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Research Article

Injective coloring of generalized Mycielskian of graphs

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Abstract: The injective chromatic number $\chi_i(G)$ of a graph G is the smallest number of colors required to color the vertices of G such that any two vertices with a common neighbor are assigned distinct colors. The Mycielskian or Mycielski graph $\mu(G)$ of a graph G, introduced by Jan Mycielski in 1955 has the property that, these graphs have large chromatic number with small clique number. The generalized Mycielskian $\mu_m(G), m > 0$ (also known as cones over graphs) are the natural generalizations of the Mycielski graphs. In this paper, sharp bounds are obtained for the injective chromatic number of generalized Mycielskian of any graph G. Further, the injective chromatic number of generalized Mycielskian of some special classes of graphs such as paths, cycles, complete graphs, and complete bipartite graphs are obtained.

Keywords: injective coloring, injective chromatic number, generalized Mycielskian.

AMS Subject classification: 05C15, 05C76

1. Introduction

All graphs considered in this paper are simple, finite, and undirected. The vertex set and edge set are indicated as $V(G)$ and $E(G)$. Also the maximum degree, clique number of a graph, and neighborhood set of a vertex $u \in V(G)$ are denoted by respectively $\Delta(G)$, $\omega(G)$ and $N(u)$. For further graph-theoretic notations and terminologies refer [7] and [21].

The concept of the injective coloring of a graph is introduced by Hahn et al. [6] in 2002 and hence injective chromatic number. An *injective coloring* of a graph is a coloring of the vertices, that assigns different colors to pair of vertices that have a common neighbor. The least number of colors required for attaining an injective coloring for a graph G is called the *injective chromatic number*, $\chi_i(G)$ of a graph G. In [6], the authors suggested the bounds in

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the case of injective chromatic number in general and computed $\chi_i(Q_n)$, where Q_n represents the hypercubes. The authors routed, $\chi_i(Q_n)$ to the context of error correcting codes also. Further, it is also known that finding the injective k-coloring for a given graph is NP-complete. Later, for a chordal graph G, Hell et al. [8], determined $\chi_i(G)$ with few conditions. The authors also designed a polynomial time algorithm that will compute the injective coloring for a given chordal graph. Nevertheless, the authors extended their study about the injective coloring of split graphs also. In 2021 the injective coloring of chordal graphs is also well studied in [17]. Further, in 2013, A. Kishore and Sunitha [10] introduced the concept of injective chromatic sum and injective chromatic polynomial and discussed that in detail in [11]. Later in 2015, for join, union, direct product, Cartesian product, graph composition, and disjunction of graphs, Song and Yue [18] computed sharp bounds (or the exact values) for the injective chromatic number.
Kim et al. [9], in 2009 showed that, for a graph G, $\chi_i(G) \geq \frac{1}{2}\chi(G^2)$, where G^2 is the square of G and $\chi(G)$ represents the chromatic number of G. Later the injective coloring of planar graphs were well studied in [1–4, 14].

In 1955, Jan Mycielski [15], defined the *Mycielskian* or *Mycielski graph* $\mu(G)$ of a graph G. If a graph G is triangle free, by the construction, its Mycielskian is also, and it is a larger graph than G itself. The graph $\mu(G)$ of a graph G with vertex set $\{u_1, u_2, \ldots, u_n\}$ is a graph obtained from G by adding $n + 1$ new vertices $\{v_1, v_2, \ldots, v_n, w\}$, joining w to each vertex $v_i (1 \leq i \leq n)$ and joining v_i to each neighbor of u_i in G. The circular chromatic number, star chromatic number, fractional chromatic number of Mycielski graphs are well studied in [5, 12, 13, 20].
The generalized Mycielskian (also known as cones over graphs) are the natural generalization of the Mycielski graphs [19] which preserves some nice properties of a good interconnection network. Let G be a graph with vertex set $V^0 = \{v_1^0, v_2^0, \ldots, v_n^0\}$ and edge set E^0. Given an integer $m \geq 1$ the m-Mycielskian of G, denoted by $\mu_m(G)$, is the graph with vertex set $V^0 \cup V^1 \cup V^2 \cup \cdots \cup V^m \cup \{w\}$, where $V^i = \{v_j^i \mid v_j^0 \in V^0\}$ is the i^{th} distinct copy of V^0 for $i = 1, 2, \ldots, m$, and edge set $E^0 \cup \left(\bigcup_{i=0}^{m-1} \{v_j^i v_{j'}^{i+1} \mid v_j^0 v_{j'}^0 \in E^0\} \right) \cup \{v_j^m w \mid v_j^m \in V^m\}$. It is clear that the so-called Mycielskian of a graph G is simply $\mu_1(G)$. Also note that the m-Mycielskian graph $\mu_m(G)$ of G contains G itself as a subgraph.
The Mycielskian of a graph retains a small clique number, while its chromatic number grows, as shown in [21]. In this paper we prove that this is also true for the injective chromatic number of generalized Mycielskians in the case of paths (Theorem 1), cycles (Theorem 3), complete bipartite graphs (Theorem 4) and stars (Corollary 2). We also determine the injective chromatic number of the generalized Mycielskian of complete graphs (Theorem 2).

2. Injective chromatic number of generalized Mycielskian of certain graphs

In this section we compute the injective chromatic number of paths, complete graphs, cycles, complete bipartite graphs and stars. Except for the second family all these graphs have small clique number and large injective chromatic number. Additionally, we derive precise upper

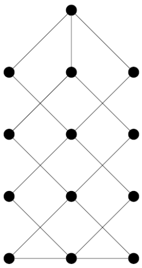

Figure 1. $\mu_3(P_3)$.

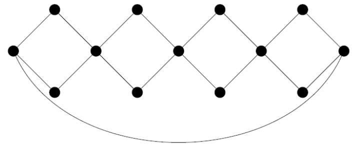

Figure 2. **Planar embedding of** $\mu_3(P_3)$.

and lower bounds for the injective chromatic number of generalized Mycielskian graphs. In Proposition 1 we quote some results from [6] that will be used throughout the paper.

Proposition 1. [6] *Let $P_n(C_n)$ be a path (cycle) of length n and G be a connected graph with maximum degree $\Delta(G)$. Then*

i. $\chi_i(P_n) = \begin{cases} 1 \text{ for } n = 1, 2 \\ 2 \text{ otherwise} \end{cases}$

ii. $\chi_i(C_n) = \begin{cases} 2 \text{ for } n \equiv 0 \bmod 4 \\ 3 \text{ otherwise} \end{cases}$

iii. $\chi_i(G) \geq \Delta(G)$

iv. Let G be an arbitrary graph of order at least four. Then, $\chi_i(G) = |V(G)|$ if and only if either G is a complete graph, or G has diameter 2 and every edge of G is contained in a triangle.

v. Let H be a subgraph of a graph G, then $\chi_i(H) \leq \chi_i(G)$.

Now let G be a graph with n vertices. By definition of generalized Mycielskian graphs, $\Delta(\mu_m(G)) = \max\{2\Delta(G), n\}$. Thus we have the following:

Corollary 1. *For each integer $m \geq 1$ and a graph G with $|V(G)| = n$, the $m-$Mycielskian of G satisfies $\chi_i(\mu_m(G)) \geq \max\{2\Delta, n\}$.*

The Corollary 1 gives a sharp bound for $\chi_i(\mu_m(G))$, for any graph G. In the next theorem the injective chromatic number of generalized Mycielskian of a path P_n for any n is obtained. Here $\chi_i(\mu_m(P_n))$ is obtained as n or $n+1$, but the clique number is just two. Figure 1 displays the graph $\mu_3(P_3)$, while its planar embedding is illustrated in Figure 2. The injective coloring of $\mu_m(P_3)$ is then presented in its planar embedding.

Theorem 1. *For $n > 1$, the injective chromatic number of m-Mycielskian of a path P_n on n vertices is,* $\chi_i(\mu_m(P_n)) = \begin{cases} n+1 & \text{for } \ n = 2, 3, 4 \\ n & \text{otherwise.} \end{cases}$

Proof. Let the vertices of P_n be $V^0 = \{v_i^0 \mid 1 \leq i \leq n\}$ and the vertices of $\mu_m(P_n)$ be $\bigcup_{j=0}^{m} V^j \cup \{w\}$, where $V^j = \{v_i^j \mid 1 \leq i \leq n\}$.

Case 1. $n = 2$.
By the construction of m-Mycielskian, it is clear that $\mu_m(P_2) = C_{2m+3}$, an odd cycle of length $2m + 3$. Then by Proposition 1(ii), the result follows.

Case 2. $n = 3$.
Since $\Delta(\mu_m(P_3)) = 4$, and by Proposition 1(iii), $\chi_i(\mu_m(P_3)) \geq 4$. Now providing an injective coloring using 4 colors shows that $\chi_i(\mu_m(P_3)) = 4$. The coloring is shown in Figure 3 and Figure 4.

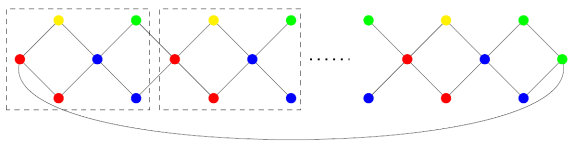

Figure 3. **Injective 4-coloring of $\mu_m(P_3)$ when m is odd**

Repeating the same coloring as that of the subgraph in the rectangle for the similar subgraphs with six vertices appearing in Figure 3 and the last vertex is colored with the color Green provides an injective coloring of $\mu_m(P_3)$ for $m-$ odd.

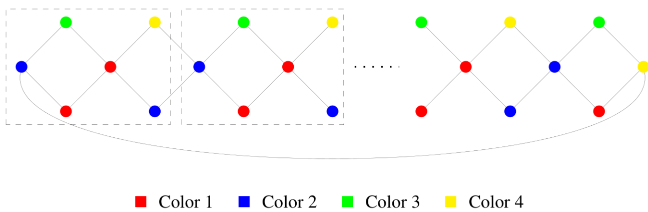

Figure 4. **Injective 4-coloring of $\mu_m(P_3)$ when m is even**

Similarly, repeating the same coloring as that of the subgraph in the rectangle for the similar subgraphs with six vertices appearing in Figure 4 and the last vertex is colored with the color Yellow provides an injective coloring of $\mu_m(P_3)$ for $m-$ even.

Case 3. $n = 4$.

Coloring of the vertices in V^m: The vertex w is a common neighbor for the vertices in V^m. Thus the vertices in V^m are colored with four distinct colors. Let j be the color of v_j^m, $1 \leq j \leq m$.

Coloring the vertices in V^{m-2k} **for** $m - 2k \neq 0, 1$: Since $v_1^m - v_2^{m-1} - v_1^{m-2}$ and $v_1^m - v_2^{m-1} - v_3^{m-2}$ are paths of length 2 and the vertex v_1^m is of Color 1 then Color 1 cannot be assigned to the

vertices v_1^{m-2} and v_3^{m-2}. Now the remaining vertices in V^{m-2} are v_2^{m-2} and v_4^{m-2}. Also v_3^{m-1} is a common neighbor for these vertices. Therefore the vertices v_2^{m-2} and v_4^{m-2} are assigned distinct colors. Thus Color 1 is assigned to exactly one vertex in V^{m-2}, let it be v_2^{m-2}.
Now $v_2^m - v_3^{m-1} - v_2^{m-2}$ and $v_2^m - v_3^{m-1} - v_4^{m-2}$ are paths of length 2 and the vertex v_2^m is of Color 2. Thus Color 2 cannot be given to the vertices v_2^{m-2} and v_4^{m-2}. Now the remaining vertices in V^{m-2} are v_1^{m-2} and v_3^{m-2}. Also v_2^{m-1} is a common neighbor for these vertices. Therefore the vertices v_1^{m-2} and v_3^{m-2} are assigned distinct colors. Thus Color 2 is assigned to exactly one vertex in V^{m-2}, let it be v_3^{m-2}. With the similar arguments, Color 3 and Color 4 are assigned to exactly one vertex of V^{m-2}. Let Color 3 and Color 4 be assigned to the vertices v_4^{m-2} and v_1^{m-2} respectively. Thus the vertices in V^{m-2} are assigned distinct colors. That is the vertices $v_1^{m-2}, v_2^{m-2}, v_3^{m-2}$ and v_4^{m-2} are colored with Color 4,1,2 and 3.
As V^{m-2} are colored with distinct colors and the vertices in V^{m-4} have common neighbors with the vertices in V^{m-2}, with similar arguments by considering the vertices in V^{m-2} and V^{m-4}, the vertices in V^{m-4} are assigned distinct colors. Continuing like this, the vertices in V^{m-2k} for some k are colored with distinct colors as the vertices in $V^{m-2(k-1)}$ are colored with distinct colors.

Coloring the vertices in V^{m-1} and the vertex w: Two vertices v_1^{m-1} and v_3^{m-1} in V^{m-1} have common neighbor v_2^m. Therefore at least two colors are needed to color the vertices in V^{m-1}. Also all the vertices in V^{m-1} have a common neighbor in V^m with vertex w. Thus no colors of the vertices in V^{m-1} can be given to the vertex w.

- Possibility 1: Coloring the vertices in V^{m-1} with two different colors.
  The vertices v_1^{m-1} and v_3^{m-1} have common neighbor v_2^m and the vertices v_2^{m-1} and v_4^{m-1} have common neighbor v_3^m. Now the coloring of vertices be:
  - The vertices v_1^{m-1} and v_2^{m-1} are of Color 1 and v_3^{m-1} and v_4^{m-1} are of Color 2. Or
  - The vertices v_1^{m-1} and v_4^{m-1} are of Color 1 and v_2^{m-1} and v_3^{m-1} are of Color 2.

  In this case vertex w need to be colored with color 3.
- Possibility 2: Coloring the vertices in V^{m-1} with three different colors.
  - The vertices $v_1^{m-1}, v_2^{m-1}, v_3^{m-1}$, and v_4^{m-1} are colored with Color 1,1,2 and 3 respectively. Or
  - The vertices $v_1^{m-1}, v_2^{m-1}, v_3^{m-1}$, and v_4^{m-1} are colored with Color 2,1,1 and 3 respectively.

  Any other 3 coloring is similar to this by reversing the order of coloring. In this case the vertex w need to be colored with color 4.
- Possibility 3: Coloring the vertices in V^{m-1} with four different colors.
  In this case vertex w need to be colored with color 5.

Thus when we use Possibility 3, five distinct colors are already needed for an injective coloring of $\mu_m(P_4)$. Therefore we do not continue the coloring for Possibility 3, and try with the other two possibilities to use less colors.

Coloring the vertices in V^{m-3}: Coloring the vertices in V^{m-3} corresponding to each of the above possibilities:

- Possibility 1: $v_1^{m-3} - v_2^{m-2} - v_1^{m-1}$, $v_2^{m-3} - v_1^{m-2} - v_2^{m-1}$, $v_3^{m-3} - v_2^{m-2} - v_1^{m-1}$ and $v_4^{m-3} - v_3^{m-2} - v_2^{m-1}$ are paths of length 2 and the vertices v_1^{m-1} and v_2^{m-1} are of Color 1. Thus no vertices in V^{m-3} can be colored with Color 1. Similarly, $v_1^{m-3} - v_2^{m-2} - v_3^{m-1}$, $v_2^{m-3} - v_3^{m-2} - v_4^{m-1}$, $v_3^{m-3} - v_2^{m-2} - v_3^{m-1}$ and $v_4^{m-3} - v_3^{m-2} - v_4^{m-1}$ are paths of length 2 and the vertices v_3^{m-1} and v_4^{m-1} are of Color 2. Thus no vertices in V^{m-3} can be colored with Color 2. Thus the color that is used twice in V^{m-1} cannot be used for the vertices in V^{m-3}. This is true for both colorings given in Possibility 1. Now the remaining colors are used to color the vertices in V^{m-3}, as follows:
  - The vertices $v_1^{m-3}, v_2^{m-3}, v_3^{m-13}$, and v_4^{m-13} are colored with Color 3,3,4 and 4 respectively. Or
  - The vertices $v_1^{m-3}, v_2^{m-3}, v_3^{m-3}$, and v_4^{m-3} are colored with Color 3,4,4 and 3 respectively.
- Possibility 2: The color used twice in V^{m-1} cannot be given to any vertex in V^{m-3} and a color used once can be given to exactly one vertex in V^{m-3}. Thus Color 2 and Color 3 are used once and Color 4 is used twice to color the vertices in V^{m-3}. Let the coloring be:
  - The vertices $v_1^{m-3}, v_2^{m-3}, v_3^{m-13}$, and v_4^{m-13} are colored with Color 4,4,3 and 2 respectively. Or
  - The vertices $v_1^{m-3}, v_2^{m-3}, v_3^{m-13}$, and v_4^{m-13} are colored with Color 3,4,4 and 2 respectively.

Colors of the vertices in V^{m-3} depends on the coloring of the vertices in V^{m-1}. Similarly colors of the vertices in V^{m-l} for an odd l depends on the colors of the vertices in $V^{m-(l-2)}$ and $V^{m-(l+2)}$. As the coloring is chosen for the vertices in the order $V^m, V^{m-2}, V^{m-4}, \cdots, V^{m-1}, V^{m-3}, \cdots$, clearly the coloring of the vertices in V^{m-l} depends on the colors of the vertices in $V^{m-(l-2)}$.

Coloring the vertices in V^{m-l}, where l is odd, $m - l \neq 0, 1$ and $l \neq 1, 3$: Continue the coloring in the similar way as done for V^{m-1} and V^{m-3}.

Coloring the vertices in V^0 and V^1 together:

Case i. m is even.

Then $0 = m - 2k$ for some k, the vertices in V^0 are colored with four different colors. The vertices in V^2 also colored with four different colors. Say, the vertices v_1^2, v_2^2, v_3^2 and v_4^2 are colored with Color 4, 1, 2 and 3 respectively. Then the vertices v_1^0, v_2^0, v_3^0 and v_4^0 are colored with Color 1, 2, 3 and 4 respectively (if $m \equiv 0 \mod 4$). (Similarly the vertices v_1^2, v_2^2, v_3^2 and v_4^2 are colored with Color 1, 2, 3 and 4 respectively and the vertices v_1^0, v_2^0, v_3^0 and v_4^0 are colored with Color 4, 1, 2 and 3 respectively (if $m \equiv 2 \mod 4$)). Now there are two possibilities for the coloring the vertices in V^3.

- The vertices in V^3 are colored with two different colors.
  Let the vertices v_1^3, v_2^3, v_3^3 and v_4^3 are colored with Color 1,1,2 and 2 respectively or v_1^3, v_2^3, v_3^3 and v_4^3 are colored with Color 1,2,2 and 1 respectively (if $m \equiv 0 \mod 4$). The colors that are used in V^3 cannot be assigned to the vertices in V^1, since each color is used twice in V^3. Also, in both cases for the vertices v_1^1 and v_3^1, $v_1^1 - v_2^0 - v_3^0$ and

$v_3^1 - v_2^0 - v_3^0$ be paths of length 2 and the vertex v_3^0 is of Color 3. Thus the vertices v_1^1 and v_3^1 cannot be assigned Color 3. Now the vertex v_2^0 is a common neighbor for the vertices v_1^1 and v_3^1. Therefore the vertices v_1^1 and v_3^1 are colored with distinct colors. Thus one of the vertex is assigned Color 4 and other with Color 5. Thus the coloring of the vertices in V^1 is as follows. The vertices v_1^1, v_2^1, v_3^1 and v_4^1 are colored with Color 4,3,5 and 5 respectively. (Similarly, if the vertices v_1^3, v_2^3, v_3^3 and v_4^3 are colored with Color 3,3,4 and 4 respectively or Color 3,4,4 and 3 respectively (if $m \equiv 2 \mod 4$), then the vertices v_1^1, v_2^1, v_3^1 and v_4^1 are colored with Color 1,2,5 and 5 respectively)

- The vertices in V^3 are colored with three different colors.
  The vertices v_1^3, v_2^3, v_3^3 and v_4^3 are colored with Color 1,1,2 and 3 respectively (if $m \equiv 0 \mod 4$). Consider the vertex v_2^1. Since $v_2^1 - v_1^2 - v_2^3$, $v_2^1 - v_1^0 - v_2^0$, $v_2^1 - v_3^2 - v_4^3$ and $v_2^1 - v_3^0 - v_4^0$ are paths of length 2. Also the vertices v_2^3, v_2^0, v_4^3 and v_4^0 are of Color 1,2,3 and 4 respectively. Thus the vertex v_2^1 is assigned Color 5. Also $v_4^1 - v_1^2 - v_2^3$, $v_4^1 - v_1^0 - v_2^0$, $v_4^1 - v_3^2 - v_4^3$, $v_4^1 - v_3^0 - v_4^0$ and $v_4^1 - v_3^0 - v_2^1$ are paths of length 2 and the vertices v_2^3, v_2^0, v_4^3, v_4^0 and v_2^1 are of Color 1,2,3,4 and 5 respectively shows that the vertex v_4^1 is colored with Color 6. Hence the vertices v_1^1, v_2^1, v_3^1 and v_4^1 are colored with Color 4,5,5 and 6 respectively. (Similarly, if the vertices v_1^3, v_2^3, v_3^3 and v_4^3 are colored with Color 4,4,3 and 2 respectively (if $m \equiv 2 \mod 4$), then the vertices v_1^1, v_2^1, v_3^1 and v_4^1 are colored with Color 1,5,5 and 6 respectively). Therefore this possibility needs six distinct colors for an injective coloring.

Case ii. m is odd.
Then $1 = m - 2k$ for some k, the vertices in V^1 are colored with four different colors. The vertices in V^3 also colored with four different colors. The coloring is as follows. The vertices v_1^1, v_2^1, v_3^1 and v_4^1 are colored with Color 1, 2, 3 and 4 respectively. Also the vertices v_1^3, v_2^3, v_3^3 and v_4^3 are colored with Color 4, 1, 2 and 3 respectively. Now there are two possibilities for the coloring of V^2.

- The vertices in V^2 are colored with two different colors.
  Let the coloring of vertices in V^2 be v_1^2, v_2^2, v_3^2 and v_4^2 be colored with Color 1,1,2 and 2 respectively or v_1^2, v_2^2, v_3^2 and v_4^2 be colored with Color 1,2,2 and 1 respectively (if $m \equiv 3 \mod 4$). The colors that are used in V^2 cannot be assigned to the vertices in V^0, since each color is used twice in V^2. Thus Color 1 and Color 2 cannot be assigned to the vertices in V^0. In both cases for the vertices v_2^0 and v_4^0, $v_2^0 - v_3^0 - v_4^1$ and $v_4^0 - v_3^0 - v_4^1$ are paths of length 2 and the vertex v_4^1 is of Color 3. Thus the vertices v_2^0 and v_4^0 cannot be assigned Color 3. Also the vertex v_3^0 is a common neighbor for the vertices v_2^0 and v_4^0. Thus one of the vertex is assigned Color 4 and other with Color 5. Thus the vertices v_1^0, v_2^0, v_3^0 and v_4^0 be colored with Color 3,4,5 and 5 respectively. (Similarly, if the vertices v_1^2, v_2^2, v_3^2 and v_4^2 are colored with Color 3,3,4 and 4 respectively (if $m \equiv 1 \mod 4$), then the vertices v_1^0, v_2^0, v_3^0 and v_4^0 are colored with Color 2,1,5 and 5 respectively).
- The vertices in V^2 are colored with three different colors.
  The vertices v_1^2, v_2^2, v_3^2 and v_4^2 are colored with Color 1,1,2 and 3 respectively (if $m \equiv 3 \mod 4$). With the similar arguments, it is clear that one of the vertex in V^0 is colored with Color 5. Thus the vertices v_1^0, v_2^0, v_3^0 and v_4^0 are colored with Color 3,2,5 and 4 respectively. (Similarly, if the vertices v_1^2, v_2^2, v_3^2 and v_4^2 are colored with Color 4,4,3

and 2 respectively (if $m \equiv 1 \mod 4$), then the vertices v_1^0, v_2^0, v_3^0 and v_4^0 are colored with Color 2,1,5 and 3 respectively.)

In any case, five different colors are necessary for an injective coloring of $\mu_m(P_4)$. A five coloring of $\mu_m(P_4)$ for $m = 3, 4, 5, 6$ is illustrated in Figure 5.

Case 4. $n > 4$.
For the vertices $v_i^m, 1 \leq i \leq n$, the vertex w is a common adjacent vertex. Then a total of n different colors are required for coloring the vertices v_i^m, $1 \leq i \leq n$. Therefore $\chi_i(\mu_m(P_n)) \geq n$. Now providing an injective coloring with n colors shows that $\chi_i(\mu_m(P_n)) = n$. The coloring is as follows.

- For $m \equiv 0$ mod 4 or $m = 2$
  - for $j = m$, allot the vertices v_k^j with Color $k, 1 \leq k \leq n$.
  - for $j = m - 1$, allot the vertices v_k^j with Color k for $1 \leq k \leq n - 1$ and allot the vertex v_n^j with Color 1.
- For $m \equiv 1$ mod 4 and $m \neq 1$
  - for $j = m$, allot the vertex v_1^j with Color n and allot the vertices v_k^j with the Color $k - 1$ for $2 \leq k \leq n$ and .
  - for $j = m - 1$, color the vertices v_k^j with Color k for $1 \leq k \leq n - 1$ and allot the vertex v_n^j with the Color 1.
- For $m \equiv 2$ mod 4 and $m \neq 2$
  - for $j = m$, allot the vertex v_1^j with Color n and allot the vertices v_k^j with the Color $k - 1$ for $2 \leq k \leq n$ and .
  - for $j = m - 1$, allot the vertex v_1^j with Color $n - 1$, color the vertices v_k^j with the Color $k - 1$ for $1 \leq k \leq n - 1$ and allot the vertex v_n^j with the Color 1.
- For $m \equiv 3$ mod 4 or $m = 1$
  - for $j = m$, allot the vertices v_k^j with the Color $k, 1 \leq k \leq n$.
  - for $j = m - 1$, allot the vertex v_1^j with the Color $n - 1$, allot the vertex v_k^j with the Color $k - 1$ for $2 \leq k \leq n - 1$ and allot the vertex v_n^j with the Color 1.

Next, color the remaining vertices as follows.

- For $j \equiv 0, 3$ mod 4 and $j \neq m, m-1$, color the vertices v_i^j with the Color i for $1 \leq i \leq n$. Except for $m = 2$ color the vertices v_1^0 with Color n and v_i^0 with the Color $i - 1$ for $1 \leq i \leq n$.
- For $j \equiv 1, 2$ mod 4 and $j \neq m, m - 1$, color the vertices v_1^j with the Color n and v_i^j with the Color $i - 1$ for $1 \leq i \leq n$.
- Color the vertex w with Color n.

□

The following theorem, determines $\chi_i(\mu_m(K_n))$, for all $n > 2$.

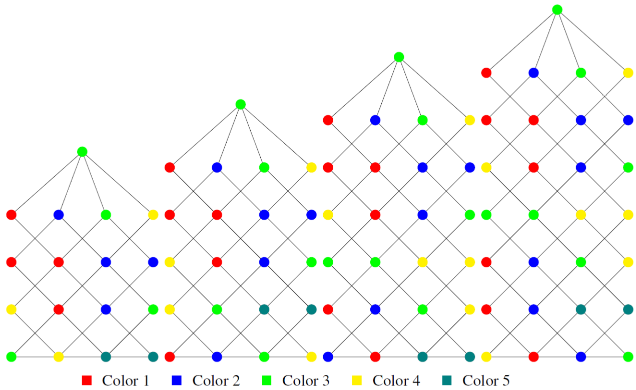

Figure 5. **Injective 5-coloring of $\mu_3(P_4)$, $\mu_4(P_4)$, $\mu_5(P_4)$ and $\mu_6(P_4)$**

Theorem 2. *For $n > 2$, the injective chromatic number of m−Mycielskian of K_n on n vertices is, $\chi_i(\mu_m(K_n)) = 2n$.*

Proof. Let $V^0 = \{v_i^0,\ 1 \le i \le n\}$ be the vertices of K_n and let $\bigcup_{j=0}^{m} V^j \cup \{w\}$ be the vertices of $\mu_m(K_n)$, where $V^j = \{v_i^j,\ 1 \le i \le n\}$. Consider the subgraph H induced by the vertices in $V^0 \cup V^1$, H is a graph with diameter 2 and every edge of H is contained in a triangle. Then by Proposition 1(iv), $\chi_i(H) = |V(H)| = 2n$. Thus $\chi_i(\mu_m(K_n)) \ge \chi_i(H) = 2n$. Now providing an injective coloring using $2n$ colors shows that $\chi_i(\mu_m(K_n)) = 2n$. The coloring is as follows.

- For $j \equiv 0, 3 \bmod 4$, allot the vertices v_k^j with the Color k, for $1 \le k \le n$.
- For $j \equiv 1, 2 \bmod 4$, allot the vertices v_k^j with the Color $n + k$, for $1 \le k \le n$.
- The vertex w is allotted the Color $n+1$ if $m \equiv 0, 1 \bmod 4$ and Color 1 if $m \equiv 2, 3 \bmod 4$.

□

Theorem 3. *For $n > 3$ the injective chromatic number of m-Mycielskian of a cycle on n vertices is*
$$\chi_i(\mu_m(C_n)) = \begin{cases} n+1 & \text{for } n = 4, 5, 6 \\ n & \text{for } n \ge 7. \end{cases}$$

Proof. Let the vertices of C_n be $V^0 = \{v_i^0,\ 1 \le i \le n\}$ and the vertices of $\mu_m(C_n)$ be $\bigcup_{j=0}^{m} V^j \cup \{w\}$, where $V^j = \{v_i^j,\ 1 \le i \le n\}$. Before moving to the proof see that the vertices of C_n can be ordered either as $v_1^0, v_2^0, v_3^0, v_4^0, \cdots, v_{n-1}^0, v_n^0$ or $v_2^0, v_3^0, v_4^0, v_5^0, \cdots, v_{n-1}^0, v_n^0, v_1^0$ or $v_3^0, v_4^0, v_5^0, \cdots, v_n^0, v_1^0, v_2^0$ etc, that is the starting point and end point does not matter. (ie., the vertices of C_5 can be ordered as $v_1^0, v_2^0, v_3^0, v_4^0, v_5^0$ or $v_2^0, v_3^0, v_4^0, v_5^0, v_1^0$ or $v_3^0, v_4^0, v_5^0, v_1^0, v_1^0$ etc). Therefore each vertex in V^0 have the same properties. Thus in the graph $\mu_m(C_5)$, each vertex in V^k have the same properties for $k = 0, 1, 2, \cdots, m$.

Case 1. $n = 4, 5, 6$.
Consider the following situations.

Subcase i. $n = 4$.
The graph $\mu_m(P_4)$ is a subgraph of $\mu_m(C_4)$ and $\chi_i(\mu_m(P_4)) = 5$ from Theorem 1. Hence from Proposition 1(v), $\chi_i(\mu_m(C_4)) \geq 5$. Now providing an injective coloring of $\mu_m(C_4)$ with 5 colors shows that $\chi_i(\mu_m(C_4)) = 5$. The coloring is as follows.

I) m is even.

- For $i = m - k$, $k \equiv 0, 1 \bmod 4$ and $i \neq 0$, color the vertices v_1^i, v_2^i, v_3^i and v_4^i with Color 1,2,3 and 4 respectively.
- For $i = m - k$, $k \equiv 2, 3 \bmod 4$ and $i \neq 0$, color the vertices v_1^i, v_2^i, v_3^i and v_4^i with Color 2,3,4 and 1 respectively.
- Color the vertex w with Color 5.

II) m is odd.

- For $i = m - k$, $k \equiv 0 \bmod 4$ and $i \neq 0$, color the vertices v_1^i, v_2^i, v_3^i and v_4^i with Color 1,2,3 and 4 respectively.
- For $i = m - k$, $k \equiv 1 \bmod 4$ and $i \neq 0$, color the vertices v_1^i, v_2^i, v_3^i and v_4^i with Color 1,1,2 and 2 respectively.
- For $i = m - k$, $k \equiv 2 \bmod 4$ and $i \neq 0$, color the vertices v_1^i, v_2^i, v_3^i and v_4^i with Color 4,1,2 and 3 respectively.
- For $i = m - k$, $k \equiv 3 \bmod 4$ and $i \neq 0$, color the vertices v_1^i, v_2^i, v_3^i and v_4^i with Color 3,3,4 and 4 respectively.
- Color the vertices v_1^0, v_2^0, v_3^0 and v_4^0 with
  - Color 5,1,2 and 5 respectively if $m \equiv 1 \bmod 4$.
  - Color 3,4,5 and 5 respectively if $m \equiv 3 \bmod 4$.
- Color the vertex w with Color 5.

Subcase ii. $n = 5, 6$.
First we will prove this for $n = 5$.

Coloring the vertices in V^m: The vertices in V^m are colored with 5 distinct colors. Since w is a common neighbor for the vertices in V^m. Let k be the color of v_k^m for $1 \leq k \leq 5$.

Coloring the vertices in V^{m-2k}, $m - 2k \neq 0, 1$: For the vertices in V^{m-2}, Color 1 cannot be given to the vertices v_1^{m-2}, v_3^{m-2} and v_4^{m-2} as $v_1^m - v_2^{m-1} - v_1^{m-2}$, $v_1^m - v_2^{m-1} - v_3^{m-2}$ and $v_1^m - v_4^{m-1} - v_3^{m-2}$ are paths of length 2 and the vertex v_1^m is of Color 1. Now the remaining vertices are v_2^{m-2} and v_5^{m-2}. Since v_1^{m-1} is a common neighbor for these vertices. Thus only one of the vertex in V^{m-2} can be colored with color 1. Similarly Color 2,3,4 and 5 is assigned to exactly one of the vertex in V^{m-2}. Thus the vertices in V^{m-2} are assigned distinct five colors. Let the vertices $v_1^{m-2}, v_2^{m-2}, v_3^{m-2}, v_4^{m-2}$ and v_5^{m-2} are colored with Color 5,1,2,3 and 4 respectively. As

the vertices in V^{m-2} are assigned five distinct colors, by the similar arguments it is clear that the vertices in V^{m-4} is assigned distinct colors. Continuing like this, it can seen that the vertices in V^{m-2k} is assigned distinct colors.
In general

- The vertices $v_1^{m-2k}, v_2^{m-2k}, v_3^{m-2k}, v_4^{m-2k}$ and v_5^{m-2k} are colored with Color 1,2,3,4 and 5 respectively for $2k = 0, 4, 8, 12, \cdots$ and $m - 2k \neq 0, 1$.
- The vertices $v_1^{m-2k}, v_2^{m-2k}, v_3^{m-2k}, v_4^{m-2k}$ and v_5^{m-2k} are colored with Color 5,1,2,3 and 4 respectively for $2k = 2, 6, 10, 14, \cdots$ and $m - 2k \neq 0, 1$.

Coloring the vertices in V^{m-1} and the vertex w: If one of the vertex say v_1^{m-1} in V^{m-1} is assigned Color 1, and $v_1^{m-1} - v_2^m - v_3^{m-1}$, $v_1^{m-1} - v_5^m - v_4^{m-1}$ are paths of length 2. Then Color 1 cannot be assigned to the vertices v_3^{m-1} and v_4^{m-1}. Now the remaining vertices are v_2^{m-1} and v_5^{m-1}. The vertex v_1^m is a common neighbor for these vertices. Thus one of the vertex v_2^{m-1} and v_5^{m-1} is assigned Color 1. Let it be v_2^{m-1}. (If we choose v_5^{m-1}, it doesn't matter, since v_5^m is in left of v_1^m and v_2^m is in right of v_1^m in the order of the vertices.) Thus Color 1 is assigned to two vertices in V^{m-1} (they are nearby vertices in the order of the vertices $v_1^{m-1}, v_2^{m-1}, v_3^{m-1}, v_4^{m-1}, v_5^{m-1}$). Also it is not possible to color three vertices of V^{m-1} with one color. Now Color 2 is assigned to the vertices v_3^{m-1} and v_4^{m-1} and Color 3 is assigned to the vertex v_5^{m-1}. Thus at least three colors are needed to color the vertices in V^{m-1}. Now the following possibilities provide coloring of vertices in V^{m-1} with three, four and five different colors.

- Possibility 1: Coloring the vertices in V^{m-1} with three different colors.
  The vertices $v_1^{m-1}, v_2^{m-1}, v_3^{m-1}, v_4^{m-1}$ and v_5^{m-1} are colored with Color 1,1,2,2 and 3 respectively. That is, two pairs of nearby vertices (it doesn't mean they are adjacent, it means that they are adjacent in the order) in the order are assigned same color and one vertex in between the pairs is assigned the third color. Any other coloring with three colors is similar to this. And color the vertex w with Color 4. Thus the coloring is as follows.
  - The vertices $v_1^{m-1}, v_2^{m-1}, v_3^{m-1}, v_4^{m-1}$ and v_5^{m-1} are colored with Color 1,1,2,2 and 3 respectively.
- Possibility 2: Coloring the vertices in V^{m-1} with four different colors.
  The vertices $v_1^{m-1}, v_2^{m-1}, v_3^{m-1}, v_4^{m-1}$ and v_5^{m-1} are colored with Color 1,1,2,3 and 4 respectively. As in Possiblity 1 any other coloring with four colors is similar to this. And color the vertex w with Color 5.
- Possibility 3: Coloring the vertices in V^{m-1} with five different colors.
  In this case the vertex w is assigned Color 6.

Thus when we use Possibility 3, six distinct colors are needed for an injective coloring of $\mu_m(C_5)$. Therefore we are not continuing the coloring for Possibility 3.

Coloring the vertices in V^{m-3} : Coloring the vertices in V^{m-3} corresponding to each possibilities from above:

- Possibility 1: Coloring the vertices in V^{m-1} with three different colors.
  The vertices $v_1^{m-1}, v_2^{m-1}, v_3^{m-1}, v_4^{m-1}$ and v_5^{m-1} are colored with Color 1,1,2,2 and 3 respectively. Then $v_1^{m-3}-v_2^{m-2}-v_1^{m-1}$, $v_2^{m-3}-v_1^{m-2}-v_2^{m-1}$, $v_3^{m-3}-v_2^{m-2}-v_1^{m-1}$, $v_4^{m-3}-v_3^{m-2}-v_2^{m-1}$ and $v_5^{m-3}-v_1^{m-2}-v_2^{m-1}$ are paths of length 2 and the vertices v_1^{m-1} and v_2^{m-1} are of Color 1. Thus the vertices $v_1^{m-3}, v_2^{m-3}, v_3^{m-3}, v_4^{m-3}$ and v_5^{m-3} cannot be assigned Color 1, i.e., the vertices in V^{m-3} cannot be assigned Color 1. Thus the color which is used twice cannot be used to the vertices in V^{m-3}. Therefore Color 2 also cannot be assigned to the vertices in V^{m-3}. Also, any color used once in V^{m-1} can be assigned to exactly one vertex of V^{m-3}. Thus the coloring is as follows.
  - The vertices $v_1^{m-3}, v_2^{m-3}, v_3^{m-3}, v_4^{m-3}$ and v_5^{m-3} are colored with Color 3,4,4,5 and 5 respectively.

  This is the only possible coloring when the vertices in V^{m-1} are colored with three colors.
- Possibility 2: Coloring the vertices in V^{m-1} with four different colors.
  The vertices $v_1^{m-1}, v_2^{m-1}, v_3^{m-1}, v_4^{m-1}$ and v_5^{m-1} are colored with Color 1,1,2,3 and 4 respectively. Here also the color which is used twice cannot be used to the vertices in V^{m-3} and any color used once can be assigned to exactly one vertex of V^{m-3}. Thus the coloring is as follows.
  - The vertices in $v_1^{m-3}, v_2^{m-3}, v_3^{m-3}, v_4^{m-3}$ and v_5^{m-3} are colored with Color 5,2,3,4 and 5 respectively.

Coloring the vertices in V^{m-l}, l is odd, $m-l \neq 0,1$ and $l \neq 1,3$: Continue the coloring in similarly way as done for V^{m-1} and V^{m-3}. The coloring in general is:

- Possibility 1:
  - Color the vertices in $v_1^{m-l}, v_2^{m-l}, v_3^{m-l}, v_4^{m-l}$ and v_5^{m-l} are colored with Color 1,1,2,2 and 3 respectively for $l = 5, 9, 13, 17, \cdots$.
  - Color the vertices in $v_1^{m-l}, v_2^{m-l}, v_3^{m-l}, v_4^{m-l}$ and v_5^{m-l} are colored with Color 3,4,4,5 and 5 respectively for $l = 7, 11, 15, 19, \cdots$.
- Possibility 2:
  - Color the vertices in $v_1^{m-l}, v_2^{m-l}, v_3^{m-l}, v_4^{m-l}$ and v_5^{m-l} are colored with Color 1,1,2,3 and 4 respectively for $l = 5, 9, 13, 17, \cdots$.
  - Color the vertices in $v_1^{m-l}, v_2^{m-l}, v_3^{m-l}, v_4^{m-l}$ and v_5^{m-l} are colored with Color 5,2,3,4 and 5 respectively for $l = 7, 11, 15, 19, \cdots$.

Coloring the vertices in V^0 and V^1 together:

Case i. m is even.
Then $0 = m-2k$ for some k, the vertices in V^0 are colored with five different colors. The vertices in V^2 also colored with five different colors. The coloring is as follows.

- The vertices $v_1^2, v_2^2, v_3^2, v_4^2$ and v_5^2 are colored with Color 1,2,3,4 and 5 respectively. (Similar arguments will follow when the $v_1^2, v_2^2, v_3^2, v_4^2$ and v_5^2 are colored with Color 5,1,2,3 and 4 respectively).

– the vertices $v_1^0, v_2^0, v_3^0, v_4^0$ and v_5^0 are colored with Color 2,3,4,5 and 1 respectively.

Now there are two possibilities for coloring the vertices in V^3.

- The vertices in V^3 are colored with three different colors. Let the vertices $v_1^3, v_2^3, v_3^3, v_4^3$ and v_5^3 are colored with Color 1,1,2,2 and 3 respectively. Here also the color which is used twice cannot be used to the vertices in V^1 and any color used once can be assigned to exactly one vertex of V^1. That is Color 3 is assigned to exactly one vertex in V^0. Now $v_4^0 - v_5^0 - v_1^1$, $v_4^0 - v_3^0 - v_2^1$, $v_4^0 - v_3^0 - v_4^1$ are path of length 2 and the vertex v_4^0 is of Color 4. Now the remaining vertices are v_3^1 and v_5^1 and they have common neighbor v_4^0. Thus Color 4 is assigned to one of the vertex. Similarly Color 5 is also assigned to exactly one vertex. Therefore the remaining two vertices are assigned Color 6. Thus the coloring is as follows.
  - The vertices $v_1^1, v_2^1, v_3^1, v_4^1$ and v_5^1 are colored with Color 3,4,5,6 and 6 respectively.
- The vertices in V^3 are colored with four different colors. Let $v_1^3, v_2^3, v_3^3, v_4^3$ and v_5^3 with Color 1,1,2,3 and 4 respectively. Here Color 1 cannot be assigned to the vertices in V^1 and Color 2,3,4 and 5 are assigned to exactly one of the vertex in V^1 by the similar arguments of above coloring. Thus the remaining one vertex is assigned Color 6. Thus the coloring is as follows.
  - The vertices $v_1^1, v_2^1, v_3^1, v_4^1$ and v_5^1 are colored with Color 6,2,3,4 and 5 respectively.

Case ii. *m* is odd.
Then $1 = m - 2k$ for some k, the vertices in V^1 are colored with five different colors. The vertices in V^3 also colored with five different colors. The coloring is as follows.

– $v_1^1, v_2^1, v_3^1, v_4^1$ and v_5^1 are colored with Color 1,2,3,4 and 5 respectively.
– $v_1^3, v_2^3, v_3^3, v_4^3$ and v_5^3 are colored with Color 2,3,4,5 and 1 respectively.

Now there are two possibilities for the coloring of V^2.

- The vertices in V^2 are colored with three different colors. Let $v_1^2, v_2^2, v_3^2, v_4^2$ and v_5^2 are colored with Color 1,1,2,2 and 3 respectively. Here also the color which is used twice cannot be used to the vertices in V^0 and any color used once can be assigned to exactly one vertex of V^0. Now $v_4^1 - v_5^0 - v_1^0$, $v_4^1 - v_3^0 - v_2^0$ $v_4^1 - v_3^0 - v_4^0$ are paths of length 2 and the vertex v_4^1 is of Color 4. Then the remaining vertices are v_3^0 and v_5^0, they have common neighbor v_4^0. Thus Color 4 is assigned to exactly one vertex. Similarly Color 5 is assigned to exactly one vertex. Thus the remaining two vertices are assigned Color 6.
  - $v_1^0, v_2^0, v_3^0, v_4^0$ and v_5^0 are colored with Color 5,6,6,3 and 4 respectively.
- The vertices in V^2 are colored with four different colors. Let $v_1^2, v_2^2, v_3^2, v_4^2$ and v_5^2 with Color 1,1,2,3 and 4 respectively. Here Color 1 cannot be assigned to the vertices in V^0 and Color 2,3,4 and 5 are assigned to exactly one of the vertex in V^0 by the similar arguments of above coloring. Thus the remaining one vertex is assigned Color 6.
  - $v_1^0, v_2^0, v_3^0, v_4^0$ and v_5^0 are colored with Color 1,2,3,4 and 5 respectively.

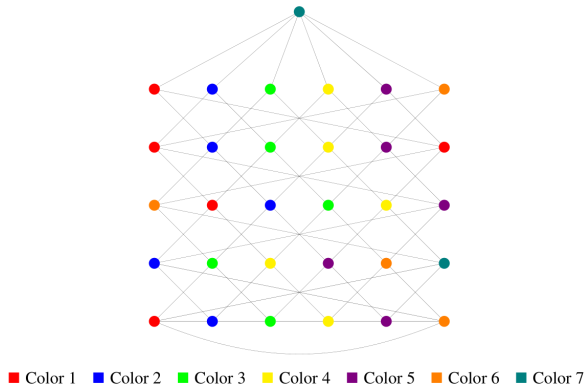

Figure 6. **Injective 7-coloring of** $\mu_4(C_6)$

Thus in any case, six different colors are necessary for an injective coloring of $\mu_m(C_5)$. Similarly $\chi_i(\mu_m(C_6)) = 7$. In Figure 6, the injective coloring of $\mu_4(C_6)$ is presented.

Case 2. $n \geq 7$

The vertex w is adjacent to all the vertices in V^m, therefore $\chi_i(\mu_m(C_n)) \geq |V^m| = n$. Now providing an injective coloring using n colors shows that $\chi_i(\mu_m(C_n)) = n$. The coloring is as follows.

Subcase i. $n = 7$.

In Table 1 the injective coloring of $\mu_m(C_7)$ using 7 colors for $m = 1, 2, 3, 4, 5, 6, 7$ is provided. Let's say the colors as 1,2,3,4,5,6 and 7. The i^{th} column represents the injective coloring of $\mu_i(C_7)$ and an $(i, j)^{th}$ cell (say 4365476) represents the colors of the vertices $v^i_1, v^i_2, v^i_3, v^i_4, v^i_5, v^i_6$ and $v^i_7 \in V^i$ as 4,3,6,5,4,7 and 6 respectively.

Table 1 : Injective 7-coloring of $\mu_m(C_7)$ for $m = 1, 2, 3, 4, 5, 6, 7$

m	1	2	3	4	5	6	7
V^7							1234567
V^6						1234567	4455465
V^5					1234567	4455465	4365476
V^4				1234567	1122132	2173173	1122132
V^3			1234567	1122132	2173173	1122132	1122132
V^2		1234567	1122132	4365476	4455465	4455465	4455465
V^1	1234567	1122132	2345234	4455665	4455465	4455465	4455465
V^0	4365476	4365476	5634567	1122132	1122132	1122132	1122132
w	1	7	7	7	7	7	7

Now for $m > 7$,

- The vertices $v^m_k, 1 \leq k \leq 7$ are allotted with the Color k.

- If $i \equiv 0,3 \bmod 4$ and $i \neq m-2$, color the vertices v_j^i, $1 \leq j \leq 7$ with Color $1,1,2,2,1,3,2$ respectively.
- If $i \equiv 1,2 \bmod 4$ and $i \neq m-2$, color the vertices v_j^i, $1 \leq j \leq 7$ with Color $4,4,5,5,4,6,5$ respectively.
- When $m = 4q+r, q \geq 2$ and $0 \leq r \leq 3$, choose $k = r+4$, the vertices in V^{m-2} of $\mu_m(C_7)$ are allotted with the colors of the vertices of V^{k-2} of $\mu_k(C_7)$ as in the table.
- Allot the vertex w with Color 7.

Subcase ii.: $n = 8$.
For $\mu_m(C_8)$, the following gives an injective coloring

- The vertices v_k^m, $1 \leq k \leq 8$ are allotted with the Color k.
- For $m = 1$, the vertices of V^0, v_i^0, $1 \leq i \leq 8$ are colored with the Color 7,6,6,1,1,2,2 and 3 respectively.
- For $m \neq 1$,
  - if $i \equiv 0,3 \bmod 4$ and $i \neq m-2$, color the vertices v_j^i, $1 \leq j \leq 8$ with Color $1,1,2,2,1,1,2,2$ respectively.
  - If $i \equiv 1,2 \bmod 4$ and $i \neq m-2$, color the vertices v_j^i, $1 \leq j \leq 8$ with Color $3,3,4,4,3,3,4,4$ respectively.
- For the vertices in V^{m-2},
  - If $m \equiv 0,3 \bmod 4$ or $m = 2$, color the vertices in v_i^{m-2}, $1 \leq i \leq 8$, with the Color 4,7,7,8,8,3,3 and 4 respectively.
  - If $m \equiv 1,2 \bmod 4$ and $m \neq 1,2$, color the vertices in v_i^{m-2}, $1 \leq i \leq 8$, with the Color 5,6,6,8,8,2,2 and 5 respectively.
- The vertex w is colored with Color n.

Subcase iii. $n \geq 9$.
There are n colors. As $n \geq 9$ and each vertex class $V^i, i \neq m$ can be colored with at least three colors, it is possible to allot the vertices of $\mu_m(C_n)$ with these n colors.

- Color the vertices in V^m with n distinct colors.
- The vertices in $V^{m-(3k+1)}, k = 0,1,2,\cdots$ are colored with three distinct colors say Color 1,2 and 3. It is possible since there is no common vertex between these vertex sets.
- The vertices in $V^{m-(3k+2)}, k = 0,1,2,\cdots$ are colored with three distinct colors say Color 4,5 and 6.
- The vertices in $V^{m-(3k)}, k = 1,3,4\cdots$ are colored with three distinct colors say Color 7,8 and 9.

□

In Theorem 4, $\chi_i(\mu_m(K_{p,q}))$ is computed. Here the clique number of $\mu_m(K_{m,n})$ is two, but the injective chromatic number is $2p$ or $2p+1$. To enhance the comprehension of the proof, the graph of generalized Mycielskian of $K_{3,3}$ for general m is presented in Figure 7.

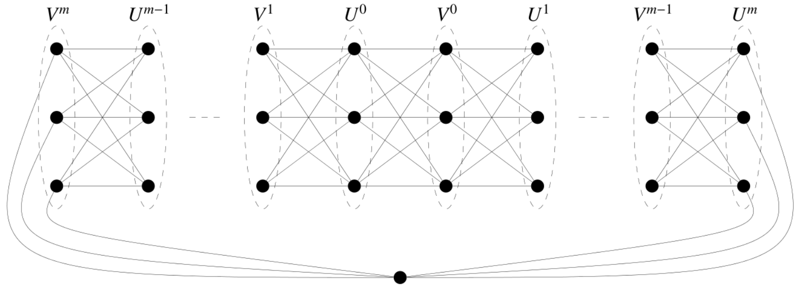

Figure 7. $\mu_m(K_{3,3})$

Theorem 4. *The injective chromatic number of* $\mu_m(K_{p,q})$ *is* $\chi_i(\mu_m(K_{p,q})) = \begin{cases} 2p & \text{for } p > q \\ 2p+1 & \text{for } p = q. \end{cases}$

Proof. Let $U^0 \cup V^0$ with $U^0 = \{u_i^0 : 1 \le i \le p\}$ and $V^0 = \{v_i^0 : 1 \le i \le q\}$ be the vertices of $K_{p,q}$. And $\bigcup_{j=0}^{m}(U^j \cup V^j) \cup \{w\}$ be the vertices of $\mu_m(K_n)$, where $U^j = \{u_i^j : 1 \le i \le p\}$, $V^j = \{v_i^j : 1 \le i \le q\}$.

Case 1. $p > q$.
A vertex in V^0 is a common adjacent vertex for the vertices in $U^0 \cup U^1$ by the construction. There are $2p$ vertices in $U^0 \cup U^1$. Thus $\chi_i(\mu_m(K_{p,q})) \ge 2p$. Now providing an injective coloring using $2p$ colors shows that $\chi_i(\mu_m(K_{p,q})) = 2p$. The coloring is as follows.

- For $j \equiv 0, 3 \bmod 4$
  - color the vertices u_i^j with Color i
  - color the vertices v_i^j with Color $p + i$
- For $j \equiv 1, 2 \bmod 4$
  - color the vertices u_i^j with Color $p + i$
  - color the vertices v_i^j with Color i
- Color the vertex w with Color $2p$.

Case 2. $p = q$.

Claim. All the vertices in a vertex class V^j (or U^j) are allotted with distinct p colors.
Proof. The following adjacency's proves the claim.

- For the vertices in U^0, any vertex in V^1 is a common adjacent vertex.
- For the vertices in V^0, any vertex in U^1 is a common adjacent vertex.
- For $j = 1, 2, \cdots, m-1$, any vertex either in V^{j-1} or in V^{j+1} is a common adjacent vertex for the vertices in U^j.

- For $j = 1, 2, \cdots, m-1$, any vertex either in U^{j-1} or in U^{j+1} is a common adjacent vertex for the vertices in V^j.
- For the vertices in U^m, vertex w is a common adjacent vertex.
- For the vertices in V^m, vertex w is a common adjacent vertex.

Also, the construction makes it possible to choose a vertex from each vertex class in a certain order, which results in an odd cycle. Say, $w - U^m - V^{m-1} - U^{m-2} - \cdots - U^1 - V^0 - U^0 - V^1 - \cdots - V^{m-2} - U^{m-1} - V^m - w$. Three colors are necessary to color an odd cycle, in a similar way three sets of colors are necessary to color $\mu_m(K_{p,q})$. Hence two sets are of size p and one is of size one. Therefore $2p+1$ colors are necessary for an injective coloring of $\mu_m(K_{p,q})$. The coloring is as follows.
Subcase i: m is odd.
Consider an ordering of the vertex classes as: $V^m, U^{m-1}, V^{m-2}, U^{m-3}, V^{m-4}, U^{m-5}, \cdots, V^1, U^0, V^0, U^1, \cdots, U^{m-2}, V^{m-1}, U^m$.

- For the vertices in V^m, allot the vertices v_k^m with Color k.
- For the vertices in U^{m-1}, allot the vertices u_k^{m-1} with Color k.
- For the vertices in V^{m-2}, allot the vertices v_k^{m-2} with Color $p+k$.
- For the vertices in U^{m-3}, allot the vertices u_k^{m-2} with Color $p+k$.

The next four vertex classes are allotted with the same coloring pattern as listed previously. Continue the same procedure for the remaining vertex classes from the ordering.

- Allot the vertex w with Color $2p+1$.

Subcase ii. m is even.
Consider an ordering of the vertex classes as: $U^m, V^{m-1}, U^{m-2}, V^{m-3}, U^{m-4}, V^{m-5}, \ldots, V^1, U^0, V^0, U^1, \ldots, V^{m-2}, U^{m-1}, V^m$.

- For the vertices in V^{m-1}, allot the vertices v_k^{m-1} with Color $p+k$.
- For the vertices in U^{m-2}, allot the vertices u_k^{m-2} with Color $p+k$.
- For the vertices in V^{m-3}, allot the vertices v_k^{m-3} with Color k.
- For the vertices in U^{m-4}, allot the vertices u_k^{m-4} with Color k.

In each one, if any of the values $m-1, m-2, m-3$ or $m-4$ is less than zero, without loss of generality consider the next class of vertices in the ordering of the vertex classes.
The next four vertex classes are allotted with the same coloring pattern as listed previously. Continue the same procedure for the remaining vertex classes from the ordering $U^m, V^{m-1}, U^{m-2}, V^{m-3}, U^{m-4}, V^{m-5}, \ldots, V^1,\ U^0, V^0, U^1, \ldots, V^{m-2}, U^{m-1}, V^m$ except for V^m and U^m.

- For the vertices in U^m, color the vertices u_k^m with Color k.
- For the vertices in V^m, color the vertices v_k^m with Color $p+k$.
- Color the vertex w with Color $2p+1$.

The injective coloring of $\mu_2(K_{3,3})$ with 7 colors is illustrated in Figure 8. □

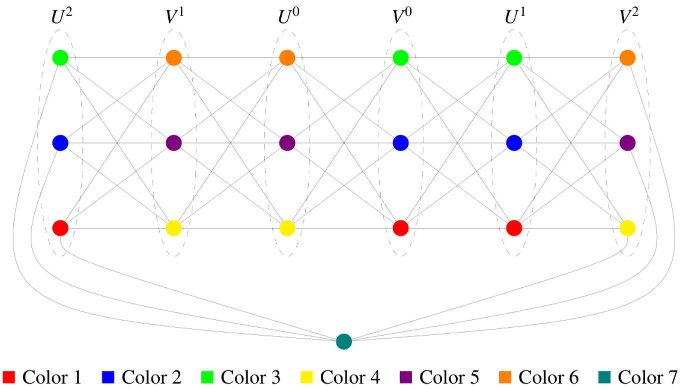

Figure 8. **Injective 7-coloring of $\mu_2(K_{3,3})$**

Corollary 2. *Injective chromatic number of generalized Mycielskian of a star graph S_{n+1} is $\chi_i(\mu_m(S_{n+1})) = 2n$ for $n > 1$.*

3. Applications

The results obtained in this paper also suggest future exploration of injective edge chromatic number, the decycling number, domination number etc. of the generalized Mycielskian of a graph. Further, it is also open to compute the injective chromatic number of generalized Mycielskian of any arbitrary graph. The properties of injective coloring plays a vital role in different areas of complexity theory, random access machine, error correcting codes etc. The n-dimensional hypercube, or n-cube, can be formed by taking one vertex for each binary n-tuple, two vertices being adjacent exactly when the Hamming distance between the corresponding n-tuples is 1.

In computer networking, hypercube networks are a type of network topology used to connect multiple processors with memory modules and accurately route data. Hypercube networks consist of 2^n nodes, which form the vertices of squares to create an internetwork connection. It is basically a multi-dimensional mesh network with two nodes in each dimension. The injective chromatic number of hypercubes is well studied in [6]. In [16], it is established that, $\chi_k(n)$ denotes the minimum number of colors necessary to color the n-dimensional hypercube Q_n so that no two vertices at a distance exactly k from each other get the same color. In other words, this is the smallest number of binary codes with minimum distance $k + 1$ that form a partition of the n-dimensional binary Hamming space. Equivalently with $k = 2$, no two vertices that are at a distance 2 from each other get the same color and $\chi_2(n)$ is the smallest number of binary codes with minimum distance $3(= k + 1)$ that form a partition of the n-dimensional binary Hamming space. Note that a coloring is viewed as a partition of Q_n into codes (color classes) and the color classes of an injective coloring attaining $\chi_2(n)$ can be viewed as binary codes with minimum distance 3.

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