to any of $\{v_2, v_3, \ldots, v_k\}$. Thus z is adjacent to x in G and $X \overset{xz}{\sim} Z$, so that $XZ \in E(H)$.
Conversely, suppose $y_1 \neq y_2$. Then, without loss of generality, say $y_2 = v_2$ and so $X = \{x, y_2, v_3, \ldots, v_k\}$, $Y = \{y_1, y_2, v_3, \ldots, v_k\}$, and $Z = \{y_1, z, v_3, \ldots, v_k\}$. Notice that $x \neq z$ since $x \sim y_1$ and $z \not\sim y_1$. Thus $|X - Z| = 2$, and it follows that $XZ \notin E(H)$. $\square$

Combining Observation 2 and Lemma 1 yields the following observation for vertices of i-graphs at distance two.

Observation 3. Let G be a graph and $H = \mathscr{I}(G)$. Then for any i-sets X and Y of G, if $d_H(X, Y) = 2$, then $|X - Y| = 2$.

Lemma 2. *Let G be a graph and $H = \mathscr{I}(G)$. Suppose H contains an induced $K_{1,m}$ with vertex set $\{X, Y_1, Y_2, \ldots, Y_m\}$ and $\deg_H(X) = m$. Let $i \neq j$. Then in G,*

(i) $X - Y_i \neq X - Y_j$,

(ii) $|Y_i \cap Y_j| = i(G) - 2$, *and*

(iii) $m \leq i(G)$.

Proof. Suppose $X \overset{x_iy_i}{\sim} Y_i$ and $X \overset{x_jy_j}{\sim} Y_j$. Then $(X - Y_i) = \{x_i\}$ and $(X - Y_j) = \{x_j\}$. From Lemma 1, since $Y_i \not\sim Y_j$, we have that $x_i \neq x_j$, which establishes Statement (i). Moreover, $Y_i \cap Y_j = X - \{x_i, x_j\}$, and so as these are i-sets, Statement (ii) also follows. Finally, for Statement (iii), again applying Lemma 1, we see that $|\bigcap_{1 \leq i \leq m} Y_i| = |X| - m = i(G) - m \geq 0$. $\square$

Induced C_4's in a target graph H play an important role in determining the i-graph realizability of H and determine a specific relationship among i-sets of a potential seed graph G, as we show next.

Proposition 1. *Let G be a graph and $H = \mathscr{I}(G)$. Suppose H has an induced C_4 with vertices X, A, B, Y, where $XY, AB \notin E(H)$. Then, without loss of generality, the set composition of X, A, B, Y in G, and the edge labelling of the induced C_4 in H, are as in Figure 1.*

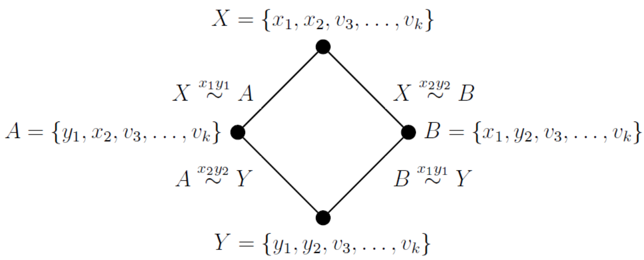


Figure 1. **Reconfiguration structure of an induced C_4 subgraph from Proposition 1.**

Proof. Suppose that the i-set X of G has $X = \{x_1, x_2, v_3, \ldots, v_k\}$. Then by Lemma 1, without loss of generality, the edge from X to A can be labelled as $X \overset{x_1y_1}{\sim} A$ for some $y_1 \in V(G) - X$, so that $A = \{y_1, x_2, v_3, \ldots, v_k\}$, while the edge from X to B can be labelled $X \overset{x_2y_2}{\sim} B$ for some y_2 and $B = \{x_1, y_2, v_3, \ldots, v_k\}$.
Consider the edge $AY \in E(H)$ labelled $A \overset{ay^*}{\sim} Y$. From Lemma 1, since $XY \notin E(G)$, $a \neq y_1$. If, say, $a = v_3$, then $Y = \{y_1, x_2, y^*, \ldots, v_k\}$. However, neither y_1 nor x_2 is in B, so $|Y - B| \geq 2$, contradicting Observation 2. Thus, $a \neq v_i$ for any $3 \leq i \leq k$. This leaves $a = x_2$, and $Y = \{y_1, y^*, v_3, \ldots, v_k\}$. Since $|Y - B| = 1$, $y^* = y_2$ and $Y = \{y_1, y_2, v_3, \ldots, v_k\}$ as required. $\square$

3. Realizability of i-Graphs

Having now established a series of observations and lemmas about the structures of i-graphs and the composition of their associate i-sets, we demonstrate that not all graphs are i-graphs by presenting three counterexamples: the diamond graph $\mathfrak{D}$, $K_{2,3}$ and κ, as pictured in Figure 2.

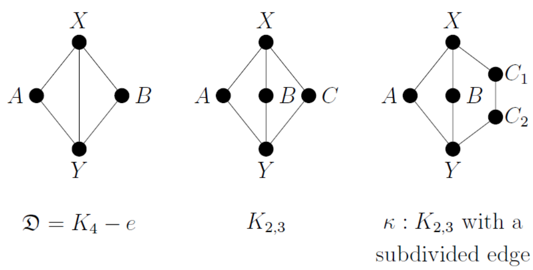


Figure 2. **Three graphs not realizable as i-graphs.**

Proposition 2. *The diamond graph $\mathfrak{D} = K_4 - e$ is not i-graph realizable.*

Proof. Suppose to the contrary there is some graph G with $\mathscr{I}(G) = \mathfrak{D}$. Let $V(\mathfrak{D}) = \{X, A, B, Y\}$ where $AB \notin E(\mathfrak{D})$. Say $X \overset{xy}{\sim} Y$. Then by Lemma 1, without loss of generality, the edges incident with A can be labelled as $X \overset{xa}{\sim} A$ and $A \overset{ay}{\sim} Y$. Likewise, $X \overset{xb}{\sim} B$ and $B \overset{by}{\sim} Y$ (see Figure 2). However, since $B \overset{bx}{\sim} X$ and $X \overset{xa}{\sim} A$, Lemma 1 implies that $AB \in E(\mathfrak{D})$, a contradiction. □

Proposition 3. *The graph $K_{2,3}$ is not i-graph realizable.*

Proof. Suppose $K_{2,3} = \mathscr{I}(G)$ for some graph G. Let $\{\{X, Y\}, \{A, B, C\}\}$ be the bipartition of $K_{2,3}$. Apply the exact labelling from Proposition 1 and Figure 1 to the i-sets and edges of X, A, B, and Y. We attempt to extend the labelling to C. By Lemma 1, since C is adjacent to X, but not A or B, without loss of generality, $X \overset{v_3c}{\sim} C$ and $C = \{x_1, x_2, c, v_4, \ldots, v_k\}$. As A is an i-set, $y_1v_3 \notin E(G)$. Since $v_3c \in E(G)$, $c \neq y_1$. Similarly, $c \neq y_2$. Now $|C - Y| = 3$ and $d(C, Y) = 1$, contradicting Observation 2. □

Proposition 4. *The graph κ is not i-graph realizable.*

Proof. Suppose $\kappa = \mathscr{I}(G)$ for some graph G and let $V(\kappa) = \{X, A, B, C_1, C_2, Y\}$ as in Figure 2, and to the subgraph induced by X, A, B, Y, apply the labelling of Proposition 1 and Figure 1. Through additional applications of Proposition 1, we can, as in the proof of Proposition 3, assume without loss of generality that $X \overset{x_3y_3}{\sim} C_1$. However, $d(C_1, Y) = 2$ but $|Y - C_1| = 3$, contradicting Observation 2. It follows that no such G exists and κ is not an i-graph. □

The observant reader will have undoubtedly noticed the common structure between the graphs in the previous three propositions – they are all members of the class of *theta graphs* (see [4]), graphs that are the union of three internally disjoint nontrivial paths with the same two distinct end vertices. The graph $\Theta\langle j, k, \ell\rangle$ with $j \leq k \leq \ell$, is the theta graph with paths of lengths j, k, and ℓ. In this notation, our three non i-graph realizable examples are $\mathfrak{D} \cong \Theta\langle 1, 2, 2\rangle$, $K_{2,3} \cong \Theta\langle 2, 2, 2\rangle$, and $\kappa \cong \Theta\langle 2, 2, 3\rangle$. Further rumination on the similarity in structure suggests that additional subdivisions of the central path in κ could yield more theta graphs that are not i-graphs. However, the proof technique used for κ no longer applies when, for example, a path between the degree 3 vertices has length greater than 4. In [6], we explore an alternative method for determining the i-graph realizability of theta graphs.

4. Some Classes of i-Graphs

Having studied several graphs that are not i-graphs, we now examine the problem of i-graph realizability from the positive direction. To begin, it is easy to see that complete graphs are i-graphs; moreover, as with γ-graphs, complete graphs are their own i-graphs, i.e., $\mathscr{I}(K_n) \cong K_n$.

Proposition 5. *Complete graphs are i-graph realizable.*

Hypercubes Q_n (the Cartesian product of K_2 taken with itself n times) are also straightforward to construct as i-graphs, with $\mathscr{I}(nK_2) \cong Q_n$. Each K_2 pair can be viewed as a $0-1$ switch, with the vertex of the i-set in each component sliding between the two states.

Proposition 6. *Hypercubes are i-graph realizable.*

Hypercubes are a special case of the following result regarding Cartesian products of i-graphs.

Proposition 7. *If $\mathscr{I}(G_1) \cong H_1$ and $\mathscr{I}(G_2) \cong H_2$, then $\mathscr{I}(G_1 \cup G_2) \cong H_1 \,\Box\, H_2$*

Proof. Let $\{X_1, X_2, \ldots, X_k\}$ be the i-sets of G_1 and let $\{Y_1, Y_2, \ldots, Y_\ell\}$ be the i-sets of G_2. Then, the i-sets of $G_1 \cup G_2$ are of the form $X_i \cup Y_j$. Clearly $X_i \cup Y_j \sim_{G_1 \cup G_2} X^*_{i^*} \cup Y^*_{j^*}$ if and only if $X_i \sim_G X^*_i$ and $Y_j = Y^*_j$, or $Y_j \sim_{G_2} Y^*_j$ and $X_i = X^*_i$. This gives a natural isomorphism to $H_1 \,\Box\, H_2$, where $X_i \cup Y_j$ is the vertex (X_i, Y_j). $\square$

We digress briefly from our discussion of i-graph classes to demonstrate that any i-graph can be realized as the i-graph of a connected seed graph. While there may be several constructions to show this, depending on the structure of the original seed graph, the construction we present here is simple to describe and works for any given seed graph G, including the case where G has isolated vertices.

Proposition 8. *Any i-graph is realizable as the i-graph of a connected seed graph.*

Proof. Let H be an i-graph; say $H \cong \mathscr{I}(G)$ for some graph G. Let k be an integer such that $k > i(G)$ and construct the graph G' from G as follows. Join a new vertex u to each vertex of G. Join another new vertex v to u and to k additional new vertices $w_1, w_2, \ldots, w_k$.
The graph G' thus obtained is connected. Any i-set of G together with v is an independent dominating set of G', and we conclude that $i(G') \leq i(G)+1$. Let Z be any independent set of G' that contains u. Then $v \notin Z$, implying that $\{w_1, w_2, \ldots, w_k\} \subseteq Z$. It follows that $|Z| \geq k+1 > i(G)+1$, so u does not belong to any i-set of G'. No subset of $V(G)$ dominates v, hence $i(G') > i(G)$, giving $i(G') = i(G)+1$. It is now clear that X is an i-set of G if and only if $X \cup \{v\}$ is an i-set of G'. A token placed on v is frozen, thus any i-sets X and X' of G are adjacent in $\mathscr{I}(G)$ if and only if $X \cup \{v\}$ and $X' \cup \{v\}$ are adjacent in $\mathscr{I}(G')$. Therefore, $\mathscr{I}(G) \cong \mathscr{I}(G') \cong H$, as required. $\square$

Moving to cycles, the constructions become markedly more difficult.

Proposition 9. *Cycles are i-graph realizable.*

Proof. The constructions for each cycle C_k for $k \geq 3$ are as described below.

(i) $\mathscr{I}(C_3) \cong C_3$
From Proposition 5.

(ii) $\mathscr{I}(2K_2) \cong C_4$
From Proposition 6.

(iii) $\mathscr{I}(C_5) \cong C_5$
Recall that $i(C_5) = 2$. A labelled C_5 and the resulting i-graph with $\mathscr{I}(C_5) \cong C_5$ are given in Figure 3 below.

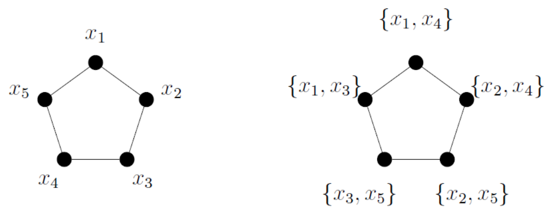


Figure 3. C_5 **and** $\mathscr{I}(C_5) \cong C_5$**.**

(iv) $\mathscr{I}(K_2 \,\square\, K_3) \cong C_6$
Label the vertices of $K_2 \,\square\, K_3$ as in Figure 4 below. The set $\{x_i, y_j\}$ is an i-set of $K_2 \,\square\, K_3$ if and only if $i \neq j$, so that $|V(\mathscr{I}(K_2 \,\square\, K_3))| = 6$, and adjacencies are as in Figure 4.

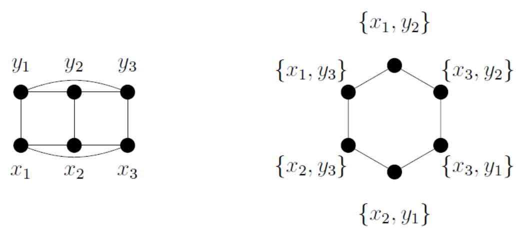


Figure 4. $K_2 \,\square\, K_3$ **and** $\mathscr{I}(K_2 \,\square\, K_3) \cong C_6$**.**

(v) *For any $k \geq 7$, construct the graph H with $V(H) = \{v_0, v_1, \ldots, v_{k-1}\}$, and $v_iv_j \in E(H)$ if and only if $j \not\equiv i-2, i-1, i, i+1, i+2 \pmod{k}$. Then $\mathscr{I}(H) \cong C_k$.*

For convenience, we assume that all subscripts are given modulo k. Thus in H, for all $0 \leq i \leq k-1$, we have the following:

(I) $N[v_i] \backslash N[v_{i+1}] = \{v_i, v_{i+3}\}$

(II) $N[v_{i+1}] \backslash N[v_i] = \{v_{i-2}, v_{i+1}\}$.

Since H is vertex-transitive, suppose that v_i is in some i-set S. Then $v_{i-2}, v_{i-1}, v_{i+1}$, and v_{i+2} are not dominated by v_i. To dominate v_{i+1}, either v_{i+1} or v_{i-2} is in S, because all other vertices in $N[v_{i+1}]$ are also adjacent to v_i, as in (II). Begin by assuming that $v_{i+1} \in S$. Now since $\{v_i, v_{i+1}\}$ dominates all of H except v_{i+2} and v_{i-1}, and $N(v_{i+2}, v_{i-1}) \subseteq N(v_i, v_{i+1})$, either v_{i+2} or v_{i-1} is in S. Thus $S = \{v_i, v_{i+1}, v_{i+2}\}$ or $S = \{v_{i-1}, v_i, v_{i+1}\}$.

Suppose now instead that $v_{i-2} \in S$. Now, only v_{i-1} is not dominated by $\{v_i, v_{i-2}\}$; moreover, since $N(v_{i-1}) \subseteq N(\{v_{i-2}, v_i\})$, we have that $v_{i-1} \in S$, and so $S = \{v_{i-2}, v_{i-1}, v_i\}$. Combining the above two cases yields that $i(H) = 3$ and that all i-sets of H have the form $S_i = \{v_i, v_{i+1}, v_{i+2}\}$, for each $0 \leq i \leq k-1$. Moreover, as there are k unique such sets, it follows that $|V(\mathscr{I}(H))| = k$.

We now consider the adjacencies of $\mathscr{I}(H)$. From our set definitions, $S_i \overset{v_{i-1}v_{i+2}}{\sim} S_{i+1}$, and $S_i \overset{v_{i+1}v_{i-2}}{\sim} S_{i-1}$. To see that S_i is not adjacent to any other i-set in H, notice that the token at v_i is frozen; $N(v_i)) \subseteq (N(v_{i-1}) \cup N(v_{i+1}))$. Moreover, by (II), the token at v_{i+1} can only slide to v_{i-2}, and likewise, the token at v_{i-1} can only slide to v_{i+2}. Thus $S_i \sim S_{i+1} \sim \cdots \sim S_{i-1} \sim S_i$, and so $\mathscr{I}(H) \cong C_k$ as required.

This completes the i-graph constructions for all cycles. $\square$

The constructions presented in Proposition 9 are not unique. Brewster, Mynhardt and Teshima show in [5] that for $k \geq 5$ and $k \equiv 2 \pmod 3$, $\mathscr{I}(C_k) \cong C_k$, and in [6] they use graph complements to construct graphs with i-graphs that are cycles.
We now present three lemmas with the eventual goal of demonstrating that all forests are i-graphs. When considering the i-graph of some graph H, if a vertex v of some i-set S of H has no external private neighbours, then the token at v is frozen. In the first of the three lemmas, Lemma 3, we construct a new seed graph for a given target graph, where each vertex of the seed graph's i-set has a non-empty private neighbourhood.

Lemma 3. *For any graph H, there exists a graph G such that $\mathscr{I}(G) \cong \mathscr{I}(H)$ and for any i-set S of G, all $v \in S$ have* $\operatorname{epn}(v, S) \neq \varnothing$.

Proof. Suppose S is an i-set of H having some $v \in S$ with $\operatorname{epn}(v, S) = \varnothing$. Construct the graph G_1 from H by joining new vertices a and b to each vertex of $N[v]$.
To begin, we show that the i-sets of G_1 are exactly the i-sets of H. Let R be some i-set of H and say that v is dominated by $u \in R$. Then $u \in N[v]$, so u also dominates a and b in G_1; therefore, R is independent and dominating in G_1, and so $i(G_1) \leq i(H)$. Conversely, suppose that Q is an i-set of G_1. If neither a nor b is in Q, then Q is an independent dominating set of H, and so $i(H) \leq i(G_1)$. Hence, suppose instead that $a \in Q$. Notice that since Q is independent and a is adjacent to each vertex in $N(v)$, $N_H[v] \cap Q = \varnothing$. Some vertex in Q dominates b; however, since $N_H(a) = N_H(b)$ and

Q is independent, it follows that b is self-dominating and so $b \in Q$. However, since $N_{G_1}[\{a,b\}] = N_{G_1}[v]$, the set $Q' = (Q - \{a,b\}) \cup \{v\}$ is an independent dominating set of G_1 such that $|Q'| < |Q|$, a contradiction. Thus, $i(G_1) = i(H)$ and the i-sets of H and G_1 are identical. In particular, S is an i-set of G_1, and moreover, $\mathrm{epn}_{G_1}(v,S) = \{a,b\}$.
By repeating the above process for each i-set of G_j, $j \geq 1$, that contains a vertex v with $\mathrm{epn}(v,S) = \varnothing$, we eventually obtain a graph $G = G_k$ such that for each i-set of S of G and each vertex $v \in S$, $\mathrm{epn}_G(v,S) \neq \varnothing$. Since the i-sets of H and G are identical and H is a subgraph of G, $\mathscr{I}(G) = \mathscr{I}(H)$ as required. □

Next on our way to constructing forests, we demonstrate that given an i-graph, the graph obtained by adding any number of isolated vertices is also an i-graph.

Lemma 4. *If H is the i-graph of some graph G, then there exists some graph G^* such that $\mathscr{I}(G^*) = H \cup \{v\}$.*

Proof. First assume that $i(G) \geq 2$. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let W be an independent set of size $i(G) = k$ disjoint from $V(G)$, say $W = \{w_1, w_2, \ldots, w_k\}$. Construct a new graph G^* by taking the join of G with the vertices of W, so that $G^* = G \vee W$.
Notice that W is independent and dominating in G^*. Moreover, if an i-set S of G^* contains any vertex w_i of W, since W is independent and each vertex of W is adjacent to all of $\{v_1, v_2, \ldots, v_n\}$, it follows that S contains all of W, and so, $S = W$. That is, if an i-set of G^* contains any vertex of W, it contains all of W. Thus, $i(G) = i(G^*)$. Furthermore, any i-set of G is also an i-set of G^*, and so the i-sets of G^* comprise of W and the i-sets of G. That is, $V(\mathscr{I}(G^*)) = V(\mathscr{I}(G)) \cup \{W\} = V(H) \cup \{W\}$.
If S is an i-set of G, then $S \cap W = \varnothing$. Thus, W is not adjacent to any other i-set in $\mathscr{I}(G^*)$. Relabelling the vertex representing the i-set W in G^* as v in $\mathscr{I}(G^*)$ yields $\mathscr{I}(G^*) = H \cup \{v\}$ as required.
If $i(G) = 1$, then G has a dominating vertex; begin with $G \cup K_1$, which has $\mathscr{I}(G) = \mathscr{I}(G \cup K_1)$ and $i(G \cup K_1) = 2$, and then proceed as above. □

As a final lemma before demonstrating the i-graph realizability of forests, we show that a pendant vertex can be added to any i-graph to create a new i-graph.

Lemma 5. *If H is the i-graph of some graph G, and H_u is the graph H with some pendant vertex u added, then there exists some graph G_u such that $\mathscr{I}(G_u) = H_u$.*

Proof. By Lemma 3 we may assume that for any i-set S of G, $\mathrm{epn}(v,S) \neq \varnothing$ for all $v \in S$. To construct G_u, begin with a copy of G. If w is the neighbour of u in H_u, then consider the i-set $W = \{v_1, v_2, \ldots, v_k\}$ in G corresponding to w. To each $v_i \in W$, attach a new vertex x_i for all $1 \leq i \leq k$. Then join each x_i to a new vertex y, and then to y, add a final pendant vertex z. Thus $V(G_u) = V(G) \cup \{x_1, x_2, \ldots, x_k, y, z\}$ as in Figure 5.

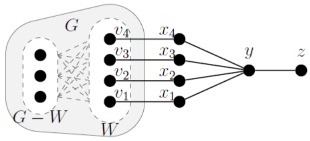


Figure 5. **The construction of G_u from G in Lemma 5.**

It is easy to see that if S is an i-set of G, then $S_y = S \cup \{y\}$ is an independent dominating set of G_u. The set $W_z = W \cup \{z\}$ is also an independent dominating set of G_u. Thus, $i(G_u) \leq i(G) + 1$. It remains only to show that these are i-sets and the only i-sets of G_u.

We claim that no x_i in $X = \{x_1, x_2, \dots, x_k\}$ is in any i-set of G_u. To show this, suppose to the contrary that S^* is an i-set with $S^* \cap X = \{x_1, x_2, \dots, x_\ell\}$ for some $1 \leq \ell \leq k$. Then, $y \notin S^*$; that is, $\{y, z\} \cap S^* = \{z\}$. To dominate the remaining $\{x_{\ell+1}, x_{\ell+2}, \dots, x_k\}$, we have that $S^* = \{x_1, x_2, \dots, x_\ell\} \cup \{v_{\ell+1}, v_{\ell+2}, \dots, v_k\} \cup \{z\}$. Recall from our initial assumption on G that there exists some $v_1^* \in \mathrm{epn}_G(v_1, W)$. Thus, $v_1^* \notin (N_{H_u}[\{v_{\ell+1}, v_{\ell+2}, \dots, v_k\}] \cap V(G))$, and so v_1^* is undominated by S^*, which implies that S^* is not an i-set.

Thus in every i-set of G_u, y is dominated either by itself or by z. If y is not in a given i-set S (and so $z \in S$), then to dominate X, $W \subseteq S$, and so $S = W \cup \{z\}$. Conversely, if $y \in S$ (and $z \notin S$), then since the vertices of G can only be dominated internally, S is an i-set of G_u if and only if $S - \{y\}$ is an i-set of G, which completes the proof of our claim.

If u^* and w^* are the vertices in $\mathscr{I}(G_u)$ associated with W_z and $W_y = W \cup \{y\}$ respectively, then clearly $\mathscr{I}(G_u) - \{u^*\} \cong \mathscr{I}(G)$. Furthermore, since W_y is the only i-set with $|W_y - W_z| = 1$ and $yz \in E(G_u)$, it follows that $\deg(u^*) = 1$ and $u^*w^* \in E(\mathscr{I}(G_u))$, and we conclude that $\mathscr{I}(G_u) \cong H_u$. $\square$

Finally, we amalgamate the previous lemmas on adding isolated and pendant vertices to i-graphs to demonstrate that forests are i-graphs.

Theorem 4. *All forests are i-graph realizable.*

Proof. We show by induction on the number of vertices that if F is a forest with m components, then F is i-graph realizable. For a base, note that $\mathscr{I}(\overline{K_2}) = K_1$. Construct the graph $\overline{K_m}$ by repeatedly applying Lemma 4. Suppose that all forests on m components on at most n vertices are i-graph realizable. Let F be some forest with $|V(F)| = n + 1$ and components $T_1, T_2, \dots, T_m$. If all vertices of F are isolated, we are done, so assume there is some leaf v with neighbour w in component T_1. Let $F^* = F - \{v\}$. By induction there exists some graph G^* with $\mathscr{I}(G^*) \cong F^*$. Applying Lemma 5 to G^* at w constructs a graph G with $\mathscr{I}(G) \cong F$. $\square$

Moreover, by adding Proposition 9 to the previous results, we obtain the following corollary.

Corollary 1. *Unicyclic graphs are* i*-graph realizable.*

Proof. Let H be a unicyclic graph with m components and let C be the unique cycle of H. By Proposition 9, C is an i-graph; say $C \cong \mathscr{I}(G)$ for some graph G. Using this as a base, and unicyclic graph(s) instead of forest(s), we follow the proof of Theorem 4 to obtain the desired result. □

With the completion of the constructions of forests and unicyclic graphs as i-graphs, we have now determined the i-graph realizability of many collections of small graphs. In particular, we draw the reader's attention to the following observation.

Observation 5. Every graph on at most four vertices except $\mathfrak{D}$ is an i-graph.

5. Building i-Graphs

In this section, we examine how new i-graphs can be constructed from known ones. We begin by presenting three very useful tools for constructing new i-graphs: the **Max Clique Replacement Lemma**, the **Deletion Lemma**, and the **Inflation Lemma**. The first among these shows that maximal cliques in i-graphs can be replaced by arbitrarily larger maximal cliques.

Lemma 6 (Max Clique Replacement Lemma). *Let* H *be an* i*-graph with a maximal* m*-vertex clique,* $\mathcal{K}_m$*. Then, the graph* H_w *formed by adding a new vertex* w^* *adjacent to all of* $\mathcal{K}_m$ *is also an* i*-graph.*

Proof. Suppose G is a graph such that $\mathscr{I}(G) = H$ and $i(G) = k+1$ where $k \geq 1$, and let $\mathcal{K}_m = \{V_1, V_2, \ldots, V_m\}$ be a maximal clique in H. From Lemma 1, the corresponding i-sets $V_1, V_2, \ldots, V_m$ of G differ on exactly one vertex, so for each $1 \leq i \leq m$, let $V_i = \{v_i, z_1, z_2, \ldots, z_k\} \subseteq V(G)$, so that $Z = \{z_1, z_2, \ldots, z_k\} = \bigcap_{1 \leq i \leq m} V_i$. Notice also from Lemma 1, for each $1 \leq i < j \leq m$, $v_i v_j \in E(G)$, and so $Q_m = \{v_1, v_2, \ldots, v_m\}$ is a (not necessarily maximal) clique of size m in G.
In addition to Q_m and Z defined above, we further weakly partition (i.e. some of the sets of the partition may be empty) the vertices of G as

$X = N(Q_m) \backslash N(Z)$, the vertices dominated by Q_m but not Z.

$Y = N(Q_m) \cap N(Z)$, the vertices dominated by both Q_m and Z.

$A = N(Z) \backslash N(Q_m)$, the vertices dominated by Z but not Q_m.

This partition (as well as the construction of G_w defined below) is illustrated in Figure 6. Before proceeding with the construction, we state the following series of claims regarding the set X:

Claim 1: *Each $x \in X$ is dominated by every vertex of Q_m.*
Otherwise, if some $x \in X$ is not adjacent to some $v_j \in Q_m$, then x is undominated in the the i-set $V_j = \{v_j\} \cup Z$.

Claim 2: $|X| \neq 1$.
If $X = 1$, say $X = \{x\}$, then $X^* = \{x\} \cup Z$ is independent, dominating, and has $|X^*| = i(G)$; that is, X^* is an i-set of G. However, since x is adjacent to all of Q_m in G, $X^* \overset{xv_j}{\sim} V_j$ for each $1 \leq j \leq m$, contradicting the maximality of the clique $\mathcal{K}_m$ in H.

Claim 3: *No $x \in X$ dominates all of X.*
If $x \in X$ dominates X, then $\{x\} \cup Z$ is an i-set of G. Following a similar argument of Claim 2, this contradicts the maximality of $\mathcal{K}_m$ in H.

Claim 4: *For any $v \in (X \cup Y \cup A)$, $\{v\} \cup Z$ is not an i-set.*
Combining Claims 2 and 3, if $v \in X$, then there exists some $x_i \in X$ such that $v \not\sim x_i$, and thus $\{v\} \cup Z$ does not dominate x_i. If $v \in (Y \cup A)$, then $v \in N(Z)$, and so $\{v\} \cup Z$ is not independent.

We construct a new graph G_w from G by joining a new vertex w to each vertex in $V(G) - Z$, as in Figure 6. We claim that $\mathscr{I}(G_w) \cong H_w$.

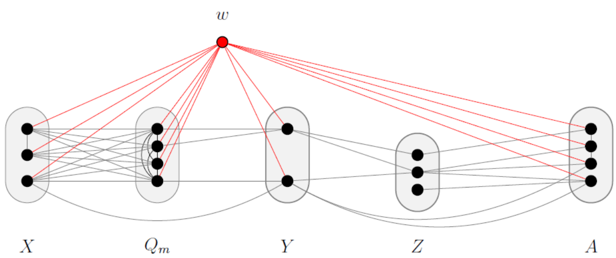


Figure 6. **Construction of G_w from G in Lemma 6.**

Let S be some i-set of G_w. If $w \notin S$, then $S \subseteq V(G)$ and so S is also independent dominating in G, implying $|S| = i(G) = k + 1$. However, if $w \in S$, then since w is adjacent to all of $V(G) - Z$ and Z is independent, we have that $S = \{w\} \cup Z$. It follows that $i(G_w) = i(G)$. Moreover, any i-set of G is also an i-set of G_w, and so $W := \{w\} \cup Z$ is the only new i-set generated in G_w. Thus $V(\mathscr{I}(G_w)) = V(\mathscr{I}(G)) \cup \{W\}$. Consider now the edges of $\mathscr{I}(G_w)$. Since w is adjacent to all of Q_m in G_w, $W \overset{wv_j}{\sim} V_j$ for each $1 \leq j \leq m$, and thus $\mathcal{K}_m \cup W$ is a clique in $\mathscr{I}(G_w)$.
Finally, we demonstrate that W is adjacent only to the i-sets of $\mathcal{K}_m$. Consider some i-set $S \notin \mathcal{K}_m$, and suppose to the contrary that $W \sim S$. As W is the only i-set

containing w, we have that $w \notin S$, and hence $W \overset{wu}{\sim} S$ for some vertex u. Since $w \sim u$, $u \notin Z$. Moreover, since W and S differ at exactly one vertex and $Z \subseteq W$, it follows that $Z \subseteq S$; that is, $S = \{u\} \cup Z$. If $u \in Q_m$ then $S \in \mathcal{K}_m$, a contradiction. If $u \in (X \cup Y \cup A)$, then by Claim 4, S is not an i-set, which is again a contraction. We conclude that $W \not\sim S$ for any i-set $S \notin \mathcal{K}_m$, and therefore $E(\mathscr{I}(G_w)) = E(\mathscr{I}(G)) \cup \left(\bigcup_{v_i \in Q_m} wv_i\right)$. If follows that $\mathscr{I}(G_w) \cong H_w$. $\square$

Our next result, the Deletion Lemma, shows that the class of i-graphs is closed under vertex deletion. It is unique among our other constructions; unlike most of our results which demonstrate how to build larger i-graphs from smaller ones, the Deletion Lemma instead shows that every induced subgraph of an i-graph is also an i-graph.

Lemma 7 (The Deletion Lemma). *If H is a nontrivial i-graph, then any induced subgraph of H is also an i-graph.*

Proof. Let G be a graph such that $H = \mathscr{I}(G)$ and $i(G) = k$. To prove this result, we show that for any $X \in V(H)$, there exists some graph G_X such that $\mathscr{I}(G_X) = H - X$. To construct G_X, take a copy of G and add to it a vertex z so that z is adjacent to each vertex of $G - X$ (see Figure 7). Observe first that since H is nontrivial, there exists an i-set $S \neq X$ of G. Then, S is also an independent dominating set of G_X, and so $i(G_X) \leq k$. Consider now some i-set S_X of G_X. Clearly $S_X \neq X$ because X does not dominate z. If $z \in S_X$, then as S_X is independent, no vertex of $G - X$ is in S_X. Moreover, since X is also independent and its vertices have all of their neighbors in $G - X$, this leaves each vertex of X to dominate itself. That is, $X \subseteq S_X$, implying that $S_X = X \cup \{z\}$ and $|S_X| = k + 1$. This contradicts that $i(G_X) \leq k$, and thus we conclude that z is not in any i-set of G_X. It follows that each i-set of G_X is composed only of vertices from G and so $i(G_X) = k$. Thus, $S_X \neq X$ is an i-set of G_X if and only if it is an i-set of G. Given that $V(\mathscr{I}(G_X)) = V(\mathscr{I}(G)) - \{X\} = V(H) - \{X\}$, we have that $\mathscr{I}(G_X) = H - X$ as required. $\square$

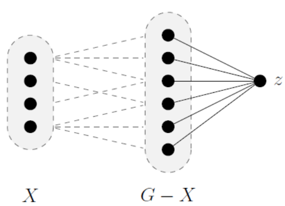


Figure 7. **Construction of G_X in Lemma 7.**

The following corollary is immediate as the contrapositive of Lemma 7.

Corollary 2. *If H is not an i-graph, then any graph containing an induced copy of H is also not an i-graph.*

This powerful corollary, although simple in statement and proof, immediately removes many families of graphs from i-graph realizability. For example, all wheels, 2-trees, and maximal planar graphs on at least five vertices contain an induced copy of the Diamond graph $\mathfrak{D}$, which was shown in Proposition 2 to not be an i-graph. Moreover, given that i-graph realizability is an inherited property, this suggests that there may be a finite-family forbidden subgraph characterization for i-graph realizability.
We now alter course to examine how one may construct new i-graphs by combining several known i-graphs. Understandably, an immediate obstruction to combining the constructions of i-graphs of, say, $\mathscr{I}(G_1) = H_1$ and $\mathscr{I}(G_2) = H_2$ is that it is possible (and indeed, likely) that $i(G_1) \neq i(G_2)$.
Two solutions to this quandary are presented in the following lemmas. In the first, Lemma 8, given a graph G, we progressively construct an infinite family of seed graphs $\mathcal{G}$ with the same number of components as G, and such that $\mathscr{I}(G) = \mathscr{I}(G_j)$ for each $G_j \in \mathcal{G}$. The second, Lemma 9 or the **Inflation Lemma**, offers a more direct solution: given an i-graph H, we demonstrate how to "inflate" a seed graph G to produce a new graph G^* such that $\mathscr{I}(G^*) = \mathscr{I}(G)$ and the i-sets of G^* are arbitrarily larger than the i-sets of G.

Lemma 8. *If G is a graph with $\mathscr{I}(G) \cong H$, then there exists an infinite family of graphs $\mathcal{G}$ such that $\mathscr{I}(G_j) \cong H$ for each $G_j \in \mathcal{G}$. Moreover, the number of components of $G_j \in \mathcal{G}$ is the same as G ($k(G) = k(G_j)$).*

Proof. Suppose $v \in V(G)$, and let G^* be the graph obtained by attaching a copy of the star $K_{1,3}$ with $V(K_{1,3}) = \{x, y_1, y_2, y_3\}$ ($\deg(x) = 3$) by joining v to y_1. As y_2 and y_3 are pendant vertices, $i(G^*) \geq i(G)+1$. If S is an i-set of G, then $S^* = S \cup \{x\}$ is dominating and independent, and so $i(G^*) = i(G) + 1$. Thus, x is in every i-set of G^*, and we can conclude that S^* is an i-set of G^* if and only if $S^* - \{x\}$ is an i-set of G. It follows that $\mathscr{I}(G^*) \cong \mathscr{I}(G)$ as required. Attaching additional copies of $K_{1,3}$ as above at any vertex of H similarly creates the other graphs of $\mathcal{G}$. □

Lemma 9 (Inflation Lemma). *If H is the i-graph of some graph G, then for any $k \geq i(G)$ there exists a graph G^* such that $i(G^*) = k$ and $\mathscr{I}(G^*) \cong H$.*

Proof. Begin with a copy of G and add to it $\ell = k - i(G)$ isolated vertices, $S = \{v_1, v_2, \dots, v_\ell\}$. Immediately, X is an i-set of G if and only if $X \cup S$ is an i-set of G^*. Moreover, if X and Y are i-sets of G such that $X \sim_G Y$ in H, then $(X \cup S) \sim_{G^*} (Y \cup S)$, and so $\mathscr{I}(G^*) \cong H$. □

Now, when attempting to combine the constructions of $\mathscr{I}(G_1) = H_1$ and $\mathscr{I}(G_2) = H_2$ and $i(G_1) < i(G_2)$, we need only inflate G_1 until its i-sets are the same size as those in

G_2. A powerful construction tool, the Inflation Lemma is used repeatedly in almost all of the following results of this section.
In the next result we show that, given i-graphs H_1 and H_2, a new i-graph H can be formed by identifying any two vertices in H_1 and H_2. The proof here uses Proposition 7, the Deletion Lemma (Lemma 7), and the Inflation Lemma (Lemma 9); a proof in which a seed graph of H is given can be found in [25, Proposition 3.30]. This result provides an alternative proof for Theorem 4.

Proposition 10. *Let H_1 and H_2 be i-graphs. Then the graph $H_{x=y}$, formed by identifying a vertex x of H_1 with a vertex y of H_2, is also an i-graph.*

Proof. Suppose G_1 and G_2 are graphs such that $\mathscr{I}(G_1) = H_1$ and $\mathscr{I}(G_2) = H_2$. Applying the Inflation Lemma we may assume that $i(G_1) = i(G_2) = k \geq 2$. By Proposition 7 there is a graph G such that $\mathscr{I}(G) = H_1 \,\square\, H_2$. Since $H_{x=y}$ is an induced subgraph of $H_1 \,\square\, H_2$, we may apply the Deletion Lemma and delete all other vertices of $H_1 \,\square\, H_2$ until only $H_{x=y}$ remains. □

We use Proposition 10 to show that two i-graphs may be connected by an edge between any two vertices to produce a new i-graph. A proof that gives a seed graph for this new i-graph is given in [25, Proposition 3.26].

Proposition 11. *Let H_1 and H_2 be disjoint i-graphs. Then the graph H_{xy}, formed by connecting H_1 to H_2 by an edge between any $x \in V(H_1)$ and any $y \in V(H_2)$, is also an i-graph.*

Proof. Let $H_3 \simeq K_2$ with $V(H_3) = \{u, v\}$. Applying Proposition 10 twice, we see that the graph H_{xu} obtained by identifying $x \in VH_1)$ with $u \in V(H_3)$, and the graph H_{xy} obtained by identifying $v \in V(H_{xu})$ with $y \in V(H_2)$ are i-graphs. □

The following corollary provides a way to connect two i-graphs with a clique rather than a bridge. A constructive proof in which a seed graph for the resulting i-graph is provided can be found in [25, Corollary 3.27].

Corollary 3. *Let H_1 and H_2 be i-graphs, and let H be the graph formed from them as in Proposition 11 by creating a bridge xy between them. Then the graph H_m formed by replacing xy with a K_m for $m \geq 2$ is also an i-graph.*

Proof. Apply the Max Clique Replacement Lemma (Lemma 6) to the edge xy in Proposition 11. □

The next proposition provides a method for combining two i-graphs without connecting them by an edge.

Proposition 12. *If H_1 and H_2 are i-graphs, then $H_1 \cup H_2$ is an i-graph.*

Proof. Suppose G_1 and G_2 are graphs such that $\mathscr{I}(G_1) = H_1$ and $\mathscr{I}(G_2) = H_2$. We assume that $i(G_1) = i(G_2) \geq 2$. Otherwise, apply the Inflation Lemma (Lemma 9) to obtain graphs with i-sets of equal size at least 2. Let $G = G_1 \vee G_2$, the join of G_1 and G_2. We claim that $\mathscr{I}(G) = H_1 \cup H_2$.
We proceed similarly to the proof of Proposition 11; namely, if S is an i-set of G_1, of G_2, then S is an independent dominating set of G. Likewise, we observe that any i-set of G is a subset of G_1 or G_2, and so, S is a i-set of G if and only if it is an i-set of G_1 or G_2.
Suppose $X \overset{xy}{\sim}_{G_1} Y$. Then in G, sets X and Y are still i-sets, and likewise, vertices X and Y are still adjacent, and so $X \overset{xy}{\sim}_{G} Y$. Now suppose instead that X is an i-set of G_1 and Y is an i-set of G_2. Within G, $X \cap Y = \varnothing$ and $|X| = |Y| \geq 2$, so X and Y are not adjacent in $\mathscr{I}(G)$. Therefore, $X {\sim}_G Y$ if and only if $X {\sim}_{G_1} Y$ or $X {\sim}_{G_2} Y$. It follows that $\mathscr{I}(G) = \mathscr{I}(G_1) \cup \mathscr{I}(G_2) = H_1 \cup H_2$ as required. □

Applying these new tools in combination yields some unexpected results. For example, the following corollary, which makes use of the previous Proposition 12 in partnership with the Deletion Lemma (a construction for combining i-graphs and a construction for vertex deletions) gives our first result on i-graph edge deletions.

Corollary 4. *Let H be an i-graph with a bridge e, such that the deletion of e separates H into components H_1 and H_2. Then*

(i) H_1 and H_2 are i-graphs, and

(ii) the graph $H^* = H - e$ is an i-graph.

Proof. Part (i) follows immediately from Lemma 7. For (ii), by Part (i), H_1 and H_2 are i-graphs. Proposition 12 now implies that $H_1 \cup H_2 = H - e$ is also an i-graph. □

Combining the results of Proposition 10 with Proposition 11 and Corollary 4, yields the following main result.

Theorem 6. *A graph G is an i-graph if and only if all of its blocks are i-graphs.*

As observed in Corollary 2, graphs with an induced $\mathfrak{D}$ subgraph are not i-realizable. If we consider the family of connected chordal graphs excluding those with an induced copy of $\mathfrak{D}$, we are left with the family of block graphs (also called clique trees): graphs where each block is a clique. As cliques are their own i-graph, the following is immediate.

Proposition 13. *Block graphs are i-graph realizable.*

Cacti are graphs whose blocks are cycles or edges. Thus, we have the following immediate corollary.

Corollary 5. *Cactus graphs are i-graph realizable.*

While the proof of Proposition 10 does provide a method for building block graphs, it is laborious to do so on a graph with many blocks, as the construction is iterative, with each block being appended one at a time. However, when we consider that the blocks of block graphs are complete graphs, and that complete graphs are their own i-graphs (and thus arguably the easiest i-graphs to construct), it is logical that there is a simpler construction. We offer one such construction below. An example of this process is illustrated in Figure 8.

Construction 1. Let H be a block graph with $V(H) = \{v_1, v_2, \ldots, v_n\}$ and let $\mathcal{B}_H = \{B_1, B_2, \ldots, B_m\}$ be the collection of maximal cliques of H. To construct a graph G such that $\mathscr{I}(G) = H$:

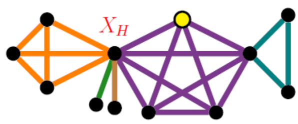
(i) Begin with a copy of each of the maximal cliques of H, labelled $A_1, A_2, \ldots, A_m$ in G, where A_i of G corresponds to B_i of H for each $1 \le i \le m$, and the A_i are pairwise disjoint. Notice that each cut vertex of H has multiple corresponding vertices in G.

(ii) Let $v \in V(H)$ be a cut vertex and $\mathcal{B}_v$ be the collection of blocks containing v in H; for notational ease, say $\mathcal{B}_v = \{B_1, B_2, \ldots, B_k\}$, and suppose that $W = \{w_1, w_2, \ldots, w_k\} \subseteq V(G)$ are the k vertices corresponding to v, where $w_i \in A_i$ for all $1 \le i \le k$.

For each distinct pair w_i and w_j of W, add to G three internally disjoint paths of length two between w_i and w_j. Since v is in k blocks of H, $3\binom{k}{2}$ vertices are added in this process. These additions are represented as the green vertices in Figure 8.

(iii) Repeat Step (ii) for each cut vertex of H.

(a)



(b)

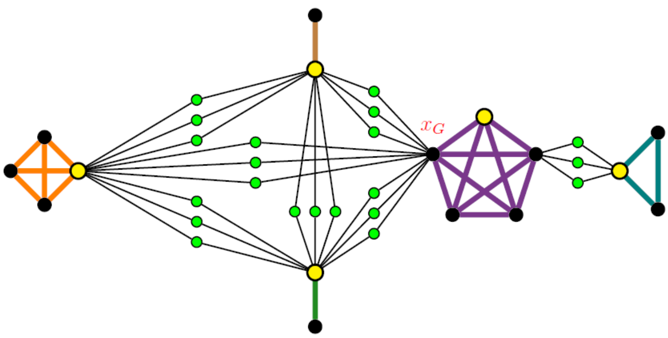


Figure 8. **The construction of G from H in the proof of Proposition 13.**

To see that the graph G from Construction 1 does indeed have $\mathscr{I}(G) = H$, notice that $i(G) = m$, where m is the number of blocks in H; if X is an i-set of G, then $|X \cap A_i|$=1 for each $A_i \in \{A_1, A_2, \ldots, A_m\}$. Moreover, no i-set of G has vertices in the added green vertices, because, as with the proof of Proposition 10, the inclusion of any one of these green vertices in an independent dominating set necessitates the addition of them all.

In Figure 8(b), the five yellow vertices form the i-set corresponding to the yellow vertex of G in Figure 8(a). Only the token on the purple K_5 can move in G; the other four tokens remain frozen, thereby generating the corresponding purple K_5 of H. It is only when the token on the purple K_5 is moved to the vertex x_G that the tokens on the orange K_4, and the brown and green K_2's, unfreeze one clique at a time. This corresponds to the cut vertex i-set X_H of H. The freedom of movement now transfers from the purple K_5 to any of the three other cliques, allowing for the generation of their associated blocks in G as required.

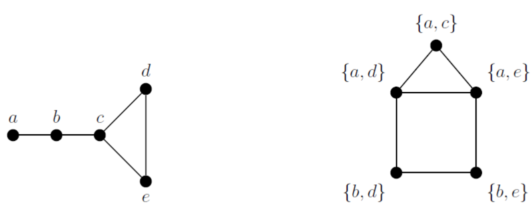
Finally, before we depart from block graphs, as chordal graphs are among the most well-studied families of graphs, we offer one additional reframing of this block graph result from the chordal graph perspective.

Corollary 6. *A chordal graph is i-graph realizable if and only if it is $\mathfrak{D}$-free.*

With the addition of Proposition 13 to the results used to build Observation 5, this leaves only the house graph (see Figure 9(b)) as unsettled with regard to its i-graph realizability among the 34 non-isomorphic graphs on five vertices. Although not strictly a result concerning the construction of larger i-graphs from known results, we include the following short proposition here for the sake of completeness.

Proposition 14. *The house graph $\mathcal{H}$ is an i-graph.*

To demonstrate Proposition 14, we provide an exact seed graph for the i-graph: the graph G in Figure 9(a) (K_3 with a P_3 tail) has $\mathscr{I}(G) = \mathcal{H}$. The i-sets of G and their adjacency are overlaid on $\mathcal{H}$ in Figure 9(b).



(a) A graph G such that $\mathscr{I}(G) = \mathcal{H}$. (b) The house graph $\mathcal{H}$ with i-sets of G.

Figure 9. **The graph G for Proposition 14 with $\mathscr{I}(G) = \mathcal{H}$.**

6. Conclusion

As we observed above, although not every graph is i-graph realizable, every graph does have an i-graph. The exact structure of the resulting i-graph can vary among families of graphs from the simplest isolated vertex to surprisingly complex structures. To illustrate this point, we determine the i-graphs of paths and cycles in [5].

We showed in Section 3 that the theta graphs $\mathfrak{D} \cong \Theta\langle 1,2,2\rangle$, $K_{2,3} \cong \Theta\langle 2,2,2\rangle$, and $\kappa \cong \Theta\langle 2,2,3\rangle$ are not i-graph realizable. In [6] we investigate the class of theta graphs and determine exactly which ones fail to be i-graph realizable – there are only finitely many such graphs. We also present a graph that is neither a theta graph nor i-graph realizable. The following question remains open.

Question 1. Does there exist a finite forbidden subgraph characterization of i-graph realizable graphs?

Proposition 9 states that every cycle is i-graph realizable. Hence, if we subdivide any edge of a cycle, the resulting graph is i-graph realizable. On the other hand, K_4 is i-graph realizable, but if we subdivide any edge, the resulting graph H has the diamond $\mathfrak{D}$ as induced subgraph. By Corollary 2, H is not i-graph realizable.

Question 2. Suppose H is i-graph realizable and let $e \in E(H)$. Under which conditions is the graph obtained by subdividing e i-graph realizable?

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References

[1] K. Adaricheva, C. Bozeman, N.E. Clarke, R. Haas, M.-E. Messinger, K. Seyffarth, and H.C. Smith, *Reconfiguration graphs for dominating sets*, Research Trends in

Graph Theory and Applications (Cham) (D. Ferrero, L. Hogben, S.R. Kingan, and G.L. Matthews, eds.), Springer International Publishing, 2021, pp. 119–135.

[2] S. Alikhani, D. Fatehi, and S. Klavžar, *On the structure of dominating graphs*, Graphs Combin. **33** (2017), no. 4, 665–672.
https://doi.org/10.1007/s00373-017-1792-5.

[3] A. Bień, *Gamma graphs of some special classes of trees*, Ann. Math. Sil. **29** (2015), 25–34.

[4] J.A. Bondy, *The "graph theory" of the Greek alphabet*, Graph theory and applications (Proc. Conf., Western Michigan Univ., Kalamazoo, Mich., 1972; dedicated to the memory of J. W. T. Youngs), Lecture Notes in Math., Vol. 303, Springer, Berlin, 1972, pp. 43–54.

[5] R.C. Brewster, C.M. Mynhardt, and L.E. Teshima, *The i-graphs of paths and cycles*, In Preparation (2023).

[6] ———, *The realizability of theta graphs as i-graphs*, In Preparation (2023).

[7] G. Chartrand, L. Lesniak, and P. Zhang, *Graphs and Digraphs*, 6th ed., Chapman & Hall, London, 2015.

[8] E. Connelly, K.R. Hutson, and S.T. Hedetniemi, *A note on γ-graphs*, AKCE Int. J. Graphs Comb. **8** (2011), no. 1, 23–31.

[9] Michelle Edwards, *Vertex-critically and bicritically for independent domination and total domination in graphs*, Ph.D. thesis, University of Victoria, 2015.

[10] G.H. Fricke, S.M. Hedetniemi, S.T. Hedetniemi, and K.R. Hutson, *γ-graphs of graphs*, Discuss. Math. Graph Theory **31** (2011), no. 3, 517–531.

[11] R. Haas and K. Seyffarth, *The k-dominating graph*, Graphs Combin. **30** (2014), no. 3, 609–617.
https://doi.org/10.1007/s00373-013-1302-3.

[12] ———, *Reconfiguring dominating sets in some well-covered and other classes of graphs*, Discrete Math. **340** (2017), no. 8, 1802–1817.
https://doi.org/10.1016/j.disc.2017.03.007.

[13] A. Haddadan, T. Ito, A.E. Mouawad, N. Nishimura, H. Ono, A. Suzuki, and Y. Tebbal, *The complexity of dominating set reconfiguration*, Theoret. Comput. Sci. **651** (2016), 37–49.
https://doi.org/10.1016/j.tcs.2016.08.016.

[14] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater, *Domination in Graphs: Advance Topics*, Marcel Dekker, New York, 1998.

[15] ———, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[16] T. Ito, E.D. Demaine, N.J.A. Harvey, C.H. Papadimitriou, M. Sideri, R. Uehara, and Y. Uno, *On the complexity of reconfiguration problems*, Theoret. Comput. Sci. **412** (2011), no. 12-14, 1054–1065.
https://doi.org/10.1016/j.tcs.2010.12.005.

[17] S.A. Lakshmanan and A. Vijayakumar, *The gamma graph of a graph*, AKCE Int. J. Graphs Comb. **7** (2010), no. 1, 53–59.

[18] C.M. Mynhardt and S. Nasserasr, *Reconfiguration of colourings and dominating sets in graphs*, 50 years of combinatorics, graph theory, and computing, Discrete

Math. Appl. (Boca Raton), CRC Press, Boca Raton, FL, 2020, pp. 171–191.

[19] C.M. Mynhardt and L. E. Teshima, *A note on some variations of the γ-graph*, J. Combin. Math. Combin. Comput. **104** (2018), 217–230.

[20] N. Nishimura, *Introduction to reconfiguration*, Algorithms **11** (2018), no. 4, Article ID: 52.
https://doi.org/10.3390/a11040052.

[21] N. Sridharan, S. Amutha, and S.B. Rao, *Induced subgraphs of gamma graphs*, Discrete Math. Algorithms Appl. **5** (2013), no. 3, Article ID: 1350012. https://doi.org/10.1142/S1793830913500122.

[22] N. Sridharan and K. Subramanian, *Trees and unicyclic graphs are γ-graphs*, J. Combin. Math. Combin. Comput. **69** (2009), 231–236.

[23] K. Subramanian and N. Sridharan, *γ-graph of a graph*, Bull. Kerala Math. Assoc. **5** (2008), no. 1, 17–34.

[24] A. Suzuki, A.E. Mouawad, and N. Nishimura, *Reconfiguration of dominating sets*, J. Comb. Optim. **32** (2016), no. 4, 1182–1195.
https://doi.org/10.1007/s10878-015-9947-x.

[25] L.E. Teshima, *The i-graph and other variations on the γ-graph*, Ph.D. thesis, University of Victoria, 2022, https://dspace.library.uvic.ca/handle/1828/14602.