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Research Article

A characterization of locating Roman domination edge critical graphs

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Abstract: A Roman dominating function (or just RDF) on a graph $G = (V, E)$ is a function $f : V \longrightarrow \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of an RDF f is the value $f(V) = \sum_{u \in V} f(u)$. An RDF f can be represented as $f = (V_0, V_1, V_2)$, where $V_i = \{v \in V : f(v) = i\}$ for $i = 0, 1, 2$. An RDF $f = (V_0, V_1, V_2)$ is called a locating Roman dominating function (or just $LRDF$) if $N(u) \cap V_2 \neq N(v) \cap V_2$ for any pair u, v of distinct vertices of V_0. The locating-Roman domination number $\gamma_R^L(G)$ is the minimum weight of an $LRDF$ of G. A graph G is said to be a locating Roman domination edge critical graph, or just γ_R^L-edge critical graph, if $\gamma_R^L(G - e) > \gamma_R^L(G)$ for all $e \in E$. The purpose of this paper is to characterize the class of γ_R^L-edge critical graphs.

Keywords: locating Roman domination number, critical graph.

AMS Subject classification: 05C69

1. Introduction

In this paper, we continue the study of a variant of Roman domination, namely, locating Roman domination. We first present some necessary terminology and notations. For notation and graph theory terminology not given here, we follow [8].

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We consider finite, undirected, and simple graphs G with vertex set $V = V(G)$ and edge set $E = E(G)$. The number of vertices $|V(G)|$ of G is called the *order* of G and is denoted by $n = n(G)$. The *open neighborhood* of a vertex $v \in V$ is $N(v) = N_G(v) = \{u \in V \mid uv \in E\}$, and the *degree* of v, denoted by $\deg_G(v)$, (or just $\deg(v)$) is the cardinality of its open neighborhood. A *leaf* of a graph G is a vertex of degree one, while a support vertex of G is a vertex adjacent to a leaf. A *strong support vertex* is a support vertex adjacent to at least two leaves. We denote the set of all support vertices of G by $S(G)$ and the set of leaves by $L(G)$. We denote $\ell(G) = |L(G)|$ and $s(G) = |S(G)|$. We also denote by $L(x)$ the set of leaves adjacent to a support vertex x, and denote $\ell_x = |L(x)|$. An edge incident with a leaf is called a pendant edge. The subgraph induced in G by a subset of vertices S is denoted $G[S]$. A subset S is an independent set if no edge exists between any two vertices of $G[S]$. If $v \in D \subseteq V$ and $w \in V - D$, then the vertex w is a private neighbor of v (with respect to D) if $N(w) \cap D = \{v\}$. We denote the set of all private neighbors of v (with respect to D) with $pn(v, D)$.
A subset $S \subseteq V$ is a *dominating set* of G if every vertex in $V - S$ has a neighbor in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A locating-dominating set $L \subseteq V(G)$ is a dominating set with the property that for each vertex $x \in V(G) - L$ the set $N(x) \cap L$ is unique. That is, any two vertices x, y in $V(G) - L$ are distinguished in the sense that there is a vertex $v \in L$ with $|N(v) \cap \{x, y\}| = 1$. The minimum size of a locating-dominating set for a graph G is the locating-domination number of G, denoted $\gamma_L(G)$. The study of locating dominating sets in graphs was pioneered by Slater [13, 14].
For a graph G, let $f : V(G) \to \{0, 1, 2\}$ be a function, and let (V_0, V_1, V_2) be the ordered partition of $V(G)$ induced by f, where $V_i = \{v \in V(G) : f(v) = i\}$ for $i = 0, 1, 2$. There is a $1-1$ correspondence between the functions $f : V(G) \to \{0, 1, 2\}$ and the ordered partition (V_0, V_1, V_2) of $V(G)$. So we will write $f = (V_0, V_1, V_2)$ (or $f = (V_0^f, V_1^f, V_2^f)$ to refer to f). A function $f : V(G) \to \{0, 1, 2\}$ is a *Roman dominating function* (or just *RDF*) if every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of an *RDF* f is $w(f) = f(V(G)) = \sum_{u \in V(G)} f(u)$. The *Roman domination number* of a graph G, denoted by $\gamma_R(G)$, is the minimum weight of an *RDF* on G. A function $f = (V_0, V_1, V_2)$ is called a γ_R-function (or $\gamma_R(G)$-function when we want to refer f to G), if it is an *RDF* and $f(V(G)) = \gamma_R(G)$. For references in Roman domination see for example [15].
Jafari Rad and Rahbani [11, 12] introduced the concept of Locating-Roman domination in graphs. An *RDF* $f = (V_0, V_1, V_2)$ is called a *locating Roman dominating function* (or just *LRDF*) if $N(v) \cap V_2 \neq N(u) \cap V_2$ for any pair u, v of distinct vertices of V_0. The locating Roman domination number $\gamma_R^L(G)$ is the minimum weight of an *LRDF*. Note that $\gamma_R^L(G)$ is defined for any graph G, since $(\emptyset, V(G), \emptyset)$ is an *LRDF* for G. We refer to a $\gamma_R^L(G)$-function as an *LRDF* of G with minimum weight. If $f = (V_0, V_1, V_2)$ is an *LRDF* in G then for any vertex $v \in V_2$, we define $pn(v, V_2) = \{u \in V_0 : N(u) \cap V_2 = \{v\}\}$.
For many graph parameters, criticality is a fundamental question. The concept of criticality with respect to various operations on graphs has been studied for several

domination parameters. Much has been written about graphs where a parameter increases or decreases whenever an edge or vertex is removed or added. This concept has been considered for several domination parameters such as domination, total domination, global domination, secure domination and Roman domination, by several authors. This concept is now well studied in domination theory. For references on the criticality concept on various domination parameters see, for example [1–7, 9, 10]. In this paper we consider this concept for locating Roman domination number.
When we remove an edge e from a graph G, $G-e$ the Locating-Roman domination number can increase or remain unchanged , e.g., if G is a P_5 then $\gamma_R^L(G) = 4$ and $\gamma_R^L(G-e) = 5$ for all e edge of $E(G)$. If G is a P_3 then $\gamma_R^L(G) = \gamma_R^L(G-e) = 3$ for all e edge of $E(G)$. A graph G is said to be a Locating-Roman domination edge critical graph, or just a γ_R^L-edge critical graph, if $\gamma_R^L(G) < \gamma_R^L(G-e)$ for all e edge of $E(G)$. The purpose of this paper is to give a descriptive characterization of the class of γ_R^L-edge critical graphs.

2. Results

We first present the some properties of the γ_R^L-edge critical graphs.

Lemma 1. *For every edge $e = xy$ in a graph G, $\gamma_R^L(G-e) \leq \gamma_R^L(G) + 1$.*

Proof. Let $e = xy$ and $f = (V_0, V_1, V_2)$ be a $\gamma_R^L(G)$-function. If $V_2 \cap \{x, y\} = \emptyset$ or $\{x, y\} \subseteq V_2 \cup V_1$, then f is a *LRDF* of graph $G-e$, as desired. Thus we may assume that $x \in V_2$ and $y \in V_0$. Now we define the function g by $g(y) = 1$ and $g(u) = f(u)$, if $u \in V - \{y\}$. Then the function g is a *LRDF* of the graph $G-e$ and so $\gamma_R^L(G-e) \leq \gamma_R^L(G) + 1$. □

Then, Lemma 1 implies the following useful corollary.

Corollary 1. *For any edge e in a γ_R^L-edge critical graph G, $\gamma_R^L(G-e) = \gamma_R^L(G) + 1$.*

Lemma 2. *Let G be a connected γ_R^L-edge critical graph of order $n \geq 3$ and $f = (V_0, V_1, V_2)$ be any $\gamma_R^L(G)-$function. Then the following hold:*
(a) *V_2 and V_0 are independent sets and $|V_1| = 0$.*
(b) *Every support vertex is weak.*
(c) *For every vertex v with $f(v) = 2$, $|pn(v, V_2)| = 1$.*
(d) *The vertex v is a support vertex if and only if $f(v) = 2$.*

Proof. **(a)** If an edge e exists in $G[V_2]$ (respectively in $G[V_0]$), then f is also an *LRDF* of $G-e$. Thus $\gamma_R^L(G-e) \leq \gamma_R^L(G)$, a contradiction. Hence V_2 and V_0 are independent sets. Next assume that there exists a vertex v in G with $f(v) = 1$ and $x \in N(v)$. Then f is a *LRDF* for the graph $G-xv$ and so $\gamma_R^L(G-xv) \leq w(f) = \gamma_R^L(G)$, a contradiction. Hence $|V_1| = 0$.

(b) Assume that v is a strong support vertex. Let u, w be two leaves adjacent to v. By part (a), $v \in V_2 \cup V_0$. If $v \in V_2$, then by part **(a)**, $\{u, w\} \subseteq V_0$, a contradiction, since f is a $\gamma_R^L(G)$-function. Next assume that $v \in V_0$. Then by part **(a)** we have $\{u, v\} \subseteq V_2$ and so f is a *LRDF* for graph $G - uv$. Hence, $\gamma_R^L(G - uv) \leq \gamma_R^L(G)$, a contradiction. Therefore every support vertex is weak.
(c) Assume that $v \in V_2$, then $|pn(v, V_2)| \leq 1$, since f is a $\gamma_R^L(G)$-function. Let $u \in N(v)$, by part **(a)**, $f(u) = 0$. If $|pn(v, V_2)| = 0$, then there exists a vertex $w \in V_2$ such that $u \in N(w) \cap N(v)$. Then f is a *LRDF* for graph $G - uw$ and so $\gamma_R^L(G - uw) \leq \gamma_R^L(G)$, a contradiction. Hence $|pn(v, V_2)| = 1$.
(d) Assume that v is a support vertex and u is a leaf adjacent to v. Let $w \in N(v) - \{u\}$. If $f(v) \neq 2$, then by part **(a)**, $f(v) = 0$ and $f(u) = f(w) = 2$. Then f is a *LRDF* for graph $G - wv$ and so $\gamma_R^L(G - wv) \leq \gamma_R^L(G)$, a contradiction. Thus we deduce that $f(v) = 2$. Now assume that for some vertex $v \in V(G)$, $f(v) = 2$. By part **(c)**, $|pn(v, V_2)| = 1$. Let $pn(v, V_2) = \{u\}$. We show that $deg(u) = 1$. If $deg(u) > 1$ and $w \in N(u) - \{v\}$, then by part **(a)**, $w \in V_2$. Then f is a *LRDF* for graph $G - uw$ and so $\gamma_R^L(G - uw) \leq \gamma_R^L(G)$, a contradiction. Hence $\deg(u) = 1$ and so v is a support vertex. □

In the next we characterization of the class of γ_R^L-edge critical graphs. For this purpose we define a family of graphs, as follows.
Let $\mathcal{G}$ be the family of all connected bipartite graphs $G = (X, Y, E)$ of order $n \geq 3$ such that for every w in Y and for every nonempty subset $X' \subseteq N(w)$ there exists a unique $w' \in Y$ such that $N(w') = X'$. See Figure 1.

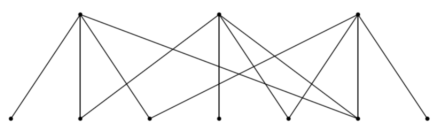

Figure 1. **An example of a graph in the family $\mathcal{G}$ with $|X| = 3$ and $|Y| = 7$.**

Lemma 3. *If $G \in \mathcal{G}$, then $\gamma_R^L(G) = 2|X|$.*

Proof. Assume that $G = (X, Y, E) \in \mathcal{G}$. Among all $\gamma_R^L(G)$-function, let $f = (V_0, V_1, V_2)$ be chosen to maximize the weight assigned to set X. We show that for any $x \in X$, $f(x) = 2$. Let $x \in X$ and $y \in Y$ be a unique leaf adjacent to x. Assume that $f(x) \neq 2$.
If $f(x) = 1$, then clearly, $f(y) = 1$. Then re-assigning to the leaf y the value 0 and re-assigning to the vertex x the value 2 produces a new *LRDF* f^* of G such that $w(f^*) \leq w(f)$ and the sum of the values assigned to set X under f^* is less than the sum of the values assigned to set X under f, a contradiction. Therefore, $f(x) \neq 1$. Next assume that $f(x) = 0$, then $f(y) = 2$ or 1. If $f(y) = 2$, then, as before, re-assigning to the leaf y the value 0 and re-assigning to the vertex x the value 2 produces

a new *LRDF* that contradicts our choice of the *LRDF* f . Therefore, $f(y) = 1$. Then there exist some vertex $u \in N(x) \cap Y$, such that $f(u) = 2$, since f is a $\gamma_R^L(G)$-function. Let $S = N(u) \cap V_0$. If $S = \{x\}$, then re-assigning to the vertex u the value 0 and re-assigning to the vertex x the value 2 produces a new *LRDF* f^* of G such that $w(f^*) \leq w(f)$ and the sum of the values assigned to set X under f^* is less than the sum of the values assigned to set X under f, a contradiction. Therefore, $|S| \geq 2$. Assume that $|S| = 2$. Let $S = \{x, w\}$. If v is the unique leaf adjacent to w, then clearly $f(v) = 1$. Then re-assigning to the vertices u, y and v the value 0 and re-assigning to the vertices x and w the value 2 produces a new *LRDF* f^* of G such that $w(f^*) \leq w(f)$ and the sum of the values assigned to set X under f^* is less than the sum of the values assigned to set X under f, a contradiction. Therefore, $|S| \geq 3$. Then by attention to the structure of G for every set $S' \subseteq S$, there exists one unique vertex $y' \in Y$, such that $N(y') = S'$. Let Y' be set of all $y' \in Y$, such that there exists a non-empty set $S' \subseteq S$ with $N(y') = S'$. Then, $|Y'| = 2^{|S|} - 1$ and also for every $y' \in Y'$, $f(y') \geq 1$, since every vertex of S have value 0. Then re-assigning to each vertex in set S' the value 2 and re-assigning to each vertex in set Y' the value 0 produces a new IDF f^* of G. Now we have, $w(f^*) = w(f) - \sum_{y' \in Y'} f(y') + 2|S| \leq w(f) - |Y'| + 2|S| = w(f) - 2^{|S|} + 2|S| + 1 < w(f)$, a contradiction. Hence for any $x \in X$, $f(x) = 2$ and so $\gamma_R^L(G) = w(f) \geq 2|X|$. On the other hand the function $h = (Y, \emptyset, X)$ is a *LRDF* of G and so $\gamma_R^L(G) \leq w(h) = 2|X|$. Consequently, $\gamma_R^L(G) = 2|X|$ □

Theorem 1. *A nontrivial connected graph $G = (V, E)$ is a γ_R^L-edge critical graph if and only if $G \in \mathcal{G}$.*

Proof. Assume that $G = (X, Y, E) \in \mathcal{G}$. Delete any edge $e = uv$ with $u \in X$ and $v \in Y$. Let $G' = G - e$. Among all $\gamma_R^L(G')$-function, let $f = (V_0, V_1, V_2)$ be chosen to maximize the weight assigned to set X. Let $z \in Y - N(u)$. In a way resembling the proof of Lemma 3, we imply that $f(z) = 0$ and for every vertex $x \neq u$ in X, $f(x) = 2$. We consider the following cases.

Case 1. e is a pendant edge. We show that $f(u) = 2$. Suppose to the contrary that $f(u) \neq 2$. If $f(u) = 0$, then there exist some vertex $y \in N_{G'}(u) \cap Y$, such that $f(y) = 2$, since f is a $\gamma_R^L(G')$-function. Let $S = N_{G'}(y) \cap V_0$. Since f is a $\gamma_R^L(G')$-function and for every $z \in Y - N(u)$, $f(z) = 0$, we deduce that $S = \{u\}$. Then re-assigning to the vertex y the value 0 and re-assigning to the vertex u the value 2 produces a new *LRDF* f^* of G such that $w(f^*) \leq w(f)$ and the sum of the values assigned to set X under f^* is less than the sum of the values assigned to set X under f, a contradiction. Thus we assume that $f(u) = 1$. If for every vertex $y \in N(u) - \{y\}$, $f(y) = 0$, then f is not a *LRDF*, a contradiction. Thus there exist at least one vertex $y \in N(u) - \{y\}$ such that $f(y) \geq 1$. Then re-assigning to the vertex y the value 0 and re-assigning to the vertex u the value 2 produces a new *LRDF* f^* of G such that $w(f^*) \leq w(f)$ and the sum of the values assigned to set X under f^* is less than the sum of the values assigned to set X under f, a contradiction. Hence $f(u) = 2$ and so $\gamma_R^L(G - v) = w(f) \geq 2|X|$. Thus $\gamma_R^L(G - e) = \gamma_R^L(G - v) + 1 \geq 2|X| + 1$. Consequently by Lemma 3 and Corollary 1, $\gamma_R^L(G-e)=\gamma_R^L(G)+1$.

Case 2. e is a non-pendant edge. As before we can see $f(u)=2$. Let $w\in Y$ such that $N(w)=N(v)-\{u\}$. Since f is a $\gamma_R^L(G-e)$-function, $f(v)\neq 0$ or $f(w)\neq 0$. Then $\gamma_R^L(G-e)=w(f)=\sum_{x\in X}f(x)+f(v)+f(w)\geq 2|X|+1$. Consequently by Lemma 3 and Corollary 1 , $\gamma_R^L(G-e)=\gamma_R^L(G)+1$. Hence in two cases G is a γ_R^L-edge critical graph.

Now assume that G be a γ_R^L-edge critical graph and $f=(V_0,V_1,V_2)$ be a $\gamma_R^L(G)$-function. then by Lemma 2, G is a bipartite graph with bipartition $X=V_2$ and $Y=V_0$. Now, it remains to show that: for every vertex $u\in V_0$ and for every nonempty subset $S\subseteq N(u)$ there exists an unique $v\in V_0$ such that $N(v)=S$. For this, let $u\in V_0, N(u)=\{v_1,\dots,v_k\}, k\geq 1$ and $S\subseteq N(u)$. If $|S|=k$, then since f is a $\gamma_R^L(G)-$function, u is the unique vertex in V_0 with $N(u)=S$.

Next assume that $|S|\leq k-1$. Assume that there is no vertex $v\in V_0$ such that $N(v)\cap V_2=S$. Let $S'\subseteq V_2$ such that $S\subseteq S'$ and $|S'|=min\{|R|:S\subseteq R\ \&\ \exists\, w\in V_0\ s.t\ N(w)=R\}$. Let $v_i\in S'-S$ for some $1\leq i\leq k$ and for vertex $y\in V_0$, $N(y)\cap V_2=S'$. Then f is a $LRDF$ for graph $G-yv_i$ and so $\gamma_R^L(G-yv_i)\leq\gamma_R^L(G)$, a contradiction. Hence for every vertex $u\in V_0$ and for every nonempty subset $S\subseteq N(u)$ exist $v\in V_0$ such that $N(v)=S$ and so $G\in\mathcal{G}$. $\square$

Notice that a disconnected graph G is γ_R^L-edge critical graph if and only if each component of G is γ_R^L-edge critical graph. So we have the following result.

Corollary 2. *A nonempty graph $G=(V,E)$ is γ_R^L-edge critical graph if and only if G is the union of independent sets and graphs of $\mathcal{G}$.*

Conflict of interest. The authors declare that they have no conflict of interest.

Data Availability. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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