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Research Article

The first leap Zagreb coindex of some graph operations

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Abstract: In the last years, Naji et al. have introduced leap Zagreb indices conceived depending on the second degrees of vertices, where the second degree of a vertex v in a graph G is equal to the number of its second neighbors and denoted by $d_2(v/G)$. Analogously, the leap Zagreb coindices were introduced by Ferdose and Shivashankara. The first leap Zagreb coindex of a graph is defined as $\overline{L_1}(G) = \sum_{uv \notin E_2(G)}(d_2(u) + d_2(v))$, where $E_2(G)$ is the 2-distance (second) edge set of G, In this paper, we present explicit exact expressions for the first leap Zagreb coindex $\overline{L_1}(G)$ of some graph operations.

Keywords: second-degrees (of vertices), leap Zagreb indices, coindices, graph operations.

AMS Subject classification: 05C07, 05C12, 05C76

1. Introduction

In this paper, we are concerned with simple graphs, i.e., finite graphs having no loops, no multiple and directed edges. Let $G = (V, E)$ be such a graph, the number of vertices and edges of a graph G, denoted by n and m, respectively. The distance between any two vertices $u, v \in V(G)$ denoted by $d_G(u, v)$ and is equal to the length of the shortest path connecting them. For a vertex $v \in V(G)$, the open 2-neighborhood of v in a graph G is defined as $N_2(v/G) = \{u \in V(G) : d_G(u, v) = 2\}$. The set of all second (2-distance) edges of a graph G is defined by $E_2(G) = \{vu : d_G(u, v) = 2,\ u, v \in V(G)\}$, and we denote by $\mu(G)$ or simply μ to the cardinality of $E_2(G)$. The second degree of a vertex v in G is denoted by $d_2(v/G)$ (or simply $d_2(v)$, if no misunderstanding) and is the number of its second neighbors, i.e., $d_2(v/G) = |N_2(v/G)|$. For a vertex

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$v \in V(G)$, the eccentricity $e(v) = \max\{d_G(v,u) : u \in V(G)\}$. The diameter of G is $diam(G) = \max\{e(v) : v \in V(G)\}$. Let $H \subseteq V(G)$. Then the induced subgraph $\langle H \rangle$ of G is the graph whose vertex set is H and whose edge set consists of all of the edges in $E(G)$ that have both endpoints in H. A graph G is called F-free graph if no induced subgraph of G is isomorphic to F. We follow [15], for unexplained graph theoretic terminologies and notations.

A topological index of a graph is a graph invariant calculated from a graph representing a molecule. Among the most important such structure descriptors are the classical first and second Zagreb indices, which were introduced by Gutman and Trinajestic [14], in 1972, and elaborated in [12]. They are defined as:

$$M_1(G) = \sum_{v \in V(G)} d^2(v) \qquad \textit{and} \qquad M_2(G) = \sum_{uv \in E(G)} d(u)d(v).$$

For more details on Zagreb indices, see the surveys [7, 10] and the references cited therein. Analogously, the Zagreb coindices were put forward in [3], and are defined as:

$$\overline{M_1}(G) = \sum_{uv \notin E(G)} (d(u) + d(v)), \qquad \textit{and} \qquad \overline{M_2}(G) = \sum_{uv \notin E(G)} d(u)d(v).$$

The forgotten coindex of a graph were introduces in [14], and is defined as

$$\overline{F}(G) = \sum_{uv \notin E(G)} (d^2(u) + d^2(v)).$$

For more details on the Zagreb coindices, see [3, 4, 8, 10].

In (2017), Naji et al. [11] introduced a new distance-degree-based topological indices conceived depending on the second degrees of vertices, which are so-called leap Zagreb indices of a graph G and are defined as:

$$LM_1(G) = \sum_{v \in V(G)} d_2^2(v/G), \qquad LM_2(G) = \sum_{uv \in E(G)} d_2(u/G)d_2(v/G)$$

$$\textit{and} \qquad LM_3(G) = \sum_{v \in V(G)} d(v/G)d_2(v/G).$$

Also, Ali and Trinajstić [1] defined and studied a modified first Zagreb connection index depended on the second degrees of vertices. They formatted it as:

$$Z_1(G) = \sum_{v \notin V(G)} d(v)d_2(v).$$

Manzoora et al. in [18] derived formulas for calculating these modified versions of the Zagreb indices of four well known nanostructures.

Naji et al. [20, 21], computed leap Zagreb indices for some graph operations. Shao et al. [13], found the external bounds on leap Zagreb indices for trees and unicyclic

graphs. For properties and details of leap Zagreb indices of a graph the readers referred to [5, 6, 11, 13, 17, 19–23].
Recently, Ferdose and Shivashankara [9], introduced the leap Zagreb coindices of a graph. They defined it as follows

$$\overline{L_1}(G) = \sum_{uv \notin E_2(G)} \Big(d_2(u/G) + d_2(v/G)\Big), \qquad \overline{L_2}(G) = \sum_{uv \notin E(G)} \Big(d_2(u/G) d_2(v/G)\Big)$$

$$\text{and} \qquad \overline{L_3}(G) = \sum_{uv \notin E(G)} \Big(d_2(u/G) + d_2(v/G)\Big).$$

Motivated by the Zagreb coindices and the Zagreb connection coindices for product of molecular networks [2], in this current work, we present the exact expressions for the first leap Zagreb coindex of some graph operations containing union, cartesian product, composition, disjunction, symmetric difference and corona product of graphs.

The following fundamental results which will be required for many of our arguments in this paper are found in Yamaguchi [24] and Soner and Naji [23].

Theorem 1. *[23, 24] Let G be a connected graph with n vertices and m edges. Then*

$$d_2(v/G) \leq \left(\sum_{u \in N_1(v/G)} d_1(u/G) \right) - d_1(v/G).$$

and equality holds if and only if G is a $\{C_3, C_4\}$-free graph.

From this theorem, the following result follows

Corollary 1. *[23] Let G be a connected graph with n vertices and m edges. Then*

$$\sum_{v \in V(G)} d_2(v/G) \leq M_1(G) - 2m.$$

and equality holds if and only if G is a $\{C_3, C_4\}$-free graph.

The following result will be useful to prove our main results.

Proposition 1. *For a connected graph G with n vertices, m edges and μ second edges,*

$$\sum_{v \in V(G)} d_2(v/G) = 2|E_2(G)| = 2\mu.$$

Proof. Let G be a connected graph, and let G^2 denote the square graph of G, that is a graph with $V(G^2) = V(G)$ and for any two vertices u and v in G^2 are adjacent if and only if $d(u,v) = 2$ in G. It is clear that $uv \in E_2(G)$ if and only if $uv \in E(G^2)$. Hence, $|E_2(G)| = |E(G^2)|$ and so $d_2(v/G) = d(v/G^2)$, for every $v \in V(G)$. Since $\sum_{v \in V(G^2)} d(v/G^2) = 2|E(G^2)| = 2|E_2(G)| = \sum_{v \in V(G)} d_2(v/G)$, we get $\sum_{v \in V(G)} d_2(v/G) = 2|E_2(G)| = 2\mu$. □

2. Main Results

In this section, we investigate the exact formula of the first leap Zagreb coindex for some graph operations. We consider six operations, each of them is treated in a separate subsection.

2.1. Union:

Definition 1. [16] Let G and H be two connected graphs with disjoint vertex sets $V(G)$ and $V(H)$ and edge sets $E(G), E(H)$, respectively. The union graph $G \cup H$, is defined as the graph with vertex set $V(G \cup H) = V(G) \cup V(H)$, and edge set $E(G \cup H) = E(G) \cup E(H)$.

Clearly that $|V(G \cup H)| = n_1 + n_2$, $|E(G \cup H)| = m_1 + m_2$, and $|E_2(G \cup H)| = \mu_1 + \mu_2$, where $\mu = \mu(G) = |E_2(G)|$. So, the following result straightforward,

Lemma 1. *[21] Let G and H be two disjoint connected graphs with n_1 and n_2 vertices. Then for each $v \in V(G \cup H)$,*

$$d_2(v/(G \cup H)) = \begin{cases} d_2(v/G), & \text{if } v \in V(G); \\ d_2(v/H), & \text{if } v \in V(H). \end{cases}$$

Theorem 2. *Let G and H be connected graphs with n_1, n_2 vertices and $\mu(G)$, $\mu(H)$ second edges, respectively. Then*

$$\overline{L_1}(G \cup H) = \overline{L_1}(G) + \overline{L_1}(H) + 2(n_2\mu(G) + n_1\mu(H)).$$

Proof. From Lemma 1, for any two vertices $u, v \in V(G \cup H)$, $uv \notin E_2(G \cup H)$ if and only if $uv \notin E_2(G)$ and $uv \notin E_2(H)$. That means either $u, v \in V(G)$ and $uv \notin E_2(G)$, or $u, v \in V(H)$ and $uv \notin E_2(H)$, or $u \in V(G)$ and $v \in V(H)$. Thus, the first leap Zagreb coindex of $G \cup H$ is then equal to the sum of the first leap Zagreb coindices of G and H, plus the contributions from the missing edges between vertex sets of G and H. Where there are $n_1 n_2$ of them. Then we get

$$\begin{aligned}
\overline{L_1}(G \cup H) &= \sum_{uv \notin E_2(G \cup H)} \Big(d_2(u/(G \cup H)) + d_2(v/(G \cup H))\Big) \\
&= \sum_{uv \notin E_2(G)} \Big(d_2(u/(G \cup H)) + d_2(v/(G \cup H))\Big) \\
&\quad + \sum_{uv \notin E_2(H)} \Big(d_2(u/(\cup H)) + d_2(v/(G \cup H))\Big) \\
&\quad + \sum_{u \in V(G)} \sum_{v \in V(H)} \Big(d_2(u/(G \cup H)) + d_2(v/(G \cup H))\Big) \\
&= \sum_{uv \notin E_2(G)} \Big(d_2(u/G) + d_2(v/G)\Big) + \sum_{uv \notin E_2(H)} \Big(d_2(u/H) + d_2(v/H)\Big) \\
&\quad + \sum_{u \in V(G)} \sum_{v \in V(H)} \Big(d_2(u/G) + d_2(v/H)\Big)
\end{aligned}$$

$$= \overline{L_1}(G) + \overline{L_1}(H) + n_2 \sum_{u \in V(G)} d_2(u/G) + n_1 \sum_{v \in V(H)} d_2(v/H)$$
$$= \overline{L_1}(G) + \overline{L_1}(H) + 2n_2\mu(G) + 2n_1\mu(H). \qquad \square$$

From Corrolary 1, the following result directly follows

Corollary 2. *Let G and H be connected (C_3, C_4)-free graphs with n_1, n_2 vertices and m_1, m_2 edges, respectively. Then*

$$\overline{L_1}(G \cup H) = \overline{L_1}(G) + \overline{L_1}(H) + 2(n_2 M_1(G) + n_1 M_1(H)) - 4(n_2 m_1 + n_1 m_2).$$

Let $G_1, ..., G_k$ be connected graphs with disjoint vertex sets $V(G_i)$ and disjoint edge sets $E(G_i)$ of orders and size n_i, m_i, respectively. Their union is a graph $G = G_1 \cup ... \cup G_k$. Starting from Theorem 2, by induction method, the following result follows.

Proposition 2. *For $k \geq 2$, let $G_1, \ldots, G_k$ be connected graphs with n_i vertices and m_i edges, respectively. Then*

$$\overline{L_1}(\bigcup_{i=1}^{k} G_i) = \sum_{i=1}^{k} \overline{L_1}(G_i) + 2 \sum_{i=1}^{k} \left(\mu(G_i) \sum_{\substack{j=1 \\ j \neq i}}^{k} n_j \right).$$

Proposition 3. *For $k \geq 2$, let $G_1, \ldots, G_k$ be connected (C_3, C_4)-free graphs with n_i vertices and m_i edges, respectively. Then*

$$\overline{L_1}(\bigcup_{i=1}^{k} G_i) = \sum_{i=1}^{k} \overline{L_1}(G_i) + 2 \sum_{i=1}^{k} \left(n_i \sum_{\substack{j=1 \\ j \neq i}}^{k} (M_1(G_j) - 2m_j) \right).$$

2.2. Join:

Definition 2. [16] For given graphs G and H with n_1 and n_2 order and m_1 and m_2 size, respectively. The join graph $G+H$, is defined as the graph with vertex set $V = V(G) \cup V(H)$, and edge set $E(G + H) = E(G) \cup E(H) \cup \{uv : \forall u \in V(G) \text{ and } \forall v \in V(H)\}$.

Clearly that $|V(G + H)| = n_1 + n_2$ and $|E(G + H)| = m_1 + m_2 + n_1 n_2$.

Lemma 2. *[21] Let G and H be two connected graph with n_1 and n_2 vertices. Then*

$$d_2(v/(G + H)) = \begin{cases} n_1 - 1 - d(v/G), & \text{if } v \in V(G); \\ n_2 - 1 - d(v/H), & \text{if } v \in V(H). \end{cases}$$

Theorem 3. *Let G and H be two nontrivial connected graphs with n_1, n_2 vertices and m_1, m_2 edges, respectively. Then*

$$\overline{L_1}(G + H) = \overline{M_1}(G) + \overline{M_1}(H) + n_1 n_2 (n_1 + n_2 - 2) - 2(n_1 m_2 + n_2 m_1).$$

Proof. Since for any two nontrivial graphs G and H, the join graph $G+H$ has diameter two. Then for any vertices $u,v \in V(G+H)$, $uv \notin E_2(G+H)$, if and only if $uv \in E(G+H)$. Then by Lemma 2 and by using the fact that $\overline{M_1}(G) = 2m(n-1) - M_1$, see [3], we obtain

$$\begin{aligned}
\overline{L_1}(G+H) &= \sum_{uv \notin E_2(G+H)} \Big(d_2(u/(G+H)) + d_2(v/(G+H)) \Big) \\
&= \sum_{uv \in E(G+H)} \Big(d_2(u/(G+H)) + d_2(v/(G+H)) \Big) \\
&= \sum_{uv \in E(G)} \Big(d_2(u/(G+H)) + d_2(v/(G+H)) \Big) \\
&\quad + \sum_{uv \in E(H)} \Big(d_2(u/(G+H)) + d_2(v/(G+H)) \Big) \\
&\quad + \sum_{u \in V(G)} \sum_{v \in V(H)} \Big(d_2(u/(G+H)) + d_2(v/(G+H)) \Big) \\
&= \sum_{uv \in E(G)} \Big(2(n_1-1) - (d(u/G) + d(v/G)) \Big) \\
&\quad + \sum_{uv \in E(H)} \Big(2(n_2-1) - (d(u/H) + d(v/H)) \Big) \\
&\quad + \sum_{u \in V(G)} \sum_{v \in V(H)} \Big((n_1-1) - d(u/G) + (n_2-1) - d(v/H) \Big) \\
&= 2m_1(n_1-1) - M_1(G) + 2m_2(n_2-1) - M_1(H) \\
&\quad + n_1n_2(n_1-1) - 2n_2m_1 + n_1n_2(n_2-1) - 2n_1m_2 \\
&= \overline{M_1}(G) + \overline{M_1}(H) + n_1n_2(n_1+n_2-2) - 2(n_1m_2 + n_2m_1).
\end{aligned}$$
□

From the definition of $\overline{L_1}(G)$, and by using the fact in [9], that $\overline{L_1}(\overline{G}) = \overline{L_1}(G)$, and by note that, for $u,v \in V(G+H)$, $uv \notin E_2(G+H)$, if and only if $uv \in E(G+H)$. Since, $d_2(v/(G+H)) = d(v/(\overline{G+H}))$. Then the following result is straightforward,

Corollary 3. *For any connected graphs G and H, $\overline{L_1}(G+H) = \overline{M_1}(G+H)$.*

For generalization, let $G_1, \ldots, G_k$ be connected graphs with disjoint vertex sets $V(G_i)$ with n_i vertices and edge sets $E(G_i)$ of size m_i. Their join is a graph $G = G_1 + \cdots + G_k$. Starting from Theorem 3, by induction method, the following result straightforward.

Proposition 4. *Let $G_1, \ldots, G_k$ be graphs with n_i vertices and m_i edges, respectively. Then*

$$\overline{L_1}(\sum_{i=1}^{k} G_i) = \sum_{i=1}^{k} \overline{M_1}(G_i) + \sum_{i=1}^{k} \left(n_i(n_i-1) \sum_{\substack{j=1 \\ j \neq i}}^{k} n_j \right) - \sum_{i=1}^{k} \left(2n_i \sum_{\substack{j=1 \\ j \neq i}}^{k} m_j \right).$$

The join of the graph G, of order n and size m, with itself k times is given by

$$\overline{L_1}(\sum_{i=1}^{k} G) = k\Bigg[\overline{M_1}(G) + n(k-1)\Bigg(n(n-1) - 2m\Bigg)\Bigg].$$

As especial case $\overline{L_1}(\sum_{i=1}^{k} K_n) = 0$, as it directly computed in [9]. Also for the complete bipartite graph $K_{r,r}$, which is a join of two copies of the total disconnected graphs $\overline{K_r}$ with $n = 2r$ vertices. We have $\overline{L_1}(K_{r,r}) = 2r^2(r-1)$. In general, we consider the case of the complete k-partite graph $K_{n_1,\ldots,n_k}$ with classes of partitions of sizes $n_1, \ldots, n_k$. This graph is a join of k copies of the total disconnected graphs K_n. We have

$$\overline{L_1}(K_{n,\ldots,n}) = kn^2(k-1)(n-1) = 4n\binom{k}{2}\binom{n}{2}.$$

2.3. Cartesian product:

Definition 3. [16] For given graphs G and H their cartesian product, denoted $G\Box H$, is the graph with vertex set $V(G) \times V(H)$, and any two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $V(G\Box H)$ are connected by an edge if and only if either $(u_1 = v_1 \text{ and } u_2v_2 \in E(H))$ or $(u_2 = v_2 \text{ and } u_1v_1 \in E(G))$.

It is a well known fact that the cartesian product of graphs is commutative and associative up to isomorphism. $|V(G\Box H)| = |V(G)||V(H)|$, the distance between any two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $G\Box H$ is given by $d_{G\Box H}(u, v) = d_G(u_1, v_1) + d_H(u_2, v_2)$.

Lemma 3. *[21] Let G and H be connected graphs of orders n_1 and n_2, respectively. Then for any vertex $(u, v) \in V(G\Box H)$, $d_2((u, v)/(G\Box H)) = d_2(u/G)+d_1(u/G)d_1(v/H)+d_2(v/H)$.*

The following result required to prove our main result,

Theorem 4. *[20] Let G and H be two nontrivial connected graphs with n_1, n_2 vertices and m_1, m_2 edges, respectively. Then*

$$L_1(G\Box H) = n_2L_1(G) + 4m_2L_3(G) + M_1(G)M_1(H) + 4\mu(G)\mu(H) + 4m_1L_3(H) + n_1L_1(H).$$

Theorem 5. *Let G and H be two nontrivial connected graphs with n_1, n_2 vertices and $\mu(G)$, $\mu(H)$ second edges, respectively. Then*

$$\overline{L_1}(G\Box H) = 2(n_1n_2 - 1)\Big[n_2\mu(G) + n_1\mu(H) + 2m_1m_2\Big] - \Big[n_2L_1(G) + n_1L_1(H)\Big]$$
$$- 4\Big[m_2L_3(G) + m_1L_3(H)\Big] - M_1(G)M_1(H) - 4\mu(G)\mu(H).$$

Proof. From Lemma 3, Theorem 4 and by using the fact that state for any graph G, $\overline{L_1}(G) = (n-1)\sum_{v\in V(G)} d_2(v/G) - L_1(G)$, we obtain

$$\begin{aligned}
\overline{L_1}(G\Box H) &= (n_1n_2-1)\sum_{(u,v)\in V(G\Box H)}\Big(d_2((u,v)/(G\Box H))\Big) - L_1(G\Box H)\\
&= (n_1n_2-1)\sum_{u\in V(G)}\sum_{v\in V(H)}\Big(d_2((u,v)/(G\Box H))\Big) - L_1(G\Box H)\\
&= (n_1n_2-1)\sum_{u\in V(G)}\sum_{v\in V(H)}\Big[d_2(u/G) + d_1(u/G)d_1(v/H) + d_2(v/H))\Big] - L_1(G\Box H)\\
&= (n_1n_2-1)\Big[n_2\sum_{u\in V(G)} d_2(u/G) + 2m_1m_2 + n_1\sum_{u\in V(G)} d_2(v/H)\Big]\\
&- \Big[n_2L_1(G) + 4m_2L_3(G) + M_1(G)M_1(H) + 4\mu(G)\mu(H) + 4m_1L_3(H) + n_1L_1(H)\Big]\\
&= (n_1n_2-1)\Big[2n_2\mu(G) + 4m_1m_2 + 2n_1\mu(H)\Big]\\
&- \Big[n_2L_1(G) + 4m_2L_3(G) + M_1(G)M_1(H) + 4\mu(G)\mu(H) + 4m_1L_3(H) + n_1L_1(H)\Big]\\
&= 2(n_1n_2-1)\Big[n_2\mu(G) + n_1\mu(H) + 2m_1m_2\Big] - n_2L_1(G) - 4m_2L_3(G) - n_1L_1(H)\\
&- 4m_1L_3(H) - M_1(G)M_1(H) - 4\mu(G)\mu(H).
\end{aligned}$$

□

From Theorem 5 above and Corollary 1, the following result follows.

Corollary 4. *If G and H are connected (C_3, C_4)-free graphs with n_1, n_2 vertices and m_1, m_2 edges, respectively. Then*

$$\begin{aligned}
\overline{L_1}(G\Box H) &= (n_1n_2^2 - n_2 + 4m_2)M_1(G) + (n_1^2n_2 - n_1 + 4m_1)M_1(H) - \Big[n_2L_1(G) + n_1L_1(H)\Big]\\
&- 3M_1(G)M_1(H) - 4\Big[m_2L_3(G) + m_1L_3(H)\Big] + 2(n_1n_2-1)\Big[2m_1m_2 - n_2m_1 - n_1m_2\Big].
\end{aligned}$$

As an application of the above results, we list explicit formulae for the first leap Zagreb coindex for the cartesian product of two complete graphs with p and q vertices and the rectangular grid $P_p\Box P_q$, the C_4 nanotube $P_p\Box C_q$, and the C_4 nanotorus $C_p\Box C_q$, respectively. The formulae follow from Theorem 5, by plugging in the expressions the following values:

- $M_1(K_p) = p(p-1)^2$, $L_1(K_p) = 0$ and $L_3(K_p) = 0$,
- $M_1(P_p) = 4n-6$, $L_1(P_p) = 4(n-3)$ and $L_3(P_p) = 2(2n-5)$,
- $M_1(C_p) = 4p$, $L_1(C_p) = 4p$ and $L_3(C_p) = 4p$.

Observation 6. For the integers number $p, q \geq 5$, the following results holds:

- $\overline{L_1}(K_p\Box K_q) = 4\binom{p}{2}\binom{q}{2}\Big[3(pq-1) + (p+q)\Big]$.
- $\overline{L_1}(P_p\Box P_q) = 4pq\Big[2pq - 2(p+q) + 1\Big] - 68pq + 108(p+q) - 136$.
- $\overline{L_1}(P_p\Box C_q) = 8q(pq-1)(p-1) - 4q\Big[24p - 4q - 13\Big] + 8(2p-3)(p-1)$.
- $\overline{L_1}(C_p\Box C_q) = 8pq(pq-13) - +16(p^2+q^2)$. 0

2.4. Composition:

Definition 4. [16] The composition $G[H]$ of graphs G and H with disjoint vertex sets and edge sets is a graph on vertex set $V(G)\times V(H)$ in which (u_1,v_1) is adjacent with (u_2,v_2) whenever $[u_1$ is adjacent with $u_2]$ or $[u_1=u_2$ and v_1 is adjacent with $v_2]$.

The composition is not commutative. The easiest way to visualize the composition $G[H]$ is to expand each vertex of G into a copy of H, with each edge of G replaced by the set of all possible edges between the corresponding copies of H. Hence, by letting $|E(G[H])| = n_1m_2 + n_2^2m_1$.

Lemma 4. *[21] Let G and H be two graphs with disjoint vertex sets with n_1 and n_2 vertices and edges sets with m_1 and m_2 edges, respectively. Then*

$$d_2((u,v)/G[H])) = n_2d_2(u/G) + d_1(v/\overline{H}).$$

The following result required to prove our main result,

Theorem 7. *[20] Let G and H be two nontrivial connected graphs with n_1, n_2 vertices and m_1, m_2 edges, respectively. Then*

$$L_1(G[H]) = n_2^3L_1(G) + n_1M_1(H) + (2n_2^3 - 2n_2^2 - 4n_2m_2)\mu(G) + n_1(n_2-1)(n_2^2 - n_2 - 4m_2).$$

Theorem 8. *Let G and H be two nontrivial connected graphs with n_1, n_2 vertices and $\mu(G)$, $\mu(H)$ second edges, respectively. Then*

$$\begin{aligned}\overline{L_1}(G[H]) &= 2n_2\mu(G)\Big(n_1n_2^2 - 2n_2^2 + n_2 + 8m_2\Big) + n_1n_2^2\Big(n_1n_2 - n_1 - n_2 + 1\Big)\\ &\quad - n_2^3L_1(G) - n_1M_1(H) - 2n_1m_2\Big(n_1n_2 - 2n_2 - 1\Big).\end{aligned}$$

Proof. From Lemma 4, Theorem 7 and by using the fact that state for any graph G, $\overline{L_1}(G) = (n-1)\sum_{v\in V(G)} d_2(v/G) - L_1(G)$, we obtain

$$\begin{aligned}\overline{L_1}(G[H]) &= (n_1n_2-1)\sum_{(u,v)\in V(G[H])} d_2((u,v)/G[H]) - L_1(G[H])\\
&= (n_1n_2-1)\sum_{u\in V(G)}\sum_{v\in V(H)} d_2((u,v)/G[H]) - L_1(G[H])\\
&= (n_1n_2-1)\sum_{u\in V(G)}\sum_{v\in V(H)} [d_2(u/G) - (n_2-1) - d(v/H)] - L_1(G[H])\\
&= (n_1n_2-1)\Big[2n_2^2\mu(G) + n_1n_2(n_2-1) - 2n_1m_2\Big] - \Big[n_2^3L_1(G) + n_1M_1(H)\\
&\quad + 2\mu(G)(2n_2^3 - 2n_2^2 - 4n_2m_2) + n_1(n_2-1)(n_2^2 - n_2 - 4m_2)\Big]\\
&= 2\mu(G)\Big[n_2^2(n_1n_2-1) - (2n_2^3 - 2n_2^2 - 4n_2m_2)\Big] + n_1n_2(n_2-1)(n_1n_2-1)\\
&\quad - 2n_1m_2(n_1n_2-1) - \Big[n_2^3L_1(G) + n_1M_1(H) + n_1n_2(n_2-1)^2 - 4m_2n_1(n_2-1)\Big]\\
&= 2\mu(G)\Big[n_2^2(n_1n_2-1) - 2n_2(n_2^2 - n_2 - 4m_2)\Big] + n_1n_2^2\Big(n_1n_2 - n_1 - n_2 + 1\Big)\\
&\quad - n_2^3L_1(G) - n_1M_1(H) - 2n_1m_2\Big(n_1n_2 - 2n_2 - 1\Big)\end{aligned}$$

$$= 2n_2\mu(G)\Big(n_1n_2^2 - 2n_2^2 + n_2 + 8m_2\Big) + n_1n_2^2\Big(n_1n_2 - n_1 - n_2 + 1\Big)$$
$$- n_2^3L_1(G) - n_1M_1(H) - 2n_1m_2\Big(n_1n_2 - 2n_2 - 1\Big). \qquad \square$$

2.5. Disjunction:

Definition 5. [16] The disjunction $G \vee H$ of two graphs G and H with disjoint vertex sets and edge sets is the graph with vertex set $V(G) \times V(H)$ in which (u_1, v_1) is adjacent with (u_2, v_2) whenever u_1 is adjacent with u_2 in G or v_1 is adjacent with v_2 in H.

The disjunction $G \vee H$ is commutative, the number of vertices is $|V(G \vee H)| = n_1n_2$, the diameter is $diam(G \vee H) \leq 2$ and the number of edges is $|E(G \vee H) = n_1^2m_2 + n_2^2m_1 - 2m_1m_2$. [3].

Lemma 5. *[21] Let G and H be two graphs with n_1 and n_2 vertices and m_1 and m_2 edges, respectively. Then*

1. $d_1((u,v)/(G \vee H)) = n_2d_1(u/G) + n_1d_1(v/H) - d_1(u/G)d_1(v/H)$
2. $d_2((u,v)/(G \vee H)) = (n_1n_2 - 1) - n_2d_1(u/G) - n_1d_1(v/H) + d_1(u/G)d_1(v/H)$.

The following result requared to show the expression of the first leap coindex of $G \vee H$.

Theorem 9. *[20] Let G and H be two graphs with n_1 and n_2 vertices and m_1 and m_2 edges, respectively, such that G or H not a complete graph. Then*

$$L_3(G \vee H) = (4n_2m_2 - n_2^3)M_1(G) + (4n_1m_1 - n_1^3)M_1(H) - M_1(G)M_1(H)$$
$$+ (n_1n_2 - 1)(2n_1^2m_2 + 2n_2^2m_1 - 4m_1m_2) - 2m_1m_2(4n_1n_2 - 1).$$

Since,for any two graphs G and H, the diameter of $G \vee H$ is at most two. Then by application the fact (Theorem 4.3, in [9]), if $diam(G) \leq 2$, then $\overline{L_1}(G) = L_3(G)$. Hence from Theorem 11, the following expression of the first leap coindex of $G \vee H$ straightforward.

Theorem 10. *Let G and H be two graphs with n_1 and n_2 vertices and m_1 and m_2 edges, respectively, such that G or H not a complete graph. Then*

$$\overline{L_1}(G \vee H) = (4n_2m_2 - n_2^3)M_1(G) + (4n_1m_1 - n_1^3)M_1(H) - M_1(G)M_1(H)$$
$$+ (n_1n_2 - 1)(2n_1^2m_2 + 2n_2^2m_1 - 4m_1m_2) - 2m_1m_2(4n_1n_2 - 1).$$

2.6. Symmetric difference:

Definition 6. [16] The Symmetric difference $G \oplus H$ of two graphs G and H with disjoint vertex sets and edge sets is the graph with vertex set $V(G) \times V(H)$ in which (u_1, v_1) is adjacent with (u_2, v_2) whenever u_1 is adjacent with u_2 in G or v_1 is adjacent with v_2 in H but not both.

The Symmetric difference is commutative, with $|V(G \oplus H)| = n_1 n_2$ vertices, $diam(G \oplus H) \leq 2$ and $|E(G \oplus H) = n_1^2 m_2 + n_2^2 m_1 - 4m_1 m_2$ edges.

Lemma 6. *[21] Let G and H be two graphs with n_1 and n_2 vertices and m_1 and m_2 edges, respectively. Then*

1. $d_1((u,v)/(G \oplus H)) = n_2 d_1(u/G) + n_1 d_1(v/H) - 2d_1(u/G)d_1(v/H)$
2. $d_2((u,v)/(G \oplus H)) = (n_1 n_2 - 1) - n_2 d_1(u/G) - n_1 d_1(v/H) + 2d_1(u/G)d_1(v/H)$.

We need the following result to show our next result.

Theorem 11. *[20] Let G and H be two graphs with n_1 and n_2 vertices and m_1 and m_2 edges, respectively, such that G or H not a complete graph. Then*

$$\begin{aligned} L_3(G \oplus H) = & (n_1 n_2^2 - 8n_2 m_2)M_1(G) + 4M_1(G)M_1(H) + (n_2 n_1^2 - 8n_1 m_1)M_1(H) \\ & + 8n_1 n_2 m_1 m_2 + n_1 n_2 (n_1 n_2 - 1)^2 - 4(n_1 n_2 - 1)(n_2^2 m_1 + n_1^2 m_2 - 4m_1 m_2). \end{aligned}$$

Since the diameter of $G \vee H$ is at most two. Then by Theorem 4.3, in [9], the following result follows,

Theorem 12. *Let G and H be two graphs with n_1 and n_2 vertices and m_1 and m_2 edges, respectively, such that G or H not a complete graph. Then*

$$\begin{aligned} \overline{L_1}(G \oplus H) = & (n_1 n_2^2 - 8n_2 m_2)M_1(G) + 4M_1(G)M_1(H) + (n_2 n_1^2 - 8n_1 m_1)M_1(H) \\ & + 8n_1 n_2 m_1 m_2 + n_1 n_2 (n_1 n_2 - 1)^2 - 4(n_1 n_2 - 1)(n_2^2 m_1 + n_1^2 m_2 - 4m_1 m_2). \end{aligned}$$

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