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Research Article

A Simple-Intersection Graph of a Ring Approach to Solving Coloring Optimization Problems

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Abstract: In this paper, we introduce a modified version of the simple-intersection graph for semisimple rings, applied to a ring R with unity. The findings from this modified version are subsequently utilized to solve several coloring optimization problems. We demonstrate how the clique number of the simple-intersection graph can be used to determine the maximum number of possibilities that can be selected from a set of n colors without replacement or order, subject to the constraint that any pair shares only one common color. We also show how the domination number can be used to determine the minimum number of possibilities that can be selected, such that any other possibility shares one color with at least one of the selected possibilities, is $n-1$.

Keywords: simple-intersection graph, semisimple rings, ideals, cliques, girth, domination number, coloring, optimization.

AMS Subject classification: 05C07, 05C25, 05C38, 05C40, 05C69.

1. Introduction

One of the active research fields of algebraic graph theory which draws the attention of researchers is the association of a graph with an algebraic structure, see for example [3, 8, 10, 14, 16, 17, 19]. The significance of this topic have led many authors to study the interplay between the graph theoretic properties (such as diameter, girth, cliques, connectedness, dominating sets, etc) and the algebraic properties of the underlined algebraic structure. A large literature has been devoted to the study of algebraic structures and their associated graphs, as shown for example in [1, 2, 4, 5, 11–13, 15, 20].

In [18], the authors introduced a new type of intersection graphs for rings defined as follows:

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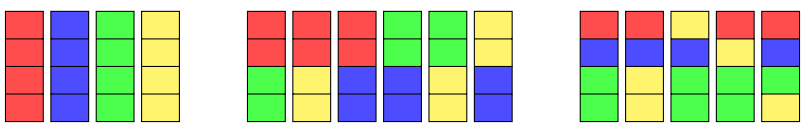
Definition 1. Let R be a ring with unity $1 \neq 0$. The simple-intersection graph of R, denoted by $GS(R)$, is defined to be a simple graph whose vertices are the nonzero ideals of R, and two vertices I and J are adjacent, and we write $I \leftrightarrow J$, if and only if $I \cap J$ is a nonzero simple ideal.

For example, consider the ring $\mathbb{Z}_4$. The nonzero ideals of $\mathbb{Z}_4$ are $\mathbb{Z}_4$ and $2\mathbb{Z}_4 \cong \mathbb{Z}_2$. Obviously, $GS(\mathbb{Z}_4)$ is $\mathbb{Z}_4 \leftrightarrow 2\mathbb{Z}_4$. Clearly, the simple-intersection graph is not a subgraph of the intersection graph of ideals of R [7] because $R \in V(GS(R))$ but $R \notin V(G(R))$. However, the subgraph of $GS(R)$ consisting of all nonzero proper ideals of R is a subgraph of $G(R)$. The authors studied several properties of this graph such as connectedness, Euler circuits, regularity, girth, cliques, the "bipartite" property, dominating sets. Moreover, they related these concepts with various algebraic properties of the ring R. In this paper, we study a modified type of the simple-intersection graph for a special class of rings which is the class of semisimple rings. The results obtained in [16] about the simple-intersection graphs are still valid for the modified simple-intersection graph of semisimple rings. The semisimple rings provide a finite number of vertices and edges for this modified simple-intersection graph, which we denote by the same notation $GS(R)$. We shall develop many results about the simple-intersection graph of semisimple rings. Moreover, we compute precisely the diameter, clique number, domination number, and the girth. Furthermore, the results obtained about $GS(R)$ of a semisimple ring R can be interpreted into coloring optimization problems and their solutions. For instance, let us consider all possible selections (rectangles) from four colors without repetition and without order as shown below:



The maximum number of rectangles selected from the above sets such that any two rectangles have one color in common is 4 (i.e. the number of colors). Also, the minimum number of rectangles selected from the above set such that any other rectangle shares only one color with at least one of the selected rectangles is 3 (The number of colors minus one).

We shall use graph properties of $GS(R)$ prove and generalize these coloring optimization problems and other problems to any n different colors, where $n \in \mathbb{N}$. More precisely, using the clique number of the modified simple-intersection graph of a semisimple ring, we show that if we have n different colors and we make all possible selections without replacement from these colors at once (i.e., we can select 1, 2, ..., or n colors at once from the n different colors), then the maximum number of possibilities we can take such that any pair of them have only one color in common is n. In addition, we use the domination number to prove that the minimum number of possibilities that can be taken such that any other possibility shares one color with at least one of the selected possibilities is $n - 1$.

2. Background

This section is devoted for a review of basic concepts of rings and graphs. All results presented here can be found in [6] or [9]. In this paper, all rings are assumed to be nonzero rings with unity $1 \neq 0$ and are not necessarily commutative. Also, the ideals are considered to be left ideals. We first start with some preliminaries from Ring Theory.

Definition 2. An ideal I of a ring R is said to be simple (or minimal) if I and $\{0\}$ are the only ideals included in I.

Definition 3. The direct sum of simple ideals of a ring R is called a semisimple ideal. We call each simple ideal in the decomposition of a semisimple ideal a component.

Obviously, a simple ideal is semisimple with one component. On the other hand, every ideal of a semisimple ideal is semisimple.

Definition 4. The socle of R, denoted by $Soc(R)$, is defined to be the sum of all nonzero simple ideals of R. If $R = Soc(R)$, we call R a semisimple ring.

Definition 5. A proper ideal I of a ring R is said to be maximal if I is not contained in another proper ideal of R.

Definition 6. A ring R is said to be Artinian if it satisfies the descending chain condition on ideals.

Theorem 1. *A ring R is Artinian if and only if Every nonzero ideal contains a nonzero simple ideal.*

It is easy to see that the semisimple rings are examples of Artinian rings.

Next, we turn to preliminaries from graph theory concerning undirected graphs. In what follows, G denotes an undirected graph. The number of vertices of G is called the order the graph G. The set of vertices of G is denoted by $Ver[G]$. If two vertices u and v are adjacent, we express that symbolically by $u \leftrightarrow v$.

Definition 7. Let v be a vertex in G. The neighborhood $N(v)$ of v is the set of all vertices adjacent to v, i.e., the set of all vertices each of which is linked to v by an edge.

If G is a simple undirected graph, then $v \notin N(v)$. If $N(v) = \emptyset$, then v is an isolated vertex.

Definition 8. The degree of a vertex v of G is the number of edges incident to v,i.e., going out of v. The degree of v is denoted by $deg_G(v)$ (or $deg(v)$ if there is no confusion with the underlined graph).

When G is a simple graph, then $deg(v) = |N(v)|$, where $|N(v)|$ means the cardinality of $N(v)$. Hence, v is isolated if and only if $deg(v) = 0$.

Definition 9. A graph whose vertices have equal degrees is called a regular graph.

Definition 10. Let v and u be two vertices of G. The length of a path between v and u is the number of edges forming the path. The distance $d(u, v)$ between v and u is the length of a shortest path between them. The diameter of G, denoted by $diam(G)$, is defined to be the supremum of the set $\{d(u, v) : u, v \in Ver[G]\}$.

Definition 11. A graph G is path connected if there is a path between any two vertices of G.

Definition 12. A graph is said to be complete if it is a simple graph and every pair of vertices are adjacent. The complete graph on n vertices is denoted by K_n.

Definition 13. A subgraph of G which is a complete graph is called a clique of G. The order of a largest clique (i.e., a clique with the largest number of vertices) is called the clique number of G and it is denoted by $\omega(G)$.

Definition 14. By the girth of G, we mean the length of a shortest cycle in G. The girth of G is denoted by $g(G)$. If G has no cycles, then we write $g(G) = \infty$.

Definition 15. An Euler path of G is a path consisting of all edges of G without repetition. If an Euler path is closed, then it is called an Euler cycle.

Theorem 2. *A graph has an Euler cycle if and only if every vertex has an even degree. However, a graph has an Euler path if and only if at most two vertices have an odd degree.*

Definition 16. A dominating set D of G is a nonempty subset of $Ver[G]$ such that each vertex of G is either in D or adjacent to a vertex in D. The infimum of the set $\{|D| : D \text{ is a dominating set of } G\}$ is called the domination number of G and is denoted by $\gamma(G)$.

Definition 17. A simple graph G is called bipartite if we can partition $Ver[G]$ into two disjoint nonempty subsets (each subset is called a part) such that the vertices belonging to the same subset are not adjacent to each other. A complete bipartite graph is a bipartite graph where each vertex in one part is adjacent to each vertex in the other part. A complete bipartite graph is denoted by $K_{m,n}$ or $K_{n,m}$, where m is the cardinality of one part and n is the cardinality of the other part.

More preliminaries will be added in the next sections as needed.

3. The Simple-Intersection Graph of A Ring

This section is concerned with the simple-intersection graph $GS(R)$ of a ring R represented in [18].

Definition 18. [18] Let R be a ring with unity $1 \neq 0$. The simple-intersection graph of R, denoted by $GS(R)$, is defined to be a simple graph whose vertices are the nonzero ideals of R, and two vertices I and J are adjacent, and we write $I \leftrightarrow J$, if and only if $I \cap J$ is a nonzero simple ideal.

Remark 1. [18] In $GS(R)$, $I \leftrightarrow R$ if and only if I is a nonzero simple ideal of R. So the subgraph consisting of R with all nonzero simple ideals of R is a star graph with center R, and hence $deg(R)$ equals the number of nonzero simple ideals of R. Thus, if R is semisimple with n components, then $deg(R) = n$. On the other hand, if I is a nonzero simple ideal of R and J is an ideal of R, then $I \leftrightarrow J$ if and only if $I \subsetneq J$. Moreover, Every pair of different nonzero simple ideals is not adjacent, or equivalently, the subgraph of $GS(R)$ consisting of nonzero simple ideals is a null graph.

Definition 19. A dominating set X is non-shrinkable in $GS(R)$, if removing a vertex from X makes the new set a non-dominating set in $GS(R$.

Corollary 1. *[18] If R is semisimple, then*

$$\gamma(GS(R)) = \inf\{|Y| : Y \textit{ is a non-shrinkable dominating set consisting of the isolated vertices and semisimple ideals of } R \textit{ such that } R \in Y \textit{ or } Y \textit{ contains at least a nonzero simple ideal}\}.$$

If R is not semisimple, then

$$\gamma(GS(R)) = \inf\{|Y| : Y \textit{ is a non-shrinkable dominating set consisting of the isolated vertices, semisimple ideals of } R, \textit{ and at least one nonzero simple ideal}\}.$$

Example 1. [18] Let $R = I \oplus J$ be a semisimple ring. Then $Y = \{R\}$ is a non-shrinkable dominating set of $GS(R) : I \leftrightarrow R \leftrightarrow J$ with the least cardinality. Thus, $\gamma(GS(R)) = 1$.

Example 2. [18] Let $R = I \oplus J \oplus K$ be semisimple ring. Then $Y = \{I \oplus J, K\}$ is a non-shrinkable dominating set of $GS(R)$ with the least cardinality. Thus, $\gamma(GS(R)) = 2$.

Example 3. [18] Consider the ring $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}$. Then $Y = \{0 \oplus 0 \oplus 0 \oplus \mathbb{Z}, 0 \oplus 0 \oplus \mathbb{Z} \oplus 0, \mathbb{Z}_2 \oplus 0 \oplus 0 \oplus 0, 0 \oplus \mathbb{Z}_2 \oplus 0 \oplus 0\}$ is a non-shrinkable dominating set of $GS(R)$ with the least cardinality. Thus, $\gamma(GS(R)) = 4$.

Definition 20. [18] Let I be a nonzero simple ideal of R. We say that the nonzero ideals J and K are adjacent through I if $J \cap K = I$.

Proposition 1. *[18] Every clique of $GS(R)$ contains at most one nonzero simple ideal.*

It follows from Proposition 1 that there are two types of cliques in $GS(R)$. The first type of cliques contains no simple ideals, while the second type of cliques contains exactly one nonzero simple ideal. The next example exhibits these types of cliques.

Example 4. [18] Let $R = I \oplus J \oplus K$ be a semisimple ring. Then, the subgraph $I \oplus J \leftrightarrow J \oplus K \leftrightarrow I \oplus K \leftrightarrow I \oplus J$ is a clique whose vertices are not simple ideals. However, the subgraph $I \oplus J \leftrightarrow I \leftrightarrow I \oplus K \leftrightarrow I \oplus J$ is a clique with one simple nonzero ideal as Proposition 1 emphasizes.

In the next result, we are going to study each type of cliques in order to discover the clique number of $GS(R)$. The next theorem states that a clique containing a unique nonzero simple ideal I is a subgraph of $GS(R, I)$ whose vertices are adjacent through I.

Theorem 3. *[18] Let Λ be a clique containing one nonzero simple ideal I. Then Λ consists, beside the vertex I, vertices in $N(I)$ that are adjacent to each other through I.*

Corollary 2. *[18] Let Λ be a clique containing one nonzero simple ideal I. Then $|Ver[\Lambda]| > 2$ if and only if $R \notin Ver[\Lambda]$ and $Ver[\Lambda]$ has at least two proper non-simple ideals (that are adjacent through I).*

Definition 21. [18] Let I be a nonzero simple ideal of R. Then, the largest clique of $GS(R)$ containing I is called the maximal clique induced by I.

Let I be a nonzero simple ideal of R. Then the clique $I \leftrightarrow R$ is always a maximal clique induced by I, which we call the trivial maximal clique induced by I. It is not difficult to see from Corollary 2 that if $|N(I)| = 1$, then the trivial maximal clique induced by I is the only maximal clique induced by I. However, if $|N(I)| > 1$, then there is another maximal clique induced by I which consists, in addition to I, of all proper non-simple ideals in $N(I)$ that are adjacent to each other through I. We denote this non-trivial maximal clique by $\Lambda(I)$. Notice that $|Ver[\Lambda(I)]| \geq 2$.

Example 5. [18] In $GS(\mathbb{Z}_4)$, the maximal cliques induced by the ideal $2\mathbb{Z}_4$ are only the trivial maximal clique $2\mathbb{Z}_4 \leftrightarrow \mathbb{Z}_4$.

Example 6. [18] In Example 4, $\Lambda(I)$ is $I \leftrightarrow I \oplus J \leftrightarrow I \oplus K \leftrightarrow I$. Therefore $|\Lambda(I)| = 3$.

In the next result, we study the second type of cliques which does not contain nonzero simple ideals. The following definition will be handy.

Definition 22. [18] Let S be a nonempty set of nonzero simple ideals. By a clique of $GS(R)$ induced by S, we mean a clique of $GS(R)$ such that any two of its vertices are adjacent through a member of S, and every member of S is the intersection of two vertices of this clique.

Example 7. [18] In Example 4, the subgraph $\Lambda : I \oplus K \leftrightarrow I \oplus J \leftrightarrow J \oplus K \leftrightarrow I \oplus K$ is the unique clique induced by $S = \{I, J, K\}$. However, the subgraphs $\Lambda_1 : I \leftrightarrow R$, $\Lambda_2 : I \oplus K \leftrightarrow I$, $\Lambda_3 : I \leftrightarrow I \oplus K \leftrightarrow I \oplus J \leftrightarrow I$ are some cliques induced by $S = \{I\}$. On the other hand, there is no clique in $GS(R)$ induced by $S = \{I, J\}$.

As displayed in the previous example, the set of all cliques induced by a nonempty set S of nonzero simple ideals may be empty, singleton, or contain more than one clique.

Remark 2. [18] If S contains at least two nonzero simple ideals, then all vertices of a clique induced by S are non-simple ideals. We leave it to the reader to check out that the last statement is true.

Next, we show that if a nonempty set S of nonzero simple ideals induces cliques, then there is a maximum clique induced by S, i.e., a clique induced by S that is not a subgraph of another clique induced by S.

Theorem 4. *[18] Let S be a nonempty set of nonzero simple ideals of R which induces cliques in $GS(R)$. Then there is a maximal clique induced by S.*

Notation 5. [18] A maximal clique of $GS(R)$ induced by a nonempty set S of nonzero simple ideals of R is denoted by $\Lambda(S)$.

If $S = \{I\}$, where I is a nonzero simple ideal of R, then either $\Lambda(S)$ is the trivial maximal clique $R \leftrightarrow I$ or $\Lambda(S) = \Lambda(I)$. In general, a maximal clique of $GS(R)$ induced by S is not necessarily unique, as we shall see in the next example.

Example 8. [18] In Example 4, there is a unique maximal clique induced by $S = \{I, J, K\}$ given by $\Lambda(S) : I \oplus K \leftrightarrow I \oplus J \leftrightarrow J \oplus K \leftrightarrow I \oplus K$. Also, $\Lambda_1 : I \leftrightarrow R$, and $\Lambda_3 : I \leftrightarrow I \oplus K \leftrightarrow I \oplus J \leftrightarrow I$ are two maximal cliques induced by $S = \{I\}$. Notice that $\Lambda_2 : I \oplus K \leftrightarrow I$ is not a maximal clique induced by $S = \{I\}$ in $GS(R)$ since it is a subgraph of Λ_3.

Theorem 6. *[18] If Λ is a clique of $GS(R)$, then there is a unique nonempty set S of nonzero simple ideals of R inducing Λ.*

Remark 3. [18] In $GS(R)$, the following statements are true:

1. $\omega(GS(R))$ equals the supremum of the set

$$\{|Ver[\Lambda(S)]| : \ S \text{ is a non empty set of nonzero simple ideals of } R\}.$$

2. $\omega(GS(R)) \geq 2$.

3. If the order of $GS(R)$ is finite, then $2 \leq \omega(GS(R)) \leq |Ver[GS(R)]|$.

Example 9. [18] We have

1. $\omega(GS(\mathbb{Z}_4)) = 2$.

2. Let R be a semisimple ring with 2 components. A direct analysis leads to that $\omega(GS(R)) = 2$.

3. Let R be a semisimple ring with 3 components. A direct analysis leads to that $\omega(GS(R)) = 3$.

4. Consider $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, a semisimple ring with 4 components. Let $I = \mathbb{Z}_2 \oplus 0 \oplus 0 \oplus 0$. Then $Ver[\Lambda(I)] = \{I, \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus 0 \oplus 0, \mathbb{Z}_2 \oplus 0 \oplus \mathbb{Z}_2 \oplus 0, \mathbb{Z}_2 \oplus 0 \oplus 0 \oplus \mathbb{Z}_2\}$. Thus, $4 \leq \omega(GS(R)) \leq 15$, where 15 is the order of $GS(R)$ which is equal to the number of nonzero ideals of R. As a matter of fact, taking all possible cliques of R, it can be shown that the maximal clique with the largest number of vertices has an order of 4. That is $\omega(GS(R)) = 4$.

5. Let $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \ldots$ be a semisimple ring with infinitely many components. For every $n \in \mathbb{N}$, let $I_n = 0 \oplus 0 \oplus \ldots \oplus \mathbb{Z}_2 \oplus 0 \oplus \ldots$, where $\mathbb{Z}_2$ is located in the component number n. Then $Ver[\Lambda(I_1)] = \{I_1, I_1 \oplus I_2, I_1 \oplus I_3, \ldots\}$. Thus, $\omega(GS(R)) = \infty$.

For the rest of this paper, we consider a modified type of the simple- intersection graph of a ring R. The vertices in this special graph are the nonzero two-sided ideals of R, and two vertices are adjacent if they intersect at a nonzero simple two-sided ideal of R. We keep the same notation $GS(R)$ for this special simple-intersection graph of a ring.

Theorem 7. *[9] If R is a semisimple ring, then the two-sided ideals of R are precisely the direct sums of components of R. Besides, the nonzero simple two-sided ideals of R are precisely the components of R.*

It follows from the previous theorem that the special simple-intersection graph of a semisimple ring with finite number of components possesses a finite number of veritces and edges.

Theorem 8. *Let $R = I_1 \oplus \ldots \oplus I_n$ be a semisimple ring. Then*

1. $\deg(R) = n$.

2. $\deg(I_k) = 2^{n-1} - 1$, *where* $k = 1, \ldots, n$.

3. $\deg(J) = m \cdot 2^{n-m}$, *where* J *is an ideal of* R *with* m *components and* $2 \leq m \leq n$.

Proof. 1. See Remark 1.

2. Fix $k = 1, \ldots, n$. Since every ideal of R is a direct sum of components of R, and I_k is adjacent to every ideal with at least 2 components containing it, then $deg(I_k) = |N(I_k)| = C(n-1,1) + C(n-1,2) + \cdots + C(n-1,n-1) = 2^{n-1} - 1$. Notice that for every $m = 1, \ldots, n-1$, $C(n-1,m)$ represents the number of ideals with $m+1$ components, one of them is I_k and the rest are different from I_k.

3. Fix $m = 0, 1, \ldots, n$. Let J be an ideal of R with m components. Fix a component of J, and let us count the ideals I of R that are adjacent to J through this component. Since I contains this component and its other components are different from the m components of J, we obtain that, for each $t = 0, 1, \ldots, n-m$, there are $C(n-m,t)$ ideals I with $t+1$ components, one of them is the fixed component of J, while the remaining t components of I are not among the components of J. So, the number of ideals adjacent to J through the fixed component equals $C(n-m,0) + C(n-m,1) + \ldots + C(n-m,n-m) = 2^{n-m}$. Repeating the procedure for each component of J, yields $\deg(J) = m \cdot 2^{n-m}$.

□

Remark 4. The degree of any component of a semisimple ring is odd, while the degree of any proper ideal of a semisimple ring with at least two components is even. Therefore if R is a semisimple ring with at least two components then $GS(R)$ is not a regular graph. If R is not semisimple, then $GS(R)$ may be regular like $GS(\mathbb{Z}_4)$.

Theorem 9. *Let* R *be a semisimple ring with* n *components. Then*

1. *If* $n \geq 3$, *then* $GS(R)$ *has neither Euler circuits nor Euler paths.*

2. *If* $n = 2$, *then* $GS(R)$ *has only one Euler path of length* 2 *which connects* R *and its two components (up to the start vertex).*

Proof. The proof follows directly from combining Theorem 2 and Remark 4. □

4. Dominating sets and domination number

In this section, for a semisimple ring R with finite number of components, we discuss the dominating sets of $GS(R)$ and compute the domination number of $GS(R)$. We then apply the domination number to solve the coloring optimization problem: "If we have n different colors and we make all possible selections without replacement from these colors, what is the minimum number of possibilities we can take such that any other possibility has one color in common with at least one of these selected possibilities?".

Lemma 1. *Let R be a ring with unity $1 \neq 0$. Assume that $Soc(R)$ consists of at least two components. Let B be a dominating set of $GS(R)$ consisting of R, the isolated vertices, and nonzero semisimple ideals. Then there exists a dominating set $\check{B}$ such that $\check{B}$ consists of nonzero simple ideals, the isolated vertices, and nonzero semisimple ideals different from R; and $|\check{B}| \leq |B|$.*

Proof. We consider three cases. In the first case, assume each nonzero simple ideal of R is contained in a semisimple ideal in B different from R. Replace R in B with any nonzero simple ideal, call it I. Denote the new set by $\check{B}$. The set $\check{B}$ is a dominating set of $GS(R)$. To demonstrate the last statement, let V be a vertex outside $\check{B}$. If $V = R$, then $V \leftrightarrow I$. If $V \neq R$ is nonzero simple, by the assumption of the first case, $V \leftrightarrow U$, where U is a vertex containing V. If $V \neq R$ is a semisimple ideal that is not simple. Then $V \not\leftrightarrow R$. Again, by the assumption of the first case, $V \leftrightarrow U$ for some $U \in B - \{R\} \subset \check{B}$. It's obvious that $|\check{B}| \leq |B|$. In the second case, assume the existence of only one nonzero simple ideal I of R not contained in any vertex of $B - \{R\}$. Thus I is only adjacent to R. Now, as in the first case, replace R in B with I and call the new set $\check{B}$. A similar argument to that in the first case yields the domination of the set $\check{B}$. Also, notice that $|\check{B}| = |B|$. In the third case, assume there exist more than one nonzero simple ideal, for instance I and J, not contained in any vertex in $B - \{R\}$ (Of course, in this portion of the proof, we assume $Soc(R)$ contains at least 3 components). We show that this case is impossible to happen. Now, $I \oplus J \notin B$ and $I \oplus J$ is not adjacent to any vertex in B, which contradicts the domination of B. □

Theorem 10. *Assume $Soc(R)$ consists of at least two components. Then*

$$\gamma(GS(R)) = \inf\{|Y| : Y \text{ is an non-shrinkable dominating set consisting of the isolated vertices, semisimple ideals of } R, \text{ and at least a nonzero simple ideal of } R\}.$$

Proof. Apply Lemma 1. □

Lemma 2. *Let R be a semisimple ring with n components where $n > 1$. Then $\gamma(GS(R)) \leq n - 1$.*

Proof. Assume $R = I_1 \oplus \ldots I_n$ is a semisimple ring. If $n = 2$, then $B = \{R\}$ is a non-shrinkable dominating set of $GS(R)$ with the least cardinality. So, $\gamma(GS(R)) = 1$. If $n > 2$, let $B = \{I_1, I_2, \ldots, I_{n-2}, I_{n-1} \oplus I_n\}$. We have $|B| = n - 1$. Let V be a vertex out of B. If $V = R$ or V includes a component I_j where $j = 1, \ldots, n-2$, then $V \leftrightarrow I_j \in B$. If $V \neq R$ and V does not contain any of the components I_j, where $j = 1, \ldots, n-2$, then $V = I_{n-1}$, $V = I_n$, or $V = I_{n-1} \oplus I_n$. The third possibility of V is a vertex in B, while the first and second possibilities of V are adjacent to $I_{n-1} \oplus I_n$ which is a vertex in B. Therefore, we obtain that B is a non-shrinkable dominating set of $GS(R)$ with cardinality equal to $n - 1$. Consequently, we obtain that $\gamma(GS(R)) \leq n - 1$. □

Theorem 11. *Let R be a semisimple ring with n components where $n > 1$. Then $\gamma(GS(R)) = n - 1$.*

Proof. The proof is carried by the mathematical induction on n. For $n = 2$, $\{R\}$ is a non-shrinkable dominating set of $GS(R)$ with the least cardinality. So, $\gamma(GS(R)) = 1 = n - 1$. Assume $\gamma(GS(R)) = n - 1$ for any semisimple ring R with n components. Let R be a semisimple ring with $n+1$ components. By Lemma 2, we have $\gamma(GS(R)) \leq n$. Suppose $\gamma(GS(R)) < n$. Then there exists a non-shrinkable dominating set B such that $|B| = \gamma(GS(R)) < n$. By Lemma 1, we can assume, without lose of generality, that B contains a nonzero simple ideal I of R. Let V be a vertex outside B and $I \not\subset V$. Then $V \not\leftrightarrow I$. So, there exists a vertex $W \in B$ such that V is adjacent to W through a nonzero simple ideal $J \neq I$. Now, let Γ be the same set as B but with the component I is removed from each vertex of B containing it. Then $|\Gamma| < |B| < n$ which implies $|\Gamma| \leq n - 2$. Moreover, Γ is a dominating set for $GS(R')$ where R' is the semisimple ring with n components which has the same components of R except I. Since $|\Gamma| < n-1$, we obtain $\gamma(GS(R')) < n-1$ which is a contradiction to the hypothesis that states "the domination number of any semisimple ring of n components is equal to n". Thus, we obtain that $n \leq \gamma(GS(R))$ or equivalently $\gamma(GS(R)) = n$. □

Remark 5. If R is a semisimple ring with n components, then a typical dominating set of $GS(R)$ with $\gamma(GS(R)) = n - 1$ vertices is the set B mentioned in the proof of Lemma 2.

Remark 6. Theorem 11 provides a solution to the following coloring optimization problem: Given n different colors (where n is an even natural number) and the ability to select 1, 2, ..., or n colors without replacement and without order (resulting in $2^n - 1$ possibilities), what is the minimum number of possibilities such that any other possibility that is not among the ones chosen has one color in common with at least one of the ones chosen? The answer is $n - 1$ possibilities. Moreover, this set of possibilities contains $n - 2$ single colors in addition to the pair of the two remaining colors.

5. Clique Number

In this section, for a semisimple ring R with finite number of components, we study the cliques of $GS(R)$ and compute the clique number of $GS(R)$. We also apply the results of this section to solve many coloring optimization problems.

Theorem 12. *Let $R = I_1 \oplus \ldots \oplus I_n$ be a semisimple ring with n components, and I a component of R. Then $|\Lambda(I)| = n$.*

Proof. According to the paragraph after Definition 21, $\Lambda(I)$ consists of I and the nonzero ideals of R that are pairwise adjacent through I. Hence, $Ver[\Lambda(I)] = \{I, I \oplus I_t : \text{where } I_t \neq I\}$ has the maximum number of vertices. Obviously, $|Ver[\Lambda(I)]| = n$ □

Remark 7. Theorem 12 provides a solution to the following coloring problem: Given n different colors and the ability to select 1, 2, ..., or n colors without replacement and without order (resulting in $2^n - 1$ possibilities). What is the maximum number of possibilities in which any two pairs share only one common color and only one possibility has one color only? The answer, as stated by Theorem 12, is n.

Theorem 13. *Let $R = I_1 \oplus \ldots \oplus I_n$ be a semisimple ring, where $n \in \mathbb{N}$. If $n = 2$ or $n = 3$, then $\omega(GS(R)) = n$. In general, $n \leq \omega(GS(R)) \leq 2^n - 2$.*

Proof. If $n = 2$ or $n = 3$, then $\omega(GS(R)) = n$ by direct calculations. Let $n \geq 4$. By part 3 of Remark 3, we have $2 \leq \omega(GS(R)) \leq 2^n - 1$, where $|Ver[GS(R)]| = 2^n - 1$ is found using the combinatorial method similar to that in Theorem 8. Now, Fix $1 \leq k \leq n$. By Theorem 12, $|\Lambda(I_k)| = n$. Since the zero ideal and R cannot exist in any clique, we obtain that $n \leq \omega(GS(R)) \leq 2^n - 2$. □

Corollary 3. *Let R be a semisimple ring with infinite number of components. Then $\omega(GS(R)) = \infty$.*

Proof. Let n goes to ∞ in $n \leq \omega(GS(R)) \leq 2^n - 2$. □

Example 10. We have

1. $\omega(GS(\mathbb{Z}_4)) = 2$.
2. Consider $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, a semisimple ring with 4 components. Let $I = \mathbb{Z}_2 \oplus 0 \oplus 0 \oplus 0$. Then $Ver[\Lambda(I)] = \{I, \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus 0 \oplus 0, \mathbb{Z}_2 \oplus 0 \oplus \mathbb{Z}_2 \oplus 0, \mathbb{Z}_2 \oplus 0 \oplus 0 \oplus \mathbb{Z}_2\}$. Thus, $4 \leq \omega(GS(R)) \leq 15$, where 15 is the order of $GS(R)$ which is equal to the number of nonzero ideals of R. As a matter of fact, taking all possible cliques of R, it can be shown that the maximal clique with the largest number of vertices has order equal to 4. That is $\omega(GS(R)) = 4$.
3. Let $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \ldots$ be a semisimple ring with infinitely many components. For every $n \in \mathbb{N}$, let $I_n = 0 \oplus 0 \oplus \ldots \oplus \mathbb{Z}_2 \oplus 0 \oplus \ldots$, where $\mathbb{Z}_2$ is located in the component number n. Then $Ver[\Lambda(I_1)] = \{I_1, I_1 \oplus I_2, I_1 \oplus I_3, \ldots\}$. Thus, $\omega(GS(R)) = \infty$.

Our goal is to extend the first part of Theorem 13 ($\omega(GS(R)) = n$) to include any $n \geq 4$.

Notation 14. Let R be a semisimple ring with n components. The subgraph of $GS(R)$ consisting of all proper ideals of R with exactly k components where $1 \leq k \leq n-1$ is denoted by G_k.

Proposition 2. *Let R be a semisimple ring with n components, where $n \geq 4$. Then*

1. *if $k \leq \lceil \frac{n}{2} \rceil$, then G_k and G_{n-k} have the same order and there is a canonical bijection between the sets of vertices.*
2. *G_k and G_{n-k} are isomorphic if $k = 1$ or $k = n - k$ (i.e., n is even and $k = \frac{n}{2}$).*

Proof. 1. $|Ver[G_k]| = C(n,k) = C(n,n-k) = |Ver[G_{n-k}]|$. Define $f : G_k \longrightarrow G_{n-k}$ by $f(I) = \bar{I}$ where $I \oplus \bar{I} = R$. Notice that the components of I and the components of $\bar{I}$ do not match, i.e. placed in different orders relative to the order of the components of R. It is easy to see that f is a bijection between $Ver[G_k]$ and $Ver[G_{n-k}]$. But f is not necessarily a graph isomorphism as shown next.

2. Suppose $I \leftrightarrow J$ in G_k. Then I and J have one component in common and the remaining $k-1$ components do not match. Thus, the $k-1$ components of I appear in $f(J)$, the $k-1$ components of J appear in $f(I)$, and the remaining $(n-k)-(k-1) = n-2k+1$ components of $f(I)$ and $f(J)$ are the same. Now, we distinguish among three cases. In the first case, assume $n-2k+1 = 0$ (i.e., $k = \frac{n+1}{2}$ and n is odd, or equivalently $k = \lfloor \frac{n}{2} \rfloor + 1 = \lceil \frac{n}{2} \rceil$) then $f(I)$ is not adjacent to $f(J)$. In the second case, if $n-2k+1 = 1$ (or n is even and $k = \frac{n}{2} = \lceil \frac{n}{2} \rceil$) and hence $f(I) \leftrightarrow f(J)$ which implies $G_{\frac{n}{2}}$ is isomorphic to itself (unsurprising case). In the third case if $n-2k+1 > 1$ (or $k > \frac{n}{2}$, or equivalently, $k > \lceil \frac{n}{2} \rceil$), then $f(I)$ is not adjacent to $f(J)$. A final observation is the case where $k = 1$. We have both G_1 and G_{n-1} are null spaces and hence isomorphic subgraphs. □

It follows from the third case in the proof of Proposition 2 the following corollary.

Corollary 4. *Let R be a semisimple ring with n components, where $n \geq 4$. Then any two vertices with k components where $k > \lceil \frac{n}{2} \rceil$ are not adjacent. Contrapositively, if two vertices in $GS(R)$ are adjacent, then at most one of them has at most $\lceil \frac{n}{2} \rceil$ components.*

Now, let R be a semisimple ring with n components, where $n \geq 4$, and I be an ideal of R. By $\hat{I}$ we mean the ideal consisting of the components of I plus one component not in I. Let f be the component-alternating function defined in Proposition 2. Obviously, I and $f(I)$ are not adjacent. We have the proposition.

Proposition 3. *Let R be a semisimple ring with n components, and I an ideal of R containing $\lceil \frac{n}{2} \rceil$ components. Then*

1. *If n is even, then $I \leftrightarrow f(\hat{I})$ and $\hat{I} \leftrightarrow f(I)$.*
2. *If n is odd, then I is not adjacent to $f(\hat{I})$ and $\hat{I}$ is not adjacent to $f(I)$.*

Proof. 1. $\hat{I}$ and $f(I)$ have one component in common and therefore they are adjacent. The rest follows from the fact that $f \circ f = id_R$ where id_R is the identity function on R.

2. $\hat{I}$ and $f(I)$ have two components in common, Therefore they are not adjacent. The rest follows from the fact that $f \circ f = id_R$.

□

Remark 8. Let R be a semisimple ring with n components, where $n \geq 4$, and I and an ideal of R containing at least two components. Then I is not adjacent to each of $\hat{\hat{I}}$, $\hat{\hat{\hat{I}}}$, and so on.

Theorem 15. *Let R be a semisimple ring with n components, where n is odd and $n \geq 4$. Consider the clique $I \leftrightarrow J$ where I and J are two nonzero ideals with $\lceil \frac{n}{2} \rceil = \dfrac{n+1}{2}$ components. Then*

1. *the clique $I \leftrightarrow J$ cannot be extended to a larger clique by adding a vertex with at least 3 components.*
2. *Any maximal clique containing the clique $I \leftrightarrow J$ has an order no more than 5.*

Proof.

1. Let I and J be two nonzero ideals with $\lceil \frac{n}{2} \rceil = \dfrac{n+1}{2}$ components such that $I \leftrightarrow J$. Then I and J have one component in common and the rest of the components are different from each other. Since $I + J = R$, by pigeonhole principle any vertex with at least 3 components must contain at least two components in common with I or at least two components in common with J. So, we can't extend the clique $I \leftrightarrow J$ by adding a vertex of at least 3 components.
2. By part 1, a third vertex V that extends the clique $I \leftrightarrow J$ into a clique of order 3 must has at most 2 components. If V is simple, then V is the common component between I and J. Thus the new clique $I \leftrightarrow J \leftrightarrow V \leftrightarrow I$ is maximal and hence has an order of 3. If $V = K_1 \oplus K_2$ where K_1 is a component of I and K_2 is a component of J and $K_1 \neq K_2$. Then the new clique can be extended by adding vertices with exactly two components. Since any clique cannot contain more than 3 ideals with exactly two components, we can add at most three vertices, with exactly two components, to the clique $I \leftrightarrow J$. Thus the resulting maximum clique has order not exceeding 5.

□

Remark 9. Theorem 15 provides a solution to the following coloring problem: Given n different colors (where n is an odd natural number) and the ability to select 1, 2, ..., or n colors without replacement and without order (resulting in $2^n - 1$ possibilities), what is the maximum number of possibilities in which any two pairs share only one common color and at least two possibilities have $\dfrac{n+1}{2}$ colors? The answer, as stated by Theorem 15, is 5.

Theorem 16. *Let R be a semisimple ring with n components, where n is even and $n \geq 4$. Then a maximal clique containing $I \leftrightarrow J$ where each of I and J consists of $\lceil \frac{n}{2} \rceil = \frac{n}{2}$ components has an order no more than $\lceil \frac{n}{2} \rceil + 1$.*

Proof. A similar argument to that of the proof of Theorem 15 shows that any extension of the clique $I \leftrightarrow J$ by vertices with two components does not exceed 5

vertices, if the number of components allows such a procedure. Also, any extension of the clique $I \leftrightarrow J$ containing the common component K between I and J does not exceed 3 vertices. Let T be the component that does not appear in $I+J$. If the clique $I \leftrightarrow J$ is extended to a larger clique using a vertex with three components, then this vertex must contain the component T, or else by the pegionhole principle this vertex contains at least two components of I or two components of J. Now, the first vertex V_1 with three components to be added to the clique $I \leftrightarrow J$ consists, in addition to T, of a component of $I' = (I - K) \cup 0$ and a component of $J' = (J - K) \cup 0$. The second vertex V_2 with three components (one of them is T) to be added to the clique consists of a component of I'' and a component of J'', where I'' and J'' are obtained from I' and J' by deleting the components of V_1, respectively. Continuing in this procedure the clique $I \leftrightarrow J$ is extended to a maximal clique of order $\lceil \frac{n}{2} \rceil + 1$ (by adding $\lceil \frac{n}{2} \rceil - 1$ vertices, each of which has three components, to the clique $I \leftrightarrow J$. In the final case we extend the clique $I \leftrightarrow J$ by adding vertices with two components or with three components We begin with adding a vertex with two components L from I' and N from J', and then a vertex with three components consisting of T, L, and a component taken from $(J' - N) \cup 0)$. The last clique of order 4 cannot be extended to a larger clique with more than 5 components. The reason behind the last statement, assuming the number of components allows us to continue the extension, is a new $2-$component vertex does not have enough components to be adjacent to the existed 4 components. Also, adding two vertices each of which consists of three components such that one of them is T to the clique with the four vertices does not produce a clique. Therefore in this way the maximal clique has an order of 5. Recalling that any extension to the clique $I \leftrightarrow J$ can't contain a vertex with more than 3 components (otherwise, the new extension will not be a clique) ends our proof. □

Remark 10. Theorem 16 provides a solution to the following coloring optimization problem: Given n different colors (where n is an even natural number) and the ability to select 1, 2, ..., or n colors without replacement and without order (resulting in $2^n - 1$ possibilities), what is the maximum number of possibilities such that any pair of them shares only one common color and at least two possibilities have $\dfrac{n}{2}$ colors? The answer, as stated by Theorem 16, is $\lceil \frac{n}{2} \rceil + 1$.

Theorem 17. *Let R be a semisimple ring with n components, where n is odd and $n \geq 4$. Then a maximal clique containing $I \leftrightarrow J$ where I consists of $\lceil \frac{n}{2} \rceil - 1 = \frac{n-1}{2}$ components and J consists of $\lceil \frac{n}{2} \rceil = \frac{n+1}{2}$ components has an order no more than $\lceil \frac{n}{2} \rceil + 1$.*

Proof. The proof is similar to the proof of Theorem 16. □

Remark 11. Theorem 17 provides a solution to the following coloring problem: Given n different colors (where n is an odd natural number) and the ability to select 1, 2, ..., or n colors without replacement and without order (resulting in $2^n - 1$ possibilities), what is the maximum number of possibilities in which any two pairs share only one common color,

at least one possibility has $\frac{n-1}{2}$ colors and at least one possibility has $\frac{n+1}{2}$ colors? The answer, as stated by Theorem 17, is $\lceil \frac{n}{2} \rceil + 1$.

Remark 12. Let R be a semisimple ring with at least 3 components. A clique $I \leftrightarrow J$, where $I = I_1 \oplus T$ and $J = T \oplus J_1$ consisting of two components can be extended to two types of maximal cliques. The first type is of order 3 and having the form $I_1 \oplus T \leftrightarrow T \oplus J_1 \leftrightarrow I_1 \oplus J_1$. The second type is $\Lambda(I_1)$ which is of order n.

The following remark will be important in the next work.

Remark 13. Let I be a component of a semisimple ring R. Then a clique Λ_I, whose vertices are adjacent through I, is a subclique of $\Lambda(I)$ obtained by making sums among the vertices, leaving vertices untouched, or excluding vertices. To illustrate this, consider for example a semisimple ring $R = I \oplus J \oplus K \oplus L$. We have $\Lambda(I) : I,\ I \oplus J,\ I \oplus K,\ I \oplus L$. Now $\Lambda_I : I,\ I \oplus J,\ I \oplus K \oplus L$ is obtained from $\Lambda(I)$ by adding $I \oplus K$ and $I \oplus L$ and leaving the other vertices untouched. While $\Lambda_I : I,\ I \oplus J,\ I \oplus K$ is obtained from $\Lambda(I)$ by excluding the vertex $I \oplus L$. Also, $\Lambda_I : I \oplus J,\ I \oplus K \oplus L$ is obtained by adding $I \oplus K$ and $I \oplus L$, excluding I, and leaving $I \oplus J$ untouched.

In what follows, if R is a semisimple ring with n components and $S = \{I_1, \ldots, I_m\}$ of components of R, where $1 \leq m \leq n$, then we denote $\Lambda(S)$ by $\Lambda(I_1, \ldots, I_m)$. In addition, for every $t = 1, \ldots, m$, Λ_{I_t} denotes the subclique of $\Lambda(I_1, \ldots, I_m)$ whose vertices are adjacent through I_t (or equivalently, containing I_t).

Lemma 3. *Let R be a semisimple ring with n components and $S = \{I_1, \ldots, I_m\}$ a set of components of R, where $1 \leq m \leq n$. Then any component of R outside S cannot exist in two adjacent vertices of $\Lambda(S)$.*

Proof. Let J be a component of R such that $J \notin S$. Let V and U be adjacent vertices of $\Lambda(S)$ containing J. Then, V and U are adjacent through J. Thus $J \in S$ which is a contradiction. □

Lemma 4. *Let R be a semisimple ring with n components and $S = \{I_1, \ldots, I_m\}$ a set of components of R, where $1 \leq m \leq n$. If I and J are different components of R lying in S, then $|\Lambda_I \cap \Lambda_J| \leq 1$ (i.e. Λ_I and Λ_J meet at at most one vertex).*

Proof. Assume $\Lambda_I \cap \Lambda_J \neq \emptyset$. Let V and U be two common vertices between Λ_I and Λ_J Then I and J exist in both V and U, which implies U and V cannot be adjacent, and that contradicts that $\Lambda(S)$ is a clique. □

Let R be a semisimple ring with n components and $S = I_1, \ldots, I_m$ a set of components of R, where $1 \leq m \leq n$. Without loss of generality, assume that $\Lambda(S)$ includes a vertex T of the form $T = I_1 \oplus \ldots \oplus I_k$, where $1 \leq k \leq m$. By Lemma 4, T is the unique vertex inside $\Lambda_{I_1} \cap \ldots \cap \Lambda_{I_k}$. Suppose $|\Lambda_{I_t}| \geq 2$, for every $1 \leq t \leq k$. Again, without loss of generality, suppose Λ_{I_1} contains a vertex $\Delta = I_1 \oplus J_1 \oplus \ldots \oplus J_l$ different from

T with the least number $(l+1)$ of components. Notice that the components of Δ other than I_1 are all different from I_1, I_2, ..., and I_k. That is they are selected from the remaining $n-k$ components of R. We have the following lemma, which we call the minimum lemma.

Lemma 5. *For every $2 \leq t \leq k$, the order of Λ_{I_t} is at most $l+1$.*

Proof. We keep in mind that all vertices considered during the proof exist in the clique $\Lambda = \Lambda(S)$. Let V_1 be a vertex in Λ_{I_t} different from T. Since $V_1 \leftrightarrow \Delta$, then V_1 contains, beside I_t, exactly one component of Δ different from I_1, say J_1. If V_2 is another vertex in Λ_{I_t} different from T and V_1, then V_2 contains, in addition to I_t, exactly one component of Δ different from I_1, I_t, and J_1, , say J_2. A third vertex V_3, if exists in Λ_{I_t}, must contain, in addition to I_t, exactly one component of Δ which is different from I_1, J_1, and J_2, say J_3, and so on. Any vertex of Λ_{I_t} must contain, in addition to I_t, one component of Δ different from I_1 that does not exist in another vertex in Λ_{I_t} different from T. Consequently, the maximum number of vertices that we can add to Λ_{I_t}, beside T, is l vertices (any extra vertex added to the $l+1$ vertices of Λ_{I_t} must contain I_t and one component of Δ which both exist in one of the vertices V_1, V_2, ..., or V_l. This implies Λ_{I_t} is not a clique). □

Now, we are ready to show that the clique number of the simple-intersection graph of a semisimple ring with n components is equal to n. In the following theorem, we shall stop using the floor and ceiling functions in order to shorten the proof and skip discussing different cases which are clear by appropriate approximations to the nearest integer.

Theorem 18. *Let R be a semisimple ring with n components ($n \geq 4$). Then $\omega(GS(R)) = n$.*

Proof. Let Λ be a maximum clique of $GS(R)$. Then there exists a set $S = \{I_1, \ldots, I_m\}$ of components of R, where $1 \leq m \leq n$ such that $\Lambda = \Lambda(S)$. Assume $m \geq 3$. We consider three cases.

Case 1. Each vertex of Λ contains at least one component outside S. By Lemma 3, the maximum number of vertices can Λ possess is $n-m$ which is less than n.

Case 2. There exists a vertex $T = I_1 \oplus \ldots \oplus I_k$, where $1 \leq k \leq m$, in Λ containing no components outside S, and T is the only vertex in Λ_{I_t}, for some $1 \leq t \leq k$. Since, T is adjacent to every another vertex in Λ, we obtain $k = m$ and T is the unique vertex common among Λ_{I_1}, Λ_{I_2}, ..., and Λ_{I_m}. So the other vertices of Λ different from T contain components outside S. By Lemma 3, the maximal number of vertices that Λ may contain is $n-m+1$ which does not exceed n.

Case 3. There exists a vertex $T = I_1 \oplus \ldots \oplus I_k$, where $1 \leq k \leq m$, in Λ containing no components outside S, and T is not the only vertex in Λ_{I_t}, for every $1 \leq t \leq k$. Notice that by Corollary 4 we have $k \leq \lceil \frac{n}{2} \rceil$. In Λ_{I_1}, let $\Delta = I_1 \oplus J_1 \oplus \ldots \oplus J_l$ be

a vertex different from T and contain the least number of components. Since Λ is a clique, every vertex in Λ_{I_t}, where $k+1 \leq t \leq m$ exists in some Λ_{I_t}, where $1 \leq t \leq k$. Thus, we get that $\Lambda = \Lambda_{I_1} \cup \ldots \cup \Lambda_{I_k}$. By Lemma 5, the number of vertices in each Λ_{I_t}, where $t = 2, \ldots, k$, does not exceed $l+1$. Since were are searching for the largest value among the orders of the maximal cliques, without loss of generality, let's take the extreme case where $|Ver[\Lambda_{I_2}]| = \ldots = |Ver[\Lambda_{I_k}]| = l+1$.
On the other hand, since Δ has the minimum number of components in Λ_{I_1}, and all other vertices of Λ_{I_1} must contain components different from those of $T+\Delta$ (i.e. components selected from the remaining $n-(k+l)$ components of R) such that no overlapping among the components of these vertices (except for I_1) occurs. Therefore the maximum number of vertices that Λ_{I_1} may contain is $\left\lfloor \frac{n-(k+l)}{l} \right\rfloor$ (after the division, the remaining components can be distributed on the vertices of Λ_{I_1} which are different from T and Δ). Consequently, noticing that T is common among Λ_{I_1}, $\Lambda_{I_2}, \ldots,$ and Λ_{I_k}, the maximum number of vertices that Λ may contain has an upper bound given by $\left\lfloor \frac{n-(k+l)}{l} \right\rfloor + (k-1)l + 2$, where the number 2 counts T once and Δ. Now

$$\left\lfloor \frac{n-(k+l)}{l} \right\rfloor + (k-1)l + 2 \leq \frac{n-(k+l)}{l} + (k-1)l + 2 = \frac{n-k}{l} + (k-1)l + 1 = \varphi(k,l).$$

where $1 \leq k \leq m$ and $1 \leq l \leq m-k$. Notice that the graph with $\varphi(k,l)$ vertices obtained above is not necessarily a clique. If it is a clique then it's equal to Λ because Λ is a maximum clique. Next, we use calculus to show that the upper bound function φ over the triangular region E given by $1 \leq k \leq m$ and $1 \leq l \leq m-k$ does not exceed n. Since $k, l \geq 1$, The region E can be written as $1 \leq k \leq m-1$ and $1 \leq l \leq m-k$. The solution of the system $\frac{\partial \varphi}{\partial k} = l - \frac{1}{l} = 0$ and $\frac{\partial \varphi}{\partial l} = k - 1 - \frac{n-k}{l^2} = 0$ is the critical point (k,l) where $k = \frac{n+1}{2}$ and $l = 1$. This critical point may and may not lie inside the triangle region E. Fortunately, this does not matter because $\varphi(\frac{n+1}{2}, 1) = n$. For the boundary $l = 1$ of E, $\varphi(k,1) = n$ for every $1 \leq k \leq m$. For the boundary $l = m-k$ of E, the function $\varphi(k, m-k) = \frac{n-k}{m-k} + (k-1)(m-k) + 1$ is increasing over the interval $[1, m-1]$ which takes its maximum at $k = m-1$ and the maximum value is n. For the boundary $k = 1$ of E, we have $\varphi(1,l) = \frac{n-1}{l} + 1$ which has the maximum value n when $l = 1$. Consequently, the maximum value of φ over E is n. By Theorem 12, the maximal clique induced by a nonzero simple ideal has n vertices, we conclude that $\omega(GS(R)) = n$. □

Remark 14. Theorem 18 provides a solution to the following coloring problem: Given n different colors and the ability to select 1, 2, ..., or n colors without replacement and without order (resulting in $2^n - 1$ possibilities). What is the maximum number of possibilities in which any two pairs share only one common color? The answer, as stated by Theorem 18, is n.

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