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Research Article

Sharp lower bounds on the metric dimension of circulant graphs

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Abstract: For $n \geq 2t+1$ where $t \geq 1$, the circulant graph $C_n(1, 2, \ldots, t)$ consists of the vertices $v_0, v_1, v_2, \ldots, v_{n-1}$ and the edges $v_i v_{i+1}$, $v_i v_{i+2}, \ldots, v_i v_{i+t}$, where $i = 0, 1, 2, \ldots, n-1$, and the subscripts are taken modulo n. We prove that the metric dimension $\dim(C_n(1, 2, \ldots, t)) \geq \lceil \frac{2t}{3} \rceil + 1$ for $t \geq 5$, where the equality holds if and only if $t = 5$ and $n = 13$. Thus $\dim(C_n(1, 2, \ldots, t)) \geq \lceil \frac{2t}{3} \rceil + 2$ for $t \geq 6$. This bound is sharp for every $t \geq 6$.

Keywords: Cayley graph, distance, resolving set

AMS Subject classification: 05C12, 05C35

1. Introduction

The distance $d(u, v)$ between two vertices u and v in a graph G is the number of edges in a shortest path between u and v. The diameter of G is the distance between any two farthest vertices in G. For an ordered set of z vertices $W = \{w_1, w_2, \ldots, w_z\}$, we investigate the representation of distances of v with respect to W:

$$r(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_z)).$$

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If every two vertices of G have distinct representations, then W is a resolving set of G. The number of vertices in a smallest resolving set is the metric dimension $\dim(G)$. Various modifications of the metric dimension such as the edge metric dimension (see [9]) and the 2-dimension (see [6]) have been studied, however the metric dimension is the main, most well-known and the most studied invariant in the area.
Due to their symmetries, circulant graphs are very interesting. For $n \geq 2t+1$ where $t \geq 1$, the circulant graph $C_n(1,2,\ldots,t)$ consists of the vertices $v_0, v_1, v_2, \ldots, v_{n-1}$ and the edges v_iv_{i+1}, $v_iv_{i+2}, \ldots, v_iv_{i+t}$, where $i = 0,1,2,\ldots,n-1$, and the subscripts are taken modulo n. The graphs $C_n(1,2,\ldots,t)$ are complete for $n = 2t+1$, therefore we usually consider $n \geq 2t+2$.
Let $n = 2(d-1)t+1+r$, where $d \geq 2$, $t \geq 1$ and $1 \leq r \leq 2t$. Then $C_n(1,2,\ldots,t)$ has diameter d. The metric dimension of $C_n(1,2,\ldots,t)$ for general t was studied in [2], [3], [5], [11], [10] and [12].
Let us present known results on $\dim(C_n(1,2,\ldots,t))$ for small t (and $n \geq 2t+2$). The case $t = 1$ is trivial since $C_n(1)$ is a cycle. For cycles,

$$\dim(C_n(1)) = 2 = \left\lceil \frac{2t}{3} \right\rceil + 1. \tag{1}$$

For $t = 2$, by [1] and [8],

$$\dim(C_n(1,2)) = \begin{cases} 3 = \lceil \frac{2t}{3} \rceil + 1 \text{ if } 1 \leq r \leq 3, \\ 4 = \lceil \frac{2t}{3} \rceil + 2 \text{ if } r = 4. \end{cases} \tag{2}$$

For $t = 3$, by [1] and [7],

$$\dim(C_n(1,2,3)) = \begin{cases} 4 = \lceil \frac{2t}{3} \rceil + 2 \text{ if } 1 \leq r \leq 5, \\ 5 = \lceil \frac{2t}{3} \rceil + 3 \text{ if } r = 6. \end{cases} \tag{3}$$

For $t = 4$ where $n \notin \{11, 19\}$, by [4],

$$\dim(C_n(1,2,3,4)) = \begin{cases} 4 = \lceil \frac{2t}{3} \rceil + 1 \text{ if } r = 3, \\ 5 = \lceil \frac{2t}{3} \rceil + 2 \text{ if } r = 1,2,4,5, \\ 6 = \lceil \frac{2t}{3} \rceil + 3 \text{ if } r = 6,7,8, \end{cases} \tag{4}$$

and

$$\dim(C_{11}(1,2,3,4)) = \dim(C_{19}(1,2,3,4)) = 4. \tag{5}$$

By [10], for $n \geq t^2+1$ where $t \geq 2$,

$$\dim(C_n(1,2,\ldots,t)) \geq t.$$

In [12], it was shown that for small n and $t \geq 9$, $\dim(C_n(1,2,\ldots,t))$ can be less than t by proving that for every $t \geq 7$, there exists an $n \in [2t+5, 2t+8]$ such that

$$\dim(C_n(1,2,\ldots,t)) \leq \left\lceil \frac{2t}{3} \right\rceil + 2.$$

In [12], the authors also presented the following conjecture.

Conjecture 1. [12] For every $t \geq 6$,

$$\dim(C_n(1,2,\ldots,t)) \geq \left\lceil \frac{2t}{3} \right\rceil + 2.$$

In this paper, we prove Conjecture 1 by showing that

$$\dim(C_n(1,2,\ldots,t)) \geq \left\lceil \frac{2t}{3} \right\rceil + 1$$

for $t \geq 5$, where the equality holds if and only if $t = 5$ and $n = 13$. From this result and (1) – (5), we can see that $\dim(C_n(1,2,\ldots,t)) \geq \lceil \frac{2t}{3} \rceil + 1$ holds for every $t \geq 1$.

2. Results

Let $n = 2(d-1)t + 1 + r$, where $1 \leq r \leq 2t$. Then $C_n(1,2,\ldots,t)$ has diameter d and for every vertex v_i there are exactly r vertices at distance d from v_i. We present lower bounds on $\dim(C_n(1,2,\ldots,t))$ in terms of the number of vertices and diameter.

Theorem 2. *Let $n = 2(d-1)t + 1 + r$, where $d \geq 2$, $t \geq 1$ and $1 \leq r \leq 2t$. Then*

$$\dim(C_n(1,2,\ldots,t)) \geq \begin{cases} \frac{n}{2d-2} & \text{if } r = 1, \\ \frac{n}{2d-1} & \text{if } r > 1. \end{cases}$$

Proof. Let W be a resolving set in $C_n(1,2,\ldots,t)$ and let $v_i \in W$. Then

$$v_{i-t}, v_{i-t+1}, \ldots, v_{i-1}; v_{i+1}, v_{i+2}, \ldots, v_{i+t}$$

are the vertices at distance 1 from v_i,

$$v_{i-2t}, v_{i-2t+1}, \ldots, v_{i-t-1}; v_{i+t+1}, v_{i+t+2}, \ldots, v_{i+2t}$$

are the vertices at distance 2 from v_i (if $n \geq 3t+1$), etc. Hence, vertices at distance j from v_i, where $j < d$, form two sets of consecutive vertices. The pairs of consecutive vertices

$$v_{i+t}, v_{i+t+1};\ \ v_{i+2t}, v_{i+2t+1};\ \ \ldots;\ \ v_{i+(d-1)t}, v_{i+(d-1)t+1},$$

as well as

$$v_{i-t}, v_{i-t-1};\ v_{i-2t}, v_{i-2t-1};\ \ldots;\ v_{i-(d-1)t}, v_{i-(d-1)t-1}$$

have different distances from v_i, so they are resolved by v_i, and we call them bordering pairs with respect to v_i. However, if $r = 1$, then $v_{i-(d-1)t-1} = v_{i+(d-1)t+1}$ since there is a unique vertex at distance d from v_i. In this case we do not call the pairs $v_{i+(d-1)t}, v_{i+(d-1)t+1}$ and $v_{i-(d-1)t}, v_{i-(d-1)t-1}$ bordering. Instead, the vertex $v_{i+(d-1)t+1}$ is called a singleton.
Bordering pairs define a border between them, so a border splits vertices at distance j from those at distance $j+1$, where $1 \le j < d$ (or $j < d-1$ if $r = 1$). Observe that v_i defines $2(d-1)$ borders if $r > 1$ and $2(d-2)$ borders if $r = 1$.
Consider pairs of consecutive vertices of $C_n(1, 2, \ldots, t)$. If v_x and v_{x+1} are resolved and none of them is a singleton or in W, then there must be a vertex $v_i \in W$ for which v_x, v_{x+1} is a bordering pair. Denote $|W| = \ell$. Then borders split the vertices of $C_n(1, 2, \ldots, t)$ into at most $\ell \cdot 2(d-1)$ sets (into at most $\ell \cdot 2(d-2)$ sets if $r = 1$) which we call states. If each state contains exactly one vertex, then all pairs of consecutive vertices are resolved. If there are more vertices in a state, say k, then they are resolved only if $k-1$ of them are among singletons or in W. Thus, since W is a resolving set,

$$\begin{aligned} n &\le 2\ell(d-1) + \ell = \ell(2d-1) \quad \text{if } r > 1 \quad \text{and} \\ n &\le 2\ell(d-2) + 2\ell = \ell(2d-2) \quad \text{if } r = 1, \end{aligned}$$

which gives $\ell \ge \frac{n}{2d-1}$ if $r > 1$, and $\ell \ge \frac{n}{2d-2}$ if $r = 1$. □

We present two corollaries of Theorem 2.

Corollary 1. *Let $n = 2(d-1)t + 2$, where $d \ge 2$ and $t \ge 1$. Then* $\dim(C_n(1, 2, \ldots, t)) \ge t + 1$.

Proof. We use $n = 2(d-1)t + 2$ in Theorem 2 and get $\dim(C_n(1, 2, \ldots, t)) \ge \left\lceil \frac{2(d-1)t+2}{2d-2} \right\rceil = t + 1$. □

Corollary 2. *Let $n = 2(d-1)t + 1 + r$, where $d \ge 2$, $t \ge 1$ and $2 \le r \le 2t$. Then*

$$\dim(C_n(1, 2, \ldots, t)) \ge t + \left\lceil \frac{r-t+1}{2d-1} \right\rceil.$$

Proof. By Theorem 2, we have

$$\dim(C_n(1, 2, \ldots, t)) \ge \left\lceil \frac{2(d-1)t+r+1}{2d-1} \right\rceil = \left\lceil \frac{(2d-1)t-t+r+1}{2d-1} \right\rceil = t + \left\lceil \frac{r-t+1}{2d-1} \right\rceil.$$

□

For given t, let us consider circulant graphs $C_n(1,2,\dots,t)$ with the smallest possible number of vertices. Clearly, $\dim(C_n(1,2,\dots,t)) = 2t$ for $n = 2t+1$, because $C_{2t+1}(1,2,\dots,t)$ are complete graphs. By Corollary 2.8 presented in [2], we obtain $\dim(C_n(1,2,\dots,t)) = t+1$ for $n = 2t+2$ (observe that $t+1 \geq \lceil \frac{2t}{3} \rceil + 2$ if $t \geq 3$). So, we study $C_n(1,2,\dots,t)$ for $n \geq 2t+3$.
In Lemmas 1, 2 and 3, we study the cases $r = 2$, $r = 3$ and $r = 4$, where $n = 2t+1+r$ (which means that the diameter is 2). Note that these cases were considered in [2] and the authors of [2] assumed that $\dim(C_n(1,2,\dots,t)) = t$; see their Theorem 2.16. However, in the proof of Theorem 2.16, they use their lower bound from Theorem 2.15, which does not hold. Easy counterexamples are the cases $t = 10, 11, 14$ for $2t+3 \leq n \leq 2t+5$. For these cases, we have

$$\dim(C_n(1,2,\dots,10)) = 9, \quad \dim(C_n(1,2,\dots,11)) = 10 \quad \text{and} \quad \dim(C_n(1,2,\dots,14)) = 12.$$

For example $\{v_0, v_1, v_6, v_7, v_9, v_{10}, v_{14}, v_{15}, v_{16}\}$ is a resolving set of $C_{23}(1,2,\dots,10)$ and $\{v_0, v_1, v_2, v_4, v_6, v_{10}, v_{12}, v_{13}, v_{14}, v_{22}, v_{24}, v_{26}\}$ is a resolving set of $C_{32}(1,2,\dots,14)$.
In the proofs of Lemmas 1, 2 and 3, we denote a minimum resolving set in $C_n(1,2,\dots,t)$ by W. A vertex in W is denoted by $\circ$, a vertex which is not in W is denoted by $\times$. If it is not determined whether a vertex is in W or not, then it is denoted by $\cdot$. We use parentheses in figures. Vertices inside parentheses have distance 2 from a particular vertex in W. Parentheses create borders and sets of vertices which are not separated by any parenthesis are called states. Outer states are states which are not inside any parentheses.

Lemma 1. *Let $n = 2t+3$ where $t \geq 5$. Then*

$$\dim(C_n(1,2,\dots,t)) \geq \frac{4t+4}{5}.$$

Proof. For any $v_i \in W$, there are two vertices at distance 2 from v_i, therefore there are two vertices inside one set of parentheses. Observe that a vertex outside W which is inside no parentheses, has distance 1 from all the vertices of W. Therefore, there can be at most one vertex outside parentheses which is not in W.
A vertex in W determines the situation $(\cdot\cdot)$. Since the two vertices inside parentheses must be resolved, we have either $(\circ\cdot)$ (or equivalently $(\cdot\circ)$) or $(\cdot(\cdot)\cdot)$. Hence, if we want to resolve as many vertices as possible, the first situation can be described by a diagram $(\circ\times)$ and the second situation by $(\times(\times)\times)\circ\circ$. In the first situation, we use one vertex of W and we resolve one vertex which is not in W, while in the second situation we use two vertices of W and we resolve three vertices which are not in W. So, if we want to have $|W|$ as small as possible, the second situation is preferable and it gives $n \leq \frac{5}{2}|W| + 1$ because one vertex outside the parentheses does not have to be in W. Thus

$$\dim(C_n(1,2,\dots,t)) = |W| \geq \frac{2n-2}{5} = \frac{2(2t+3)-2}{5} = \frac{4t+4}{5}.$$

□

In Lemma 2, we present a lower bound for $\dim(C_n(1,2,\dots,t))$, where $n = 2(d-1)t+1+r$, $d = 2$ and $r = 3$. Let us note that the proof of Lemma 2 for $r = 3$ is more complicated than the proof of Lemma 3 for $r = 4$.

Lemma 2. *Let $n = 2t+4$ where $t \geq 5$. Then $\dim(C_n(1,2,\dots,t)) \geq \lceil \frac{2t}{3} \rceil + 2$.*

Proof. Since $t \geq 5$, we have $n \geq 14$. By Theorem 2, $\dim(C_n(1,2,\dots,t)) \geq \frac{n}{2d-1} = \frac{2t+4}{3} = \frac{2t+1}{3} + 1$. So if $t \equiv 0 \pmod 3$, then $\dim(C_n(1,2,\dots,t)) \geq \lceil \frac{2t}{3} \rceil + 2$. If $t \equiv 1 \pmod 3$ or $t \equiv 2 \pmod 3$, then $\dim(C_n(1,2,\dots,t)) \geq \lceil \frac{2t}{3} \rceil + 1$. We prove that $\dim(C_n(1,2,\dots,t)) \geq \lceil \frac{2t}{3} \rceil + 2$.
Assume to the contrary that $\dim(C_n(1,2,\dots,t)) = \lceil \frac{2t}{3} \rceil + 1$. So $t \equiv 1 \pmod 3$ or $t \equiv 2 \pmod 3$. Since $\dim(C_n(1,2,\dots,t)) = \lceil \frac{2t}{3} \rceil + 1$, the vertices of a minimum resolving set W form $2(\lceil \frac{2t}{3} \rceil + 1)$ parentheses, and hence $2(\lceil \frac{2t}{3} \rceil + 1)$ states. In these states, there are $n - (\lceil \frac{2t}{3} \rceil + 1)$ vertices which are not in W. If $t \equiv 1 \pmod 3$, then $2(\lceil \frac{2t}{3} \rceil + 1) = 2t+4-(\lceil \frac{2t}{3} \rceil + 1)$, and so all the states must contain a vertex which is not in W. (Recall that no state can contain two vertices outside W.) On the other hand, if $t \equiv 2 \pmod 3$, then $2(\lceil \frac{2t}{3} \rceil + 1) = 2t+4-(\lceil \frac{2t}{3} \rceil + 1)+1$, and so there is at most one state without a vertex outside W. This implies that there are at most two outer states (because two vertices not in W in two different outer states cannot be resolved). Observe that since an empty state does not contain a vertex outside W, there can be at most one empty state.
In the reasoning we often use the following fact. Suppose that there is a collection of consecutive states with at least three consecutive vertices, finished by a closing parenthesis, such that every state in the collection contains exactly one vertex outside W. Moreover, suppose that there cannot be (another) empty state in the collection. Then the finishing closing parenthesis bounds an outer state. The typical examples are $(\circ \times \circ)$ or $(\times(\times \circ \times) \times)$, although in these cases also the left-hand side final parenthesis bounds an empty state.
In figures we describe situations on opposite sides of a cycle $(v_0, v_1, \dots, v_{2t+4})$. We distinguish several cases.

Case 1: *There is an empty state.*
This situation is depicted in Figure 1.1. Note that all the states except for the empty state must contain a vertex outside W and there is at most one outer state different from the empty one.
First suppose that the underlined vertices in Figure 1.1 are in W, see Figure 1.2. Since there is at most one empty state, there cannot be another parentheses between the bold ones in the upper line of Figure 1.2. So the bold parentheses bound outer states. But since there is an empty state, there is at most one outer state (otherwise two outer states contain vertices outside W and these vertices are not distinguished).

Thus the bold parentheses bound the unique outer state and all the parentheses are already present in Figure 1.2. Since $n \geq 14$, the outer state contains exactly four vertices from W and at least four vertices outside W, a contradiction.

$$\begin{array}{cccc} (\cdot\cdot\cdot)(\cdot\cdot\cdot) & \mathbf{(}\,\cdot(\cdot(\cdot)(\cdot)\cdot)\cdot\,\mathbf{)} & (\cdot\cdot(\times)(\underline{\cdot\cdot})\times) & (\underline{\cdot\cdot}(\times)(\times(\times)\times)\times\,\mathbf{)} \\ \circ\underline{\cdot\cdot}\circ & \circ\;\circ\;\circ\circ & \circ\;\circ\;\times\circ & \cdot\;\cdot\;\underline{\circ\;\circ\;\circ}\;\times\;\circ \end{array}$$

Figure 1.

Now suppose that one of the underlined vertices in Figure 1.1 is in W and the other one is not in W. By symmetry, we may assume the situation depicted in Figure 1.3. If the underlined vertices are in different states, then we have the situation depicted in Figure 1.4. Observe that the bold parenthesis bounds the unique outer state (since the empty state is not outer in this case). In the upper line there are five consecutive vertices which are not in W, so the underlined vertices in the bottom line are in an outer state. But then the underlined vertices in the upper line must be outside W, since otherwise there will be another parentheses on the right-hand side of already determined vertices in the bottom line of Figure 1.4, which means that there will be two different outer states, one starting with the bold parenthesis in the upper line of Figure 1.4, and the other containing the three consecutive vertices of W in the bottom line, a contradiction. Consequently, the nonempty outer state contains all five determined vertices in the bottom line including the vertex outside W. Moreover, the underlined vertices in the upper line must be in different states, which gives a situation depicted in Figure 2.1. And in the outer state must be all determined vertices in the bottom line including two vertices outside W, a contradiction.

$$\begin{array}{cccc} (\cdot\cdot(\times)\times(\times)(\times(\times)\times)\times\,\mathbf{)} & (\cdot\cdot(\times)(\circ\times)\times) & (\cdot\cdot(\times)(\times\circ)\times\,\mathbf{)} & (\cdot\cdot\cdot)(\cdot\cdot\cdot) \\ \cdot\cdot\circ\circ\circ\times\circ\times\circ & \mathbf{(}\,\circ\circ\times\,\mathbf{)}\circ & (\cdot\circ\circ\mathbf{)}\times\circ\underline{\cdot} & \circ\underline{\times\times}\circ \end{array}$$

Figure 2.

If the underlined vertices in Figure 1.3 are in a common state, then one is in W and one is not in W. If the left-hand side vertex is in W, then we have the situation depicted in Figure 2.2. Then the bold parentheses bound two different outer states, and each of them must contain a vertex which is not in W, a contradiction. So the right-hand side vertex is in W, see Figure 2.3. Then the bold parentheses bound the nonempty outer state. Moreover, the underlined vertex cannot be in W, since then there would be another empty state. Hence, the underlined vertex is not in W. But since no state contains two vertices which are not in W, the underlined vertex cannot exist (or in other words, it must coincide with the vertex $\times$ on the right-hand side of the upper line). Thus $n = 10$, a contradiction.

Finally, suppose that none of the underlined vertices in Figure 1.1 is in W, see Figure 2.4. Since underlined vertices cannot belong to a common state, by symmetry we

$$
\begin{array}{cccc}
(\cdot \circ \underline{\cdot})(\cdot \times \cdot) & \boldsymbol{(}\underline{\cdot} \circ \circ)(\cdot \times \cdot) & (\cdot \circ \times)(\cdot \times \cdot) & \boldsymbol{(}\underline{\cdot(\cdot}\circ)\times)(\cdot \times \cdot) \\
\circ \times (\times \circ \cdot) & \circ(\times(\times\circ)\cdot\boldsymbol{)} & \circ \times (\times \circ \underline{\cdot}) & \circ \times (\times \circ \circ\boldsymbol{)}
\end{array}
$$

Figure 3.

may assume the situation depicted in Figure 3.1. In this figure the symmetric vertex in the upper line (that one which is denoted by $\times$) cannot be in W since that would give another empty state. Now suppose that the underlined vertex is in W. That gives a situation depicted in Figure 3.2. But then the underlined vertex is not in W and by bold parentheses we have denoted boundaries of two different outer states (since every state except the unique empty one must contain a vertex outside W), a contradiction.

Hence the underlined vertex in Figure 3.1 is not in W, see Figure 3.3. Now if the underlined vertex is in W, then we get the situation depicted in Figure 3.4. Here the underlined vertices are not in W and bold parentheses bound two different outer states, a contradiction.

Hence the underlined vertex in Figure 3.3 is not in W, which means that the vertices in the parentheses in the bottom line must be in two different states. We consider three subcases.

$$
\begin{array}{cccc}
(\cdot \circ \times)(\circ \times \cdot) & (\circ \circ \times)(\times \times \cdot) & \circ(\underline{\times \circ \times})(\times \times \cdot) & \boldsymbol{(}\underline{\cdot} \circ (\times) \circ \times)(\times \times \cdot) \\
\boldsymbol{(} \circ \times (\times) \circ \times \boldsymbol{)} & \circ \times \boldsymbol{(} \times (\circ\times)\underline{\cdot}\boldsymbol{)} & \circ \times (\times \circ (\times) \cdot \cdot) & \circ \times \boldsymbol{(} \times \circ (\times) \circ \underline{\cdot}\boldsymbol{)}
\end{array}
$$

Figure 4.

The situation in the first subcase is depicted in Figure 4.1. Here the bold parentheses bound two different nonempty outer states, a contradiction.

The situation in the second subcase is depicted in Figure 4.2. Here the underlined vertex must be outside W and again the bold parentheses bound two different nonempty outer states, a contradiction.

The situation in the third subcase is depicted in Figure 4.3. Here the underlined vertices cannot be in a common state, so we have a situation depicted in Figure 4.4. The underlined vertices must be outside W and bold parentheses bound two different nonempty outer states, a contradiction.

Thus, we can exclude the empty state in the following consideration, which means that there is no $v_i \in W$ such that also $v_{i+3} \in W$.

Case 2: *There is a state containing a unique vertex which is from W.*

Then all the remaining states must contain a vertex outside W, which means also that there is at most one outer state. In this case there are three subcases. The first one is described in Figure 5.1. Here the underlined vertex must be outside W, and so the bold parentheses bound two different outer states, a contradiction.

The second subcase is in Figure 5.2. By Case 1, the underlined vertices must be

$$\begin{array}{cccc}
(\cdot\cdot(\circ)\cdot\cdot) & (\cdot\cdot\cdot)\circ(\cdot\cdot\cdot) & (\cdot\circ\cdot)\circ(\cdot\cdot\cdot) & (\cdot\circ(\cdot)\circ(\cdot)\cdot\cdot) \\
\boldsymbol{(}\circ\underline{\cdot}\circ\boldsymbol{)} & \circ(\underline{\cdot}\cdot\underline{\cdot})\circ & \circ(\times\underline{\cdot}(\times)\circ\cdot) & \circ\boldsymbol{(}\times\circ(\times)\circ\underline{\cdot}\boldsymbol{)}
\end{array}$$

Figure 5.

outside W. Hence, they must be in different states. Since there cannot be another state containing no vertex outside W, by symmetry we have a situation depicted in Figure 5.3. Now if the underlined vertex is in W, then the situation is depicted in Figure 5.4 where the underlined vertex is not in W. Hence the bold parentheses bound two different outer states, a contradiction. So the underlined vertex in Figure 5.3. is not in W. Since every state contains at most one vertex which is not in W, we have the situation depicted in Figure 6.1. Here the underlined vertices are not in W, so the bold parentheses bound two different outer states, a contradiction.

$$\begin{array}{cccc}
(\cdot\circ\cdot)\circ(\cdot\circ\cdot) & (\circ(\underline{\cdot\cdot})\cdot) & \boldsymbol{(}\circ(\times(\times)\cdot)\cdot\boldsymbol{)} & \boldsymbol{(}\cdot(\circ(\times)\times)\cdot\boldsymbol{)} \\
\boldsymbol{(}\underline{\cdot}\circ(\times)\times(\times)\circ\underline{\cdot}\boldsymbol{)} & \circ(\circ\cdot\cdot) & \circ\circ(\circ\cdot\cdot) & \circ(\circ\circ\cdot)
\end{array}$$

Figure 6.

The third subcase is depicted in Figure 6.2. If the underlined vertices are in a common state, then one of them must be in W. However, then there is another state which contains a unique vertex and this vertex is in W, a contradiction. Hence, the underlined vertices are outside W and they are in different states. We have two possibilities. The first one is depicted in Figure 6.3, where bold parentheses bound two different outer states, a contradiction. The second one is depicted in Figure 6.4, where again bold parentheses bound two different outer states, a contradiction.

Case 3: *There is a state containing exactly two vertices and both of them are from W.*

Also now, the remaining states must contain a vertex outside W, and there is at most one outer state. Again, we have three subcases. The first one is depicted in Figure 7.1. But there are two states without a vertex outside W, a contradiction.

The second subcase is in Figure 7.2. From Case 1 we know that the underlined vertices are outside W, so they must be in different states. By symmetry, we can assume the situation depicted in Figure 7.3. But then the bold parentheses bound two different outer states, a contradiction.

$$\begin{array}{cccc}
(\cdot(\circ\circ)\cdot) & (\cdot\cdot\cdot)\circ\circ(\cdot\cdot\cdot) & (\cdot\cdot\circ)\circ\circ(\cdot\cdot\cdot) & (\circ\circ(\cdot)\cdot\cdot) \\
(\cdot(\circ\circ)\cdot) & \circ(\cdot(\underline{\cdot\cdot})\cdot)\circ & \circ\boldsymbol{(}\cdot(\times(\times)\cdot)\circ\boldsymbol{)} & \circ\boldsymbol{(}\underline{\cdot}(\circ\underline{\cdot})\underline{\cdot}\boldsymbol{)}
\end{array}$$

Figure 7.

The third subcase is depicted in Figure 7.4. Here the underlined vertices must be outside W and so the bold parentheses bound two different outer states, a contradiction.

Case 4: *There is a state containing at least three vertices and all of them are from W.*
Four consecutive vertices from W imply that there is an empty state, so we need to consider a possibility of a state containing exactly three consecutive vertices, all from W. Observe that the remaining states must contain a vertex outside W, and there is at most one outer state.
We have two subcases. The first one is depicted in Figure 8.1. But there are two states without a vertex outside W, a contradiction.
The second subcase is in Figure 8.2. Here the bold parentheses bound an outer state. Consequently $n = 8$, a contradiction.

$$\begin{array}{cccc}
(\circ\circ\circ) & (\cdot\cdot\cdot)\circ\circ\circ(\cdot\cdot\cdot) & \boldsymbol{(}\cdot(\cdot(\cdot)\cdot)\cdot\boldsymbol{)} & (\cdot(\underline{\cdot\cdot})\cdot) \\
(\cdot(\cdot(\circ)\cdot)\cdot) & \circ\boldsymbol{(}\cdot(\cdot(\cdot)\cdot)\cdot\boldsymbol{)}\circ & \circ\circ\circ & \times\circ\circ\times
\end{array}$$

Figure 8.

From Cases 1, 2, 3 and 4 it follows that there is no state which does not contain vertices outside W, which implies that there is at most one outer state, and every state contains exactly one vertex which is not in W.

Case 5: *W contains three consecutive vertices.*
This situation is depicted in Figure 8.3. Then all the vertices inside the bold parentheses are outside W, and bold parentheses bound an outer state. This outer state is unique, so there are no other parentheses, and hence there are only three vertices in W. Since $n \geq 14$, the outer state contains at least ten vertices outside W, a contradiction.

Case 6: *W contains two but not three consecutive vertices.*
This situation is depicted in Figure 8.4. Then the underlined vertices are in a common state, so one of them is in W. By symmetry, we may assume that the left-hand side one is in W, see Figure 9.1. But then the bold parentheses bound two different outer states, a contradiction.

$$\begin{array}{cccc}
\boldsymbol{(}\cdot(\circ\times)\cdot\boldsymbol{)} & (\cdot\cdot(\cdot)\underline{\cdot\cdot}) & (\cdot\cdot(\times)\circ\times) & \boldsymbol{(}\cdot\circ(\times)\circ\times\boldsymbol{)} \\
\times\boldsymbol{(}\circ\circ\times\boldsymbol{)} & \times\circ\times\circ\times & (\underline{\times}\circ\underline{\times})\circ\times & \boldsymbol{(}\times\circ(\times)\circ\times\boldsymbol{)}
\end{array}$$

Figure 9.

Case 7: *W contains no pair of consecutive vertices, but $v_i, v_{i+2} \in W$ for some i.*
This situation is depicted in Figure 9.2. We distinguish three subcases with respect to the underlined vertices in Figure 9.2. First suppose that the left-hand side one is in W. Then the other is outside W, see Figure 9.3. Now the underlined vertices must be in distinct states for which we have two possibilities. The first one is described in Figure 9.4, where bold parentheses bound distinct outer states, a contradiction. The

second one is described in Figure 10.1, where bold parentheses bound an outer state which has at least four vertices since $n \geq 14$. So in this outer state there are either two consecutive vertices from W (which was considered in the previous cases) or two vertices not from W, a contradiction.

$$\begin{array}{cccc}
(\cdot\cdot(\times)\circ\times\boldsymbol{)}\circ & (\cdot\cdot(\times)\times\circ) & \boldsymbol{(}\circ\cdot(\times)\times\circ\boldsymbol{)} & \circ(\underline{\cdot\cdot}(\times)\times\circ\boldsymbol{)} \\
(\cdot\cdot(\times)\circ\times\boldsymbol{)}\circ\times & (\cdot\times\circ)\underline{\times}\circ\underline{\times} & (\cdot\times\circ)\times(\circ\times\cdot) & (\cdot\times\circ\boldsymbol{)}\times\circ\boldsymbol{(}\times\cdot\cdot)
\end{array}$$

Figure 10.

Now suppose that the right-hand side underlined vertex in Figure 9.2 is in W, see Figure 10.2. The underlined vertices cannot belong to a common state. So we have either the situation depicted in Figure 10.3, where bold parentheses bound two distinct outer states, a contradiction, or the situation depicted in Figure 10.4. Due to Cases 1 and 6, underlined vertices in Figure 10.4 must be outside W. Hence, bold parentheses bound two distinct outer states, a counterexample.

Finally, suppose that the underlined vertices in Figure 9.2 are in different states. Since consecutive vertices cannot be in W, we have the situation depicted in Figure 11.1. Now we can consider underlined vertices in Figure 11.1 analogously as those on Figure 9.2. By previous two subcases, we may assume that they are in different states. And this procedure can be repeated $\frac{n-4}{2}$ times until we obtain the other end of the upper line. Then there are $\frac{n}{2}$ vertices in W. These $\frac{n}{2}$ vertices create n parentheses which bound n states, all of which contain single vertices. But then half of the states do not contain vertices outside W, a contradiction.

$$\begin{array}{cc}
(\cdot\cdot(\cdot)\cdot(\cdot)\underline{\cdot\cdot}) & \boldsymbol{(}\cdot\cdot\cdot\boldsymbol{)} \\
\times\circ\times\circ\times\circ\times & \times\times\times\circ\times\times\times
\end{array}$$

Figure 11.

Case 8: *If $v_i \in W$, then $v_{i-3}, v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}, v_{i+3} \notin W$.*
This situation is depicted in Figure 11.2. Then the bold parentheses bound the outer state. Since $n \geq 14$, this outer state contains at least ten vertices which is not in W, a contradiction. □

In Lemma 3, we present a lower bound for $\dim(C_n(1,2,\ldots,t))$, where $n = 2(d-1)t + 1 + r$, $d = 2$ and $r = 4$.

Lemma 3. *Let $n = 2t + 5$ where $t \geq 5$. Then*

$$\dim(C_n(1,2,\ldots,t)) \geq \left\lceil \frac{2t}{3} \right\rceil + 2.$$

Proof. Since $t \geq 5$, we have $n \geq 15$. By Theorem 2, $\dim(C_n(1,2,\ldots,t)) \geq \frac{n}{2d-1} = \frac{2t+5}{3} = \frac{2t+2}{3} + 1$. So $\dim(C_n(1,2,\ldots,t)) \geq \lceil \frac{2t}{3} \rceil + 2$ if $t \equiv 0 \pmod 3$ or $t \equiv 1 \pmod 3$,

and $\dim(C_n(1,2,\ldots,t)) \geq \lceil \frac{2t}{3} \rceil + 1$ if $t \equiv 2 \pmod 3$. Thus, if $\dim(C_n(1,2,\ldots,t)) = \lceil \frac{2t}{3} \rceil + 1$, then $t \equiv 2 \pmod 3$ and every state contains exactly one vertex which is not in W. We show that such a situation cannot occur.
Assume to the contrary that $\dim(C_n(1,2,\ldots,t)) = \lceil \frac{2t}{3} \rceil + 1$. So $t \equiv 2 \pmod 3$. Since $\dim(C_n(1,2,\ldots,t)) = \lceil \frac{2t}{3} \rceil + 1$, the vertices of a minimum resolving set W form $2(\lceil \frac{2t}{3} \rceil + 1)$ states. In these states there are $n - (\lceil \frac{2t}{3} \rceil + 1)$ vertices which are not in W. Since $t \equiv 2 \pmod 3$, we have then $2(\lceil \frac{2t}{3} \rceil + 1) = 2t + 5 - (\lceil \frac{2t}{3} \rceil + 1)$, and so all the states must contain exactly one vertex which is not in W. This implies that there is at most one outer state and there cannot be an empty state.
Analogously as in the proof of Lemma 2, we use the following fact. Suppose that there is a collection of consecutive states with at least four consecutive vertices, finished by a closing parenthesis, such that every state in the collection contains exactly one vertex outside W. Then (recall that there is not an empty state) the finishing closing parenthesis bounds an outer state.
We distinguish several cases.

Case 1: *W contains three consecutive vertices.*
This situation is depicted in Figure 12.1. If there are four consecutive vertices in W, then the situation is depicted in Figure 12.2 and bold parentheses bound the outer state. That means that all parentheses are already present in Figure 12.2, since the outer state is unique. So $|W| = 4$. Since $t \geq 5$, we have $|W| < \lceil \frac{2t}{3} \rceil + 1$, a contradiction.

$$\begin{array}{cc} (\cdot(\cdot(\cdot\cdot)\cdot)\cdot) & \boldsymbol{(}\cdot(\cdot(\cdot(\cdot)\cdot)\cdot)\cdot\boldsymbol{)} \\ \circ\;\circ\;\circ & \circ\;\circ\;\circ\circ \end{array}$$

Figure 12.

So there are exactly three consecutive vertices in W, see Figure 13.1. But the underlined vertices must be resolved, so one of them is in W. By symmetry, we assume that the left-hand side one is in W, see Figure 13.2. Since every state contains one vertex outside W, the thick parentheses bound outer states. These outer states are different, since inbetween there are some other states (see the upper lilne of Figure 13.2). Thus, there are at least two outer states, contradiction.

$$\begin{array}{cccc} (\cdot(\cdot(\underline{\cdot\cdot})\cdot)\cdot) & (\times(\times(\circ\times)\times)\times) & (\cdot(\underline{\cdot\;\cdot\;\cdot})\cdot) & (\times(\underline{\cdot\cdot}(\times)\times)\cdot\cdot) \\ \times\;\circ\;\circ\;\circ\;\times & \times\boldsymbol{(}\;\circ\;\circ\;\circ\;\times\boldsymbol{)} & \times\;\circ\;\circ\times & \circ\times\circ\;\circ\;\times \end{array}$$

Figure 13.

Case 2: *W contains two but not three consecutive vertices.*
This situation is depicted in Figure 13.3. The underlined vertices on Figure 13.3 cannot belong to a common state because then the outer parentheses in the upper line of Figure 13.3 bound an outer state (recall that there is not an empty state). This

outer state is unique, so it contains all the vertices, except the five in the upper line of Figure 13.3. Since $n \geq 15$, it contains also both vertices outside W in the bottom line of Figure 13.3, a contradiction.
So there is a vertex of W neighbouring the chain $\times \circ \circ \times$ in the bottom line of Figure 13.3. By symmetry, we may assume that this vertex is on the left-hand side of the chain, see Figure 13.4. But the underlined vertices must be resolved, so we consider three subcases.

$$\begin{array}{ccc} (\cdot\cdot(\times(\times)\times(\times)\times)\cdot\cdot) & (\cdot\times(\times(\times)\times(\times)\times)\circ\times\boldsymbol{)} & \circ(\times\times(\times(\times)\times(\times)\times)\circ\times\boldsymbol{)} \\ \circ\times\underline{\circ\circ}\times\circ & (\cdot\cdot\circ\times\,\boldsymbol{)}\circ\circ\times\circ\underline{\cdot} & (\cdot\cdot\circ\times\,\boldsymbol{)}\circ\circ\times\circ\boldsymbol{(}\times \end{array}$$

Figure 14.

First, suppose that the underlined vertices in Figure 13.4 are in different states. Since there are not three consecutive vertices in W, we have a situation depicted on Figure 14.1. Here underlined vertices must be in the outer state, due to the vertices outside W in the upper line. But the outer state must contain a vertex outside W. By symmetry, we may assume that it is the right-hand side neighbour of underlined vertices, which gives a situation depicted on Figure 14.2, where bold parentheses bound the outer state. The underlined vertex in Figure 14.2 cannot be in W since that would give an empty state. But since the outer state contains exactly one vertex outside W, in the bottom line we have either $(\cdot\cdot\circ\times\,\boldsymbol{)}\circ\circ\times\,\boldsymbol{(}\underline{\circ}\times$ or $(\cdot\cdot\circ\times\,\boldsymbol{)}\circ\circ\times\circ\boldsymbol{(}\times$. The first case is impossible, since the last vertex before outer state is outside W in the upper line of Figure 14.2, while in the first case it is in W (see the underlined vertex). So we have the second case which is depicted in Figure 14.3. Here three bold parentheses bound the outer state. Since it is unique, the two bold parentheses on the right-hand side must coincide, which gives $n = 15$. Thus, all vertices are already in the diagram. Hence, when splitting the vertices between consecutive vertices of W, the situation is

$$\circ\boldsymbol{(}\times\circ(\times\times(\times(\times)\times(\times)\times)\circ\times\boldsymbol{)}\circ\times\circ$$

and there are two vertices outside W in a common state, a contradiction.
So now suppose that the underlined vertices in Figure 13.4 are in a common state. Then one of them must be in W. Suppose that it is the right-hand side one, see Figure 15.1. Since every state must contain one vertex outside W, the bold parenthesis bounds the outer state. We have two possibilities for the underlined vertex in Figure 15.1. First suppose that it is in W, see Figure 15.2. Then its neighbour must be outside W since every state must contain a vertex outside W, and for the same reason the bold parentheses bound the outer state. However, due to the bottom line, there are at least two different outer states, a counterexample. So suppose that the underlined vertex in Figure 15.1 is not in W, see Figure 15.3. Then the bold parenthesis in the bottom line bounds an outer state due to three consecutive vertices outside W in the upper line. But since there is not an empty state, the underlined vertices cannot be in W. Hence, the underlined vertices are split by the bold parenthesis of the upper line which gives $n = 13$, a contradiction.

$$\begin{array}{cccc}
\boldsymbol{(}\times(\times\circ(\times)\times)\underline{\cdot}\cdot) & \boldsymbol{(}\times(\times\circ(\times)\times)\circ\times\boldsymbol{)} & \boldsymbol{(}\times(\times\circ(\times)\times)\times\cdot) & \boldsymbol{(}\times(\circ\times(\times)\times)\cdot\cdot) \\
\circ(\times\circ\circ\times) & (\cdot\cdot\circ(\times)\circ\circ\times) & \underline{\cdot\cdot}\circ\boldsymbol{(}\times\circ\circ\times) & \underline{\cdot}\circ\times\boldsymbol{(}\circ\circ\times\cdot)
\end{array}$$

Figure 15.

Finally, suppose that the left-hand side underlined vertex on Figure 13.4 is in W, see Figure 15.4. Since every state must contain vertex outside W, the bold parentheses bound the outer state. But for the same reason underlined vertex in Figure 15.4 cannot be in W. Hence, this vertex cannot be in the outer state, since this state already has a vertex outside W. So the outer state contains exactly two vertices and $n = 11$, a contradiction.

Case 3: *W contains no pair of consecutive vertices, but $v_i, v_{i+2} \in W$ for some i.* This situation is depicted in Figure 16.1. Since W contains no consecutive vertices, underlined vertices in Figure 16.1 belong to a common state. Hence, one of them is in W. By symmetry, we may assume that the right-hand side one is in W, see Figure 16.2.

$$\begin{array}{cccc}
(\cdot\cdot(\underline{\cdot\cdot})\cdot\cdot) & (\underline{\cdot\cdot}(\times\circ)\cdot\cdot) & (\cdot\cdot\cdot(\times)\times(\times\circ)\cdot\cdot) & (\times\circ\times(\times)\times(\times\circ)\cdot\cdot) \\
\times\circ\times\circ\times & (\times\circ\times\circ)\times & (\times\circ\times\circ\boldsymbol{)}\times\underline{\cdot}\circ & (\times\circ\times\circ\boldsymbol{)}\times\boldsymbol{(}\underline{\times}\circ\underline{\times}\cdot)
\end{array}$$

Figure 16.

First suppose that the underlined vertices in Figure 16.2 belong to different states. This gives the situation depicted on Figure 16.3 since consecutive vertices are not in W and every state contains a vertex outside W. Moreover, the bold parenthesis bounds an outer state due to three consecutive vertices outside W in the upper line. Since W does not have consecutive vertices, the underlined vertex in Figure 16.3 is outside W. And since there cannot be two vertices outside W in a state, we must have a situation depicted in Figure 16.4. Due to four consecutive vertices outside W, we have an outer state with a single vertex outside W, see Figure 16.4. Since underlined vertices must be in different states, we get a situation depicted in Figure 17.1. But then the underlined vertices are in a common state, so one is in W, see Figure 17.2. Then the bold parenthesis in the upper line bounds another outer state, a contradiction.

$$\begin{array}{cc}
\cdot\circ(\times\circ\times(\times)\times(\times\circ)\cdot\cdot) & \boldsymbol{(}\times\circ(\times\circ)\times(\times)\times(\times\circ)\cdot\cdot) \\
(\times\circ\times\circ\boldsymbol{)}\times\boldsymbol{(}\times\circ(\underline{\times\cdot})\cdot\cdot) & (\times\circ\times\circ\boldsymbol{)}\times\boldsymbol{(}\times\circ(\times\circ)\times\cdot)
\end{array}$$

Figure 17.

So the underlined vertices in Figure 16.2 belong to a common state. But the left-hand side one cannot be in W, since then there will be a state without a vertex outside W. So we get the situation depicted in Figure 18.1. Now underlined vertices in Figure

18.1 cannot be in W since that will give a state without a vertex outside W. So they are outside W and hence belong to different states, see Figure 18.2. Since every state contains exactly one vertex outside W, the bold parentheses bound the outer state. And since there cannot be an empty state, the underlined vertices in Figure 18.3 are outside W. Hence, the outer state contains at least two vertices outside W, a contradiction.

$$\begin{array}{ccc} (\times \circ (\times\circ)\underline{\cdot\cdot}) & \boldsymbol{(} \times \circ(\times\circ) \times (\times) \cdot\cdot\cdot) & \underline{\cdot}\boldsymbol{(} \times \circ(\times\circ) \times (\times) \cdot\cdot\cdot) \\ (\times \circ (\times\circ) \times \cdot) & \circ \cdot \boldsymbol{(} \times \circ(\times\circ) \times \cdot) & \circ\underline{\cdot}\boldsymbol{(} \times \circ(\times\circ) \times \times) \cdot\cdot\cdot \end{array}$$

Figure 18.

Case 4: *If $v_i, v_j \in W$, then $|i-j| \geq 3$ and there is k such that $v_k, v_{k+3} \in W$.*
This situation is depicted in Figure 19.1. By previous cases, at most one of the underlined vertices in Figure 19.1 can be in W, and so these vertices cannot all belong to a common state. By our restrictions, there must be parenthesis separating the right-hand side vertex, see Figure 19.2. Now we can repeat the arguments and we get $n = 3k$ for some k. Observe that $k = |W|$. Since $n = 2t + 5$, the value k is odd, and so considering the vertices opposite to v_i (those in the upper line of Figure 19.2), we get that the situation is $\times\times(\circ)\times\times(\circ)\times\times$, a contradiction.

$$\begin{array}{ccc} (\cdot\cdot\cdot(\cdot)\underline{\cdot\cdot\cdot}) & (\cdot\cdot\cdot(\cdot)\cdot\cdot(\cdot)\cdot\cdot\cdot) & \boldsymbol{(}\cdot\cdot\cdot\cdot\boldsymbol{)} \\ \times\times\circ\times\times\circ\times\times & \times\times\circ\times\times\circ\times\times\circ\times\times & \times\times\times\circ\times\times\times \end{array}$$

Figure 19.

Case 5: *If $v_i \in W$, then $v_{i-3}, v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}, v_{i+3} \notin W$.*
This situation is depicted in Figure 19.3. Due to two collections of three consecutive vertices outside W in the bottom line, bold parentheses bound the outer state. This state is unique (so there cannot be another parentheses in the diagram) and it contains $n-4$ vertices. Since $n \geq 15$ and only one of the $n-4$ vertices is in W, the outer state contains at least ten vertices outside W, a contradiction. $\square$

In Lemma 4, we present a lower bound for $\dim(C_n(1, 2, \ldots, t))$, where $n = 2(d-1)t + 1 + r$ and $t = 5$. Lemma 4 is used in the proof of Theorem 3. Note that the set of vertices adjacent to v is the neighbourhood of v and it is denoted by $N(v)$.

Lemma 4. *Let $n = 10(d-1)+1+r$ where $d \geq 6$ and $3 \leq r \leq 4$. Then* $\dim(C_n(1,2,3,4,5)) \geq 6$.

Proof. By way of contradiction, suppose that $\dim(C_n(1,2,3,4,5)) \leq 5$. So, there exists a resolving set containing (at most) five vertices. Take one vertex from the resolving set of $C_n(1,2,3,4,5)$, say v_x, and consider its neighbourhood. In Figure 20 we have the vertices of $N(v_x)$ as a sequence $v_{x-5}, v_{x-4}, \ldots v_x, v_{x+1}, \ldots, v_{x+5}$, and those distinct from v_x are denoted by circles. By vertical bars we denote borders caused by v_x and by dots we denote places for possible borders which should appear due to the other vertices of the resolving set. However, from the two possible borders between the pairs $v_{x-1}v_x$ and v_xv_{x+1}, one is sufficient. So, to resolve all the vertices in Figure 20, we need nine borders. If another vertex from the resolving set appears in $N(v_x)$, then we can regard this vertex as a very specific border, since this specific border resolves only one vertex. Anyway, such a vertex creates only two borders in $N(v_x)$. And since we need nine borders using only four remaining vertices of the resolving set, some vertex, say v_y, must create three borders in $N(v_x)$. But this is possible only if vertices at distance d from v_y form a subset of $\{v_{x-4}, v_{x-3}, \ldots, v_{x+4}\}$.

$$|\circ.\circ.\circ.\circ.\circ.v_x.\circ.\circ.\circ.\circ.\circ|$$

Figure 20.

If $r = 4$, then vertices at distance d from v_y must be $\{v_{x-4}, v_{x-3}, v_{x-2}, v_{x-1}\}$ so that the third border caused by v_y is between v_{x+4} and v_{x+5}, or they must be $\{v_{x+1}, v_{x+2}, v_{x+3}, v_{x+4}\}$. If $r = 3$, then we have more choices. Anyway, to resolve all the vertices in $N(v_x)$, the resolving set must contain a vertex (v_y in our case) which is opposite to v_x. (It suffices to take just the vertices $v_{z+\lfloor \frac{n}{2} \rfloor -3}, v_{z+\lfloor \frac{n}{2} \rfloor -2}, \ldots, v_{z+\lfloor \frac{n}{2} \rfloor +4}$ as being oppoite to v_z.) Then also v_x is opposite to v_y. Since we suppose that there are five vertices in the resolving set (and $|N(v_x)|$ is much smaller than $\frac{n}{2}$), there must be a vertex in the resolving set, say v_a, which is opposite to two vertices of the resolving set, say v_b and v_c. Then the sets of vertices at distance d from v_b (from v_c) are contained in $v_{a-4}, v_{a-3}, \ldots, v_{a+4}$. In the following we consider the situation in $N(v_a)$ in detail. Borders caused by v_a are denoted by vertical bars, while borders caused by v_b (by v_c) are denoted by "()" (by "[]"), so that the vertex which causes the borders is not inside the brackets.

Case 1: $r = 4$.
Up to symmetry v_b and v_c cause borders as described in Figure 21. However, there remain six places for the borders which should appear due to remaining two vertices of the resolving set. But the only possibilities for vertices creating three borders in $N(v_a)$ are already occupied by v_b and v_c, so the resolving set cannot resolve all the vertices of $N(v_a)$.

$$|\circ\,[(\circ\;\circ\;\circ\circ)v_a[\circ\;\circ\;\circ\circ])\;\circ\,|$$

Figure 21.

Case 2: $r = 3$.
Before we proceed to subcases, we introduce one notion. By index distance we mean the distance of vertices (or edges, i.e. possible borders) in $C_n(1)$. So v_i and v_{i+t}

have index distance t although their distance is 1 in $C_n(1, 2, \ldots, t)$. Observe that if a vertex v_x creates two distinct borders, then their index distance is either $5k$ or $5k+1$ (if v_x is between the borders) or $5k+3$ (if all the vertices at distance d from v_x are between the borders).

We extend Figure 21 to the second neighbourhood of v_a which is $N(N(v_a))$. We use the fact that n is big enough. Observe that for $d \geq 6$, we have $n = 10(d-1)+1+r \geq 54$, so $\frac{n}{2} \geq 27$.

Up to symmetry, we have two possibilities for v_b. Thus, we consider four subcases for the position of v_b and v_c, see Figure 22.

$$\begin{array}{rc}
\text{(i)} & \big| \circ (\circ[\circ.\circ, \circ\big| \circ (\circ[\circ\circ)\circ]v_a \circ \circ\circ)\circ] \circ \big|\circ : \circ; \circ)\circ] \circ \big| \\
\text{(ii)} & \big| \circ [(\circ.\ \circ\ .\ \circ\ .\ \circ \big| \circ [(\circ \circ \circ) \circ v_a[\circ \circ \circ]) \circ \circ\big| \circ \circ\circ]) \circ \circ\big| \\
\text{(iii)} & \big| \circ (\circ[\circ.\circ, \circ\big| \circ (\circ[\circ\circ) \circ v_a \circ [\circ\circ)\circ] \circ \big|\circ : \circ; \circ)\circ] \circ \big| \\
\text{(iv)} & \big| \circ [\circ(\circ.\circ, \circ\big| \circ [\circ(\circ \circ \circ)v_a[\circ \circ \circ]\circ) \circ \big|\circ : \circ; \circ]\circ) \circ \big|
\end{array}$$

Figure 22.

First consider the subcase (i). Possible borders denoted by "." and "," are at index distances 14, 15 and 13, 14 to possible borders ":" and ";", respectively. But index distance 14 cannot occur as an index distance between two borders caused by a single vertex of the resolving set. Even if a vertex adjacent to the possible border is in the resolving set, then consider possible borders between the next five vertices (i.e. $N(N(N(v_a)))$). And if there is the fourth vertex of the resolving set between these five vertices, then consider the five vertices of $N(N(N(v_a)))$ on the other side of v_a. Observe that here we need $25 = 15 + 5 + 5 < \frac{n}{2}$ vertices, which is satisfied for $d \geq 6$. Hence, the borders "." and ";" (at mutual index distance 15) are caused by one vertex of the resolving set, say v_e, and borders "," and ":" (at mutual index distance 13) are caused by v_f. But then between "," and ":" there are vertices at distance d from v_f. Consequently, v_e has no vertex which is opposite to it and so $N(v_e)$ cannot be resolved.

Subcase (ii) is impossible since the four vertices $v_{a-9}, v_{a-8}, v_{a-7}, v_{a-6}$ cannot be resolved by the two remaining vertices of the resolving set.

Finally, subcases (iii) and (iv) can be reduced to subcase (i), since they have identical position of the possible borders ".", ",", ":" and ";", see Figure 22. □

In the proof of Theorem 3, we use also the following result of Chau and Gosselin [2].

Lemma 5. *Let $n = 10(d-1)+3$ where $d \geq 3$. Then* $\dim(C_n(1, 2, 3, 4, 5)) = 6$.

Theorem 3 is the main result of this paper.

Theorem 3. *For $t \geq 5$,*

$$\dim(C_n(1, 2, \ldots, t)) \geq \left\lceil \frac{2t}{3} \right\rceil + 1,$$

where the equality holds if and only if $t = 5$ and $n = 13$.

Proof. Let $n = 2(d-1)t + 1 + r$, where $d \geq 2$, $t \geq 5$ and $1 \leq r \leq 2t$. If $r = 1$, then by Corollary 1, we get

$$\dim(C_n(1,2,\ldots,t)) \geq t+1 = \left\lceil \frac{2t}{3} \right\rceil + \left\lfloor \frac{t}{3} \right\rfloor + 1 \geq \left\lceil \frac{2t}{3} \right\rceil + 2.$$

Let $r \geq 2$. By Corollary 2, we have

$$\begin{aligned} \dim(C_n(1,2,\ldots,t)) &\geq t + \left\lceil \frac{r-t+1}{2d-1} \right\rceil = t - \left\lfloor \frac{t-r-1}{2d-1} \right\rfloor \\ &\geq t - \left\lfloor \frac{t-3}{3} \right\rfloor = t - \left\lfloor \frac{t}{3} \right\rfloor + 1 = \left\lceil \frac{2t}{3} \right\rceil + 1, \end{aligned} \tag{6}$$

where the second inequality is equality if and only if

$$\left\lfloor \frac{t-r-1}{2d-1} \right\rfloor = \left\lfloor \frac{t-3}{3} \right\rfloor. \tag{7}$$

We consider several cases.

Case 1: $r \geq 5$.
Then from (6), we get

$$\dim(C_n(1,2,\ldots,t)) \geq t - \left\lfloor \frac{t-r-1}{2d-1} \right\rfloor \geq t - \left\lfloor \frac{t-6}{3} \right\rfloor = \left\lceil \frac{2t}{3} \right\rceil + 2.$$

Case 2: $2 \leq r \leq 4$ and $d = 2$.
Then by Lemmas 1, 2 and 3, we have $\dim(C_n(1,2,\ldots,t)) \geq \lceil \frac{2t}{3} \rceil + 2$ (note that $\dim(C_n(1,2,\ldots,t)) \geq \frac{4t+4}{5}$ implies that $\dim(C_n(1,2,\ldots,t)) \geq \lceil \frac{2t}{3} \rceil + 2$ for $t \geq 6$) except for the case $r = 2$ and $t = 5$. If $r = 2$ and $t = 5$ (so $n = 13$), then by Lemma 1, we have $\dim(C_n(1,2,\ldots,t)) \geq \frac{4t+4}{5} = \frac{24}{5}$ which implies that $\dim(C_{13}(1,2,3,4,5)) \geq 5$. Since $\{v_0,\ v_1,\ v_2,\ v_4,\ v_5\}$ is a resolving set of $C_{13}(1,2,3,4,5)$, we have $\dim(C_{13}(1,2,3,4,5)) = 5 = \lceil \frac{2t}{3} \rceil + 1$.

Case 3: $r = 2$ *and* $d \geq 3$.
Then $\lfloor \frac{t-r-1}{2d-1} \rfloor < \lfloor \frac{t-3}{3} \rfloor$ except for the cases $t = 8$ where $d = 3$, and $t = 5$ (where $d \geq 3$). Note that $\lfloor \frac{t-r-1}{2d-1} \rfloor < \lfloor \frac{t-3}{3} \rfloor$ means that we do not have equality in (7), thus $\dim(C_n(1,2,\ldots,t)) \geq \lceil \frac{2t}{3} \rceil + 2$.
If $t = 5$, then by Lemma 5, $\dim(C_n(1,2,3,4,5)) = 6 = \lceil \frac{2t}{3} \rceil + 2$. If $t = 8$ and $d = 3$, we have $n = 35$ and it can be checked by a computer that $\dim(C_{35}(1,2,\ldots,8)) = 8 = \lceil \frac{2t}{3} \rceil + 2$ (where the vertices $v_0, v_2, v_4, v_{13}, v_{26}, v_{28}, v_{32}, v_{33}$ resolve the graph $C_{35}(1,2,\ldots,8)$).

Case 4: $3 \le r \le 4$ *and* $d \ge 3$.
If $t \ge 6$, then

$$\left\lfloor \frac{t-r-1}{2d-1} \right\rfloor \le \left\lfloor \frac{t-4}{5} \right\rfloor < \left\lfloor \frac{t-3}{3} \right\rfloor,$$

so we cannot have equality in (7), which implies that $\dim(C_n(1,2,\ldots,t)) \ge \lceil \frac{2t}{3} \rceil + 2$. If $t = 5$ and $d \ge 6$, then by Lemma 4, $\dim(C_n(1,2,3,4,5)) \ge 6 = \lceil \frac{2t}{3} \rceil + 2$. If $t = 5$ and $3 \le d \le 5$, then by Table 4 given in [2], we have $\dim(C_n(1,2,3,4,5)) = 6 = \lceil \frac{2t}{3} \rceil + 2$. □

Theorem 3 in combination with (1), (2), (3), (4) and (5) yields Corollary 3.

Corollary 3. *Let $n = 2(d-1)t + 1 + r$, where $d \ge 2$, $t \ge 1$ and $1 \le r \le 2t$. Then*

$$\dim(C_n(1,2,\ldots,t)) \ge \left\lceil \frac{2t}{3} \right\rceil + 1.$$

The equality is attained if and only if $t = 1$, $t = 2$ and $1 \le r \le 3$, $t = 4$ and $r = 3$, $t = 4$ and $n = 11$, $t = 4$ and $n = 19$, and $t = 5$ and $n = 13$.

From Theorem 3, we obtain Corollary 4 which proves Conjecture 1.

Corollary 4. *For $t \ge 6$,*

$$\dim(C_n(1,2,\ldots,t)) \ge \left\lceil \frac{2t}{3} \right\rceil + 2.$$

In [12], the authors have shown that for every $t \ge 7$, there exists an $n \in [2t+5, 2t+8]$ such that $\dim(C_n(1,2,\ldots,t)) \le \lceil \frac{2t}{3} \rceil + 2$, which means that the bound presented in Corollary 4 is sharp. The bound is sharp also for $t = 6$, because there exist values of n for which $\dim(C_n(1,2,3,4,5,6)) = 6 = \lceil \frac{2t}{3} \rceil + 2$ (for example $\{v_0, v_1, v_2, v_3, v_5, v_6\}$ is a resolving set of $C_{15}(1,2,3,4,5,6)$).

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