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Research Article

Vertex-degree function index on tournaments

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Abstract: Let G be a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. For a real function f defined on nonnegative real numbers, the vertex-degree function index $H_f(G)$ is defined as

$$H_f(G) = \sum_{u \in V(G)} f(d_u).$$

In this paper we introduce the vertex-degree function index $H_f(D)$ of a digraph D. After giving some examples and basic properties of $H_f(D)$, we find the extremal values of H_f among all tournaments with a fixed number of vertices, when f is a continuous and convex (or concave) real function on $[0, +\infty)$.

Keywords: Tournaments, Vertex-degree function index, Vertex-degree-based topological index.

AMS Subject classification: 05C09, 05C20, 05C35

1. Introduction

Let G be a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. Denote by d_u the degree of a vertex u in G. For a real function f defined on nonnegative real numbers, the vertex-degree function index $H_f(G)$ was introduced in [6] as

$$H_f(G) = \sum_{u \in V(G)} f(d_u).$$

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Important examples of vertex-degree function indices are the zeroth-order general Randić index ${}^0R_\alpha(G)$, corresponding to the function $f(x) = x^\alpha$ [5]. In particular, when $\alpha = 2$ we obtain the first Zagreb index of a graph [1]

$$\mathcal{M}_1(G) = \sum_{u \in V(G)} (d_u)^2 = \sum_{uv \in E(G)} (d_u + d_v),$$

and when $\alpha = 3$, the Forgotten index [1]

$$\mathcal{F}(G) = \sum_{u \in V(G)} (d_u)^3 = \sum_{uv \in E(G)} ((d_u)^2 + (d_v)^2).$$

For recent results on the general concept of vertex-degree function index of graphs we refer to [4, 9–11].

In this paper we introduce the vertex-degree function index $H_f(D)$ of a digraph D. Let us recall some basic terminology of digraphs. Assume that D is a digraph with vertex set $V(D)$ and arc set $A(D)$. If there is an arc from the vertex u to the vertex v we denote it by uv. For a vertex u of D, N_u^+ (resp. N_u^-) is the set of vertices v of D such that uv (resp. vu) is an arc of D. The outdegree (resp. indegree) of u is denoted by d_u^+ (resp. d_u^-) and it is defined as the cardinality of the set N_u^+ (resp. N_u^-). A digraph D is called an oriented graph if whenever $uv \in A(D)$ then $vu \notin A(D)$. An oriented graph D can be obtained from a graph G by assigning a direction to each edge of G; D is called an orientation of G.

After giving in Section 2 the definition, examples and basic properties of a vertex-degree function index $H_f(D)$ of a digraph D, we consider in Section 3 the extremal value problem of H_f among all orientations of a complete graph. Recall that a tournament T on n vertices is an orientation of the complete graph K_n. The nondecreasing sequence $(s_1, s_2, \ldots, s_n)$ of outdegrees of the vertices of T is called the score vector of T. We will show that when f is a continuous and convex (or concave) real function on the interval $[0, +\infty)$, then among all tournaments on n vertices, one extremal value of H_f is attained in the transitive tournament U with score vector $(0, 1, 2, \ldots, n-1)$, and the other extremal value is attained in a regular tournament R with score vector $(\frac{n-1}{2}, \frac{n-1}{2}, \ldots, \frac{n-1}{2})$ when n is odd, or in a semiregular tournament S with score vector $(\frac{n}{2}-1, \ldots, \frac{n}{2}-1, \frac{n}{2}, \ldots, \frac{n}{2})$, when n is even (see Figure 1).

2. Vertex-degree function index of digraphs

In this section we introduce the concept of vertex-degree function index of digraphs.

Definition 1. Let f be a real function defined in the interval $[0, \infty)$. The vertex-degree function index of the digraph D, denoted as $H_f(D)$, is defined as

$$H_f(D) = \frac{1}{2} \sum_{u \in V(D)} \left[f(d_u^+) + f(d_u^-) \right].$$

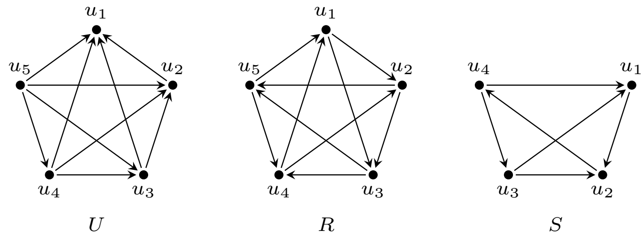

Figure 1. Transitive tournament U $(n = 5)$, regular tournament R $(n = 5)$ and semiregular tournament S $(n = 4)$.

Example 1. Consider the digraph shown in Figure 2. Then for any function f as in Definition 1,

$$\begin{aligned} H_f(D) &= \frac{1}{2}\sum_{i=1}^{5}\left[f(d^+_{v_i}) + f(d^-_{v_i})\right] \\ &= \frac{1}{2}([f(0) + f(3)] + [f(1) + f(1)] + [f(1) + f(1)] + [f(1) + f(2)] + [f(3) + f(1)]) \\ &= \frac{1}{2}[f(0) + 6f(1) + f(2) + 2f(3)]. \end{aligned}$$

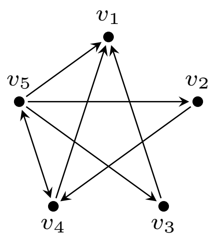

Figure 2. **Digraph used in Example 1**

Next we will see that Definition 1 extends the concept of vertex-degree function index to digraphs. If G is a graph, then G can be identified with the symmetric digraph $\widehat{G}$, which has the same vertex set as the graph G, and each edge uv of G is replaced by a pair of symmetric arcs uv and vu in $\widehat{G}$.

Proposition 1. *Let G be a graph. Then $H_f(\widehat{G}) = H_f(G)$.*

Proof. Let $u \in V(G) = V(\widehat{G})$, and denote by d_u the degree of u in G. Then $d_u^+ = d_u^- = d_u$ and so

$$H_f(\widehat{G}) = \frac{1}{2} \sum_{u \in V(D)} [f(d_u^+) + f(d_u^-)] = \frac{1}{2} \sum_{u \in V(G)} [f(d_u) + f(d_u)] = H_f(G).$$

□

A vertex-degree-based (VDB for short) topological index of a digraph D (see [7, 8]) is defined as

$$\varphi(D) = \frac{1}{2} \sum_{uv \in A(D)} \varphi_{d_u^+, d_v^-},$$

where $\varphi_{x,y}$ is a bivariate symmetric function, each variable defined over nonnegative real numbers. In our next result we show that the vertex-degree function index is a special type of VDB topological index.

Proposition 2. *Let f be a real function defined in the interval $[0, \infty)$ such that $f(0) = 0$, and let D be a digraph. Then*

$$H_f(D) = \frac{1}{2} \sum_{uv \in A(D)} \varphi_{d_u^+, d_v^-},$$

where $\varphi_{x,y}$ is the symmetric bivariate function $\varphi_{x,y} = \frac{f(x)}{x} + \frac{f(y)}{y}$, defined in $[1, +\infty) \times [1, +\infty)$.

Proof. Note that in the sum

$$\frac{1}{2} \sum_{uv \in A(D)} \left(\frac{f(d_u^+)}{d_u^+} + \frac{f(d_v^-)}{d_v^-} \right),$$

the summand $\frac{f(d_u^+)}{d_u^+}$ appears d_u^+ times for any vertex u such that $d_u^+ > 0$, and the summand $\frac{f(d_v^-)}{d_v^-}$ appears d_v^- times for any vertex v such that $d_v^- > 0$. Since $f(0) = 0$, it follows that

$$\begin{aligned} \frac{1}{2} \sum_{uv \in A(D)} \left(\frac{f(d_u^+)}{d_u^+} + \frac{f(d_v^-)}{d_v^-} \right) &= \frac{1}{2} \sum_{\{u \in V(D): d_u^+ > 0\}} d_u^+ \left(\frac{f(d_u^+)}{d_u^+} \right) + \frac{1}{2} \sum_{\{v \in V(D): d_v^- > 0\}} d_v^- \left(\frac{f(d_v^-)}{d_v^-} \right) \\ &= \frac{1}{2} \sum_{u \in V(G)} (f(d_u^+) + f(d_u^-)) = H_f(D). \end{aligned}$$

□

Example 2. Consider the function $f(x) = x^2$ defined in the interval $[0, +\infty)$. If D is a digraph then by Proposition 2,

$$H_{x^2}(D) = \frac{1}{2} \sum_{uv \in A(D)} \varphi_{d_u^+, d_v^-},$$

where $\varphi_{x,y} = \frac{x^2}{x} + \frac{y^2}{y} = x + y$. In other words,

$$H_{x^2}(D) = \frac{1}{2} \sum_{uv \in A(D)} (d_u^+ + d_v^-) = \mathcal{M}_1(D),$$

the first Zagreb index of D. Similarly, the Forgotten index

$$\mathcal{F}(D) = \frac{1}{2} \sum_{uv \in A(D)} [(d_u^+)^2 + (d_v^-)^2]$$

is the vertex-degree function index $H_{x^3}(D)$. In general, for $\alpha \in \mathbb{R}, \alpha \neq 0$, the generalized first Zagreb index

$$M_\alpha(D) = \frac{1}{2} \sum_{uv \in A(D)} [(d_u^+)^\alpha + (d_v^-)^\alpha] = H_{f(x)}(D),$$

where $f(x) = x^{\alpha+1}$ in the interval $[0, +\infty)$. Note that in the case $\alpha < -1$, we define $f(x) = x^{\alpha+1}$ for $x > 0$, and $f(0) = 0$.

3. Vertex-degree function index of tournaments

Let T be a tournament on n vertices. Recall that the nondecreasing sequence $(s_1, s_2, \ldots, s_n)$ of outdegrees of the vertices of T is called the score vector of T.

Theorem 1. *(Landau [3]) A nondecreasing sequence of integers $(s_1, s_2, \ldots, s_n)$ is a score vector of a tournament on n vertices, if and only if,*

$$\sum_{i=1}^{k} s_i \geq \frac{k(k-1)}{2} \qquad \textit{for all } 1 \leq k \leq n,$$

with equality for $k = n$.

Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ two n-tuple of real numbers such that

$$x_1 \geq x_2 \geq \cdots \geq x_n \quad \text{and} \quad y_1 \geq y_2 \geq \cdots \geq y_n. \tag{1}$$

The n-tuple x is said to majorize y, in symbols we write $x \succ y$, if

$$\sum_{i=1}^{k} x_i \geq \sum_{i=1}^{k} y_i \qquad \text{for all } 1 \leq k \leq n,$$

with equality for $k = n$.
Let $I \subseteq \mathbb{R}$ be an interval. Recall that a function $f : I \to \mathbb{R}$ is convex if

$$f((1-\lambda)a + \lambda b) \leq (1-\lambda)f(a) + \lambda f(b), \tag{2}$$

for all $a, b \in I$ and all $\lambda \in [0,1]$.

Theorem 2. *(Hardy, Littlewood, Pólya [2]) Let x and y be two n-tuples of real numbers as in (1), whose entries belong to an interval I. The following statements are equivalent:*

1. *$x \succ y$;*
2. *The inequality*
$$\sum_{i=1}^{n} f(x_i) \geq \sum_{i=1}^{n} f(y_i)$$
holds for every continuous convex function $f : I \to \mathbb{R}$.

Now we can study the extremal value problem of H_f over the set of all tournaments with n vertices.

Theorem 3. *Let f be a continuous and convex real function on $[0, +\infty)$ and let T be a tournament on n vertices. Then*

$$H_f(T) \leq \sum_{i=1}^{n} f(n-i).$$

Equality occurs if T is the transitive tournament with score vector $(0, 1, 2, \ldots, n-1)$.

Proof. Let $(s_1, \ldots, s_n)$ be the score vector of T. Clearly,

$$0 \leq s_1 \leq s_2 \leq \cdots \leq s_n \leq n-1.$$

Define $t_j = s_{n-j+1}$, for all $j = 1, \ldots, n$. Then $t_1 \geq t_2 \geq \cdots \geq t_n$ and $(t_1, \ldots, t_n)$ is the nonincreasing sequence score vector of T. We are going to show that

$$(n-1, n-2, \ldots, 1, 0) \succ (t_1, \ldots, t_n). \tag{3}$$

First note that by Theorem 1,

$$\sum_{i=1}^{n} t_i = \sum_{i=1}^{n} s_i = \frac{n(n-1)}{2} = \sum_{i=1}^{n} (n-i).$$

Again, by Theorem 1, for every $1 \leq k \leq n$,

$$\sum_{i=1}^{n-k} s_i \geq \frac{(n-k)(n-k-1)}{2} \tag{4}$$

and so

$$\sum_{i=1}^{k} t_i = \sum_{i=1}^{k} s_{n-i+1} = \frac{n(n-1)}{2} - \sum_{i=1}^{n-k} s_i \leq \frac{n(n-1)}{2} - \frac{(n-k)(n-k-1)}{2} = \sum_{i=1}^{k} (n-i).$$

Hence (3) holds.
Also,

$$(n-1, n-2, \ldots, 1, 0) \succ (n-1-s_1, n-1-s_2, \ldots, n-1-s_n). \tag{5}$$

In fact,

$$\sum_{i=1}^{n} (n-1-s_i) = n(n-1) - \frac{n(n-1)}{2} = \frac{n(n-1)}{2} = \sum_{i=1}^{n} (n-i),$$

and for $1 \leq k \leq n-1$, bearing in mind that

$$\sum_{i=1}^{k} s_i \geq \frac{k(k-1)}{2},$$

it follows that

$$\sum_{i=1}^{k} (n-1-s_i) = k(n-1) - \sum_{i=1}^{k} s_i \leq k(n-1) - \frac{k(k-1)}{2} = \sum_{i=1}^{k} (n-i).$$

Therefore, (5) holds.
Now, since f is a continuous and convex real function on $[0, +\infty)$, and t_i, $n-1-s_i$, and $n-i$ belong to the interval $[0, +\infty)$ for all $i = 1, \ldots, n$, we deduce from Theorem 2 that

$$\sum_{i=1}^{n} f(s_i) = \sum_{i=1}^{n} f(t_i) \leq \sum_{i=1}^{n} f(n-i),$$

and

$$\sum_{i=1}^{n} f(n-1-s_i) \leq \sum_{i=1}^{n} f(n-i).$$

Finally,

$$\begin{aligned} H_f(T) &= \frac{1}{2} \sum_{i=1}^{n} [f(s_i) + f(n-1-s_i)] = \frac{1}{2} \sum_{i=1}^{n} f(s_i) + \frac{1}{2} \sum_{i=1}^{n} f(n-1-s_i) \\ &\leq \frac{1}{2} \sum_{i=1}^{n} f(n-i) + \frac{1}{2} \sum_{i=1}^{n} f(n-i) = \sum_{i=1}^{n} f(n-i). \end{aligned}$$

For the last statement, assume that U is the transitive tournament with score vector $(0, 1, 2, \ldots, n-1)$. Then,

$$H_f(U) = \frac{1}{2} [f(0) + f(n-1) + f(1) + f(n-2) + \cdots + f(n-2) + f(1) + f(n-1) + f(0)] = \sum_{i=1}^{n} f(n-i).$$

$\square$

Next we will show that the regular or semiregular tournaments attain the minimal value of H_f.

Theorem 4. *Let f be a continuous and convex real function on $[0, +\infty)$ and let T be a tournament on n vertices. Then,*

1. $H_f(T) \geq nf\left(\frac{n-1}{2}\right)$ *if n is odd. Equality occurs in any regular tournament with score vector* $(\frac{n-1}{2}, \frac{n-1}{2}, \ldots, \frac{n-1}{2})$.
2. $H_f(T) \geq \frac{n}{2}[f\left(\frac{n}{2}-1\right) + f\left(\frac{n}{2}\right)]$ *if n is even. Equality occurs in any semiregular tournament with score vector* $(\frac{n}{2}-1, \ldots, \frac{n}{2}-1, \frac{n}{2}, \ldots, \frac{n}{2})$.

Proof. Let $(s_1, \ldots, s_n)$ be the score vector of T. We know that

$$0 \leq s_1 \leq s_2 \leq \cdots \leq s_n \leq n-1,$$

and so s_i and $n-1-s_i$ belong to the interval $[0, +\infty)$, for all $i = 1, \ldots, n$.

1. Let us assume first that n is odd. Since f is convex then by (2),

$$\frac{1}{2}f(s_i) + \frac{1}{2}f(n-1-s_i) \geq f\left(\frac{1}{2}s_i + \frac{1}{2}(n-1-s_i)\right) = f\left(\frac{n-1}{2}\right), \tag{6}$$

for all $i = 1, \ldots, n$. It follows from (6) that

$$H_f(T) = \frac{1}{2}\sum_{i=1}^{n}[f(s_i) + f(n-1-s_i)] \geq \sum_{i=1}^{n} f\left(\frac{n-1}{2}\right) = nf\left(\frac{n-1}{2}\right).$$

Next we see that equality occurs in regular tournaments. Assume that R is a regular tournament with score vector $(\frac{n-1}{2}, \frac{n-1}{2}, \ldots, \frac{n-1}{2})$. Clearly,

$$\begin{aligned} H_f(R) &= \frac{1}{2}\left[f\left(\frac{n-1}{2}\right) + f\left(n-1-\frac{n-1}{2}\right) + \cdots + f\left(\frac{n-1}{2}\right) + f\left(n-1-\frac{n-1}{2}\right)\right] \\ &= \frac{1}{2}2nf\left(\frac{n-1}{2}\right) = nf\left(\frac{n-1}{2}\right). \end{aligned}$$

2. Now assume that n is even. If $s_i \leq \frac{n}{2}-1$, then $n-1-s_i \geq \frac{n}{2} > \frac{n}{2}-1 \geq s_i$. Let $y = (\frac{n}{2}, \frac{n}{2}-1)$ and $x = (n-1-s_i, s_i)$. Clearly $x \succ y$, so by Theorem 2,

$$f(n-1-s_i) + f(s_i) \geq f\left(\frac{n}{2}\right) + f\left(\frac{n}{2}-1\right).$$

If $s_i \geq \frac{n}{2}$, then $n-1-s_i \leq \frac{n}{2}-1 < \frac{n}{2} \leq s_i$. Let $y = (\frac{n}{2}, \frac{n}{2}-1)$ and $x = (s_i, n-1-s_i)$. Clearly $x \succ y$, so again by Theorem 2,

$$f(s_i) + f(n-1-s_i) \geq f\left(\frac{n}{2}\right) + f\left(\frac{n}{2}-1\right).$$

Then, for all $i = 1, \dots, n$,

$$H_f(T) = \frac{1}{2}\sum_{i=1}^{n}[f(s_i) + f(n-1-s_i)] \geq \frac{n}{2}\left[f\left(\frac{n}{2}-1\right) + f\left(\frac{n}{2}\right)\right].$$

If n is even, the score vector of a semiregular tournament S with n vertices is

$$\left(\underbrace{\frac{n}{2}-1, \dots, \frac{n}{2}-1}_{\frac{n}{2}}, \underbrace{\frac{n}{2}, \dots, \frac{n}{2}}_{\frac{n}{2}}\right).$$

Hence,

$$\begin{aligned} H_f(S) &= \frac{1}{2}\sum_{i=1}^{\frac{n}{2}}\left[f\left(\frac{n}{2}-1\right) + f\left(\frac{n}{2}\right)\right] + \frac{1}{2}\sum_{i=\frac{n}{2}+1}^{n}\left[f\left(\frac{n}{2}\right) + f\left(\frac{n}{2}-1\right)\right] \\ &= \frac{n}{2}\left[f\left(\frac{n}{2}-1\right) + f\left(\frac{n}{2}\right)\right]. \end{aligned}$$

□

Example 3. Let α be a positive real number. Consider the VDB topological index

$$M_\alpha(D) = \frac{1}{2}\sum_{uv\in A(D)}((d_u^+)^\alpha + (d_v^-)^\alpha).$$

Among all tournaments with n vertices, M_α attains its maximal value in the transitive tournament on n vertices, and its minimal value in a regular tournament (if n is odd) or in a semiregular tournament (if n is even). Indeed, apply Theorems 3 and 4 to the continuous and convex function $f : [0, +\infty) \to \mathbb{R}$ defined by $f(x) = x^{\alpha+1}$.
In particular, among all tournaments with a fixed number of vertices, the extremal value problem of the first Zagreb index $\mathcal{M}_1$ ($\alpha = 1$) and the Forgotten index $\mathcal{F} = \mathcal{M}_2$ are solved.

Dually, we can easily deduce similar results for continuous and concave real functions $g : [0, +\infty) \to \mathbb{R}$, by reversing the inequalities in Theorems 3 and 4.

Example 4. Let $\alpha \in (-1, 0)$. Then $\alpha + 1 \in (0, 1)$ and so the function $g(x) = x^{\alpha+1}$ is a continuous and concave function on $[0, +\infty)$ which satisfies $g(0) = 0$. Then, among all tournaments on n vertices, the vertex degree function index (and VDB topological index)

$$H_g(T) = \frac{1}{2}\sum_{uv\in A(T)}((d_u^+)^\alpha + (d_v^-)^\alpha),$$

attains its minimal value in the transitive tournament on n vertices, and the maximal value is reached in a regular or a semiregular tournament, depending on the parity of n.

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