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Research Article

k-Secure Sets and k-Security Number of a Graph

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Abstract: Let $G = (V, E)$ be a simple connected graph. A nonempty set $S \subseteq V$ is a secure set if every attack on S is defendable. In this paper, k-secure sets are introduced as a generalization of secure sets. For any integer $k \geq 0$, a nonempty subset S of V is a k-secure set if, for each attack on S, there is a defense of S such that for every $v \in S$, the defending set of v contains at least k more elements than that of the attacking set of v, whenever the vertex v has neighbors outside S. The cardinality of a minimum k-secure set in G is the k-security number of G. Some properties of k-secure sets are discussed and a characterization of k-secure sets is obtained. Also, 1-security numbers of certain classes of graphs are determined.

Keywords: secure sets, alliances, security number, k-secure sets.

AMS Subject classification: 05C69, 05C70, 05C76.

1. Introduction

Throughout the article, $G = (V, E)$ is a simple connected graph with vertex set V and edge set E. For any $v \in V$, the set $N(v) = \{w \in V : vw \in E\}$ is the open neighborhood of v and $N[v] = N(v) \cup \{v\}$ is the closed neighborhood of v. Let $S \subseteq V$. The sets $N(S) = \cup_{s \in S} N(s)$ and $N[S] = \cup_{s \in S} N[s]$ are called the open neighborhood and the closed neighborhood of S respectively. The set $\partial S = N[S] - S$ is called the boundary of S. The subgraph of G induced by S is denoted by $\langle S \rangle$. For a graph G, $\Delta(G)$ denotes the degree of a vertex having the maximum degree in G. The basic graph theory terminologies used in the article are from [2, 17].

The concept of alliances in graphs was introduced by P. Kristiansen et al. in [12]. A nonempty subset of the vertex set is an alliance. For any vertex v in an alliance,

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a neighbor of v which lies inside the alliance is a defender of v whereas any neighbor lying outside the alliance is an attacker of v. Further, every vertex in the alliance is a defender of itself. A defensive alliance is an alliance in which every vertex in it has at least as many defenders as attackers. Till date, several varieties of alliances have been introduced and studied. The related works can be found in [5, 8, 15, 16]. As a generalization of defensive alliances, R.C. Brigham et al. introduced the concept of secure sets in [1]. More results on secure sets can be found in [1, 3, 4, 9].

Let $S \subseteq V$ be any nonempty set. A collection of mutually disjoint sets of attackers of vertices in S is called an attack [1] on S and that of defenders is called a defense of S. An attack on S is defendable if there is a defense of S such that for every vertex in S, the corresponding set of attackers in the attack has fewer elements than that of defenders in the defense. The set S is a secure set [1] if every attack on S is defendable. Since no attack on V exists, V is always a secure set. The set $\{u \in S : uv \in E \text{ for some } v \in V - S\}$ is the border of S, denoted by $Bord(S)$. The set $Int(S) = S - Bord(S)$ is called the interior of S. An attack and a defense can be re-defined by viewing them as functions.

Let $\emptyset \subsetneqq S \subsetneqq V$. An attack on S is a function $A : \partial S \to Bord(S)$ such that x and $A(x)$ are adjacent for every $x \in \partial S$. A function $D : N[Bord(S)] \cap S \to Bord(S)$ is a defense of S if y and $D(y)$ are equal or adjacent for every $y \in N[Bord(S)] \cap S$. An attack A on S is defendable if there exists a defense D of S such that $|D^{-1}(z)| \geq |A^{-1}(z)|$ for all $z \in Bord(S)$. In this case, D is said to be a successful defense against A. The set S is a secure set if every attack on S is defendable. By considering vertex set V to be secure, the above definition of secure sets coincides with that of [1]. A comprehensive demonstration of their equivalence is available in [11].

In many practical situations, it is required to have a stronger defense than each attack. In this direction, as an extension of defensive alliances, defensive k-alliances are introduced in [16] and their properties are discussed in [15]. For any integer $k \in \{-\Delta(G), \ldots, \Delta(G)\}$, a nonempty set $S \subseteq V$ is a defensive k-alliance [16] in G whenever $|N[x] \cap S| - |N[x] - S| \geq k$ for all $x \in S$. In this paper, k-secure sets are introduced analogous to defensive k-alliances.

2. k-Secure Sets

In this section, k-secure sets are defined and some of their properties are discussed.

Definition 1. For any integer k with $-\Delta(G) < k < \Delta(G)$, a set $S \subseteq V$ is a k-secure set if for any attack A on S, there exists a defense D of S with $|D^{-1}(z)| - |A^{-1}(z)| \geq k$ for all $z \in Bord(S)$.

The minimum cardinality of a k-secure set in G is the k-security number of G, denoted by $s_k(G)$. In this paper, only the case $k \geq 0$ is considered. A 0-secure set is the same as a secure set defined in [1]. The following theorem gives a characterization of secure sets.

Theorem 1. *[1] A set $S \subseteq V$ is a secure set if and only if $|N[X] \cap S| \geq |N[X] - S|$ for every $X \subseteq S$.*

Analogous to Theorem 1, a characterization of k-secure sets can be obtained. Now we recall the theorem due to P. Hall from [6, 7, 14].

Theorem 2. *Suppose $A_1, A_2, \ldots, A_n$ are sets and $a_1, a_2, \ldots, a_n$ are non negative integers. There exist pairwise disjoint sets $B_1, B_2, \ldots, B_n$ such that $B_i \subseteq A_i$, $|B_i| = a_i$ for all i with $1 \leq i \leq n$ if and only if for any $I \subseteq \{1, 2, \ldots, n\}$, $|\bigcup\limits_{i \in I} A_i| \geq \sum\limits_{i \in I} a_i$.*

Remark 1. For any attack A and defense D of S, $A^{-1}(x) \cap A^{-1}(y) = \emptyset$ and $D^{-1}(x) \cap D^{-1}(y) = \emptyset$ for all $x, y \in Bord(S)$ with $x \neq y$.

The following theorem characterizes k-secure sets.

Theorem 3. *For any integer $k \geq 0$, a nonempty set $S \subseteq V$ is a k-secure set if and only if $|N[X] \cap S| \geq |N[X] - S| + k|X|$ for every $X \subseteq Bord(S)$.*

Proof. Let S be a k-secure set and $A : \partial S \to Bord(S)$ be any attack on S. Then there is a defense D such that $|D^{-1}(z)| - |A^{-1}(z)| \geq k$ for all $z \in Bord(S)$. By Remark 1, for any $X \subseteq Bord(S)$,

$$\begin{aligned}
|N[X] \cap S| &\geq |D^{-1}(X)| \\
&= |\bigcup_{x \in X} D^{-1}(x)| \\
&= \sum_{x \in X} |D^{-1}(x)| \\
&\geq \sum_{x \in X} \Big(|A^{-1}(x)| + k\Big) \\
&= k|X| + \sum_{x \in X} |A^{-1}(x)| \\
&= k|X| + |\bigcup_{x \in X} A^{-1}(x)|.
\end{aligned}$$

Note that there exists an attack A_1 such that $\bigcup\limits_{x \in X} A_1^{-1}(x) = N[X] - S$, which can be obtained by mapping every vertex of $N[X] - S$ to any one of its neighbors lying in X. Thus $|N[X] \cap S| \geq k|X| + |N[X] - S|$.

Conversely suppose that $|N[X] \cap S| - |N[X] - S| \geq k|X|$ for every $X \subseteq Bord(S)$. Let $P_x = N[x] \cap S$ for all $x \in Bord(S)$. Then $\bigcup\limits_{x \in Bord(S)} P_x = N[Bord(S)] \cap S$. Let A be any attack on S. Let $|A^{-1}(x)| = a_x$ for every $x \in Bord(S)$. Then by Remark 1, for any $X \subseteq Bord(S)$,

$$\sum_{x \in X} a_x = \sum_{x \in X} |A^{-1}(x)| = |\bigcup_{x \in X} A^{-1}(x)| \leq |N[X] - S|.$$

Let $a_x + k = k_x$ for all $x \in Bord(S)$. Then

$$\sum_{x\in X} k_x = \sum_{x\in X} a_x + \sum_{x\in X} k \leq |N[X]-S| + k|X| \leq |N[X]\cap S| = |\bigcup_{x\in X}\Big(N[x]\cap S\Big)| = |\bigcup_{x\in X} P_x|.$$

Then by Theorem 2, there exist pairwise disjoint sets $D_x \subseteq P_x$ such that

$$|D_x| = k_x = a_x + k = |A^{-1}(x)| + k \quad \text{for all } x \in Bord(S).$$

Now define $D : N[Bord(S)] \cap S \to Bord(S)$ as follows. Let $y \in N[Bord(S)] \cap S$ be arbitrary. If $y \in D_x$ for some $x \in Bord(S)$, then define $D(y) = x$. Otherwise, define $D(y)$ to be any neighbor of y lying in $Bord(S)$. Then D is a defense of S with $|D^{-1}(x)| \geq |D_x| = k_x = |A^{-1}(x)| + k$ for all $x \in Bord(S)$. Therefore S is a k-secure set. □

Corollary 1. *For any $k \geq 1$ and a k-secure set S, $Bord(S) \subsetneq S$.*

Proof. If $Bord(S) = S$, then by Theorem 3, $|S| = |N[S] \cap S| \geq |N[S] - S| + k|S|$. Then $0 \geq |N[S] - S| + (k-1)|S|$ and hence for $k > 1$, $|S| = 0$ which is impossible. If $k = 1$, then $N[S] - S = \emptyset$ and hence $S = V$. Then $V = S = Bord(S) = Bord(V) = \emptyset$ which is impossible. □

The case $k = 0$ in Theorem 3 gives the following characterization of secure sets, which is also a sharpening of Theorem 1.

Theorem 4. *A set $S \subseteq V$ is a secure set if and only if $|N[X] \cap S| \geq |N[X] - S|$ for every $X \subseteq Bord(S)$.*

Proposition 1. *For any $k \geq 0$, if S is a minimal k-secure set, then $\langle S \rangle$ is connected.*

Proof. Suppose that $\langle S \rangle$ is not connected. Then vertex set of any component of $\langle S \rangle$ is a k-secure set, which contradicts the minimality of S. □

Theorem 5. *For $k \geq 0$, if S_1 and S_2 are k-secure sets with $S_1 \cap S_2 = \emptyset$, then so is $S_1 \cup S_2$.*

Proof. Let $X \subseteq S_1 \cup S_2$ and $X_i = X \cap S_i$ for $i \in \{1, 2\}$. Note that

$$N[X] \cap (S_1 \cup S_2) \supseteq (N[X_1] \cap S_1) \cup (N[X_2] \cap S_2)$$

and

$$N[X] - (S_1 \cup S_2) \subseteq (N[X_1] - S_1) \cup (N[X_2] - S_2).$$

Thus,

$$\begin{aligned}|N[X] \cap (S_1 \cup S_2)| &\geq |N[X_1] \cap S_1| + |N[X_2] \cap S_2| \\ &\geq |N[X_1] - S_1| + |N[X_2] - S_2| + k|X_1| + k|X_2| \\ &\geq |N[X] - (S_1 \cup S_2)| + k|X|.\end{aligned}$$

Then by Theorem 3, $S_1 \cup S_2$ is a k-secure set. □

3. 1-Secure Sets and Fractional Secure Sets

In [10], G. Isaak et al. introduced 'Fractional Secure Sets' as a variant of secure sets in graphs. Furthermore, two types of attacks and defenses were defined by considering them as functions. As a result, four types of secure sets were defined. One of the four forms of secure sets coincides with the secure sets defined in [1], while another two of them are equivalent to it.

Definition 2. [10] Let $S \subseteq V$. An attack on S is a function $A : (V - S) \times S \to [0, 1]$ such that $A(u, v) = 0$ if $uv \notin E$ and $\sum_{v \in N(u)-S} A(u, v) \leq 1 \;\forall\; u \in V - S$. A defense of S is a function $D : S \times S \to [0, 1]$ such that for any $u, v \in S$, $\sum_{v \in N[u] \cap S} D(u, v) \leq 1$ and $D(u, v) = 0$ whenever $uv \notin E$. For any $v \in S$, denote $\sum_{u \in V-S} A(u, v) = A^*(v)$ and $\sum_{u \in S} D(v, u) = D^*(v)$.

An attack A on S is said to defendable with a defense D if $A^*(v) \leq D^*(v)$ for all $v \in S$. An attack A (defense D) is said to be an integer attack(defense) if the range of attack A (defense D) is $\{0, 1\}$.

Definition 3. [10] A nonempty set $S \subseteq V$ is said to be an

1. (I, I)-secure set if every integer attack on S is defendable with an integer defense.
2. (I, F)-secure set if every integer attack on S is defendable with a defense.
3. (F, F)-secure set if every attack on S is defendable with a defense.
4. (F, I)-secure set if every attack on S is defendable with an integer defense.

The definition of an (I, I)-secure set coincides with that of a secure set given in [1]. Further it has been proved that (I, I)-security, (F, F)-security and (I, F)-security are equivalent. Every (F, I)-secure set is an (I, I)-secure set. But an (I, I)-secure set need not be an (F, I)-secure set. For any $X \subseteq S \subseteq V$, let $E[X, N[X] - S]$ be the set of edges of G between X and $N[X] - S$. Let G^X be the subgraph of G with vertex set $X \cup (N[X] - S)$ and edge set $E[X, N[X] - S]$. We state the following results for immediate reference.

Theorem 6. *[13] A set S is an (F, I)-secure set if and only if for every $X \subseteq S$, $|N[X] \cap S| \geq |N[X] - S| + |X| - c(G^X)$ where $c(G^X)$ denote the number of components of G^X.*

A set S is an ultra secure set [13] if there exists an integer defense of S which is successful against any integer attack on S.

Theorem 7. *[13] A set S is ultra secure if and only if $|N[X] \cap S| \geq \sum_{x \in X} |N[x] - S|$ for every $X \subseteq S$.*

Theorem 8. *[13] Every ultra secure set is an (F, I)-secure set.*

The following lemma is useful to obtain a characterization of (F, I)-secure sets, which is a sharpening of Theorem 6.

Lemma 1. *Let $X \subseteq S \subseteq V$. Then $|X| - c(G^X) = |X \cap Bord(S)| - c(G^{X \cap Bord(S)})$.*

Proof. Since $S = Bord(S) \cup Int(S)$, $X = \Big(X \cap Bord(S)\Big) \bigcup \Big(X \cap Int(S)\Big)$ for any $X \subseteq S$. Let $Y = X \cap Bord(S)$ and $Z = X \cap Int(S)$. Each vertex of Z is a component of G^X. Thus $c(G^X) = c(G^Y) + |Z|$. Since $Bord(S) \cap Int(S) = \emptyset$, $|X| = |Y| + |Z|$. Thus $|X| - c(G^X) = |Y| + |Z| - c(G^Y) - |Z| = |Y| - c(G^Y)$ which completes the proof. □

The following theorem gives a characterization of an (F, I)-secure set.

Theorem 9. *A set S is an (F, I)-secure set if and only if for every $X \subseteq Bord(S)$, $|N[X] \cap S| \geq |N[X] - S| + |X| - c(G^X)$.*

Proof. Follows by Lemma 1 and Theorem 6. □

Theorem 10. *Every 1-secure set is an (F, I)-secure set.*

Proof. Let S be a 1-secure set and $X \subseteq Bord(S)$. Then by Theorem 3,

$$|N[X] \cap S| - |N[X] - S| \geq |X| > |X| - c(G^X).$$

Then by Theorem 9, S is an (F, I)-secure set. □

Remark 2. By Theorem 10, every 1-secure set is (F, I)-secure. By Theorem 8, every ultra secure set is also (F, I)-secure. However, none of these two types imply the other in general.

Example 1. In the graph G of Figure 1, consider the set $S = \{b_1, b_2, b_3, i_1, i_2, ..., i_6\}$ which is 1-secure. Suppose there is a defense $D : \{i_1, i_2, ..., i_6\} \to \{b_1, b_2, b_3\}$ which defends

S against any attack. Without loss of generality, D can be assumed to be a maximal defense. Consider the attacks A_1 and A_2 given by

$$A_1(a_1) = A_1(a_2) = A_1(a_3) = b_2,$$
$$A_2(a_1) = A_2(a_2) = A_2(a_3) = b_3.$$

Since D is successful against A_1, $|D^{-1}(b_2)| \geq 3$. Also since D is successful against A_2, $|D^{-1}(b_3)| \geq 3$. Thus $|D^{-1}(b_2)| + |D^{-1}(b_3)| \geq 6$ which is impossible. Because, $|D^{-1}(b_2)| + |D^{-1}(b_3)| = |D^{-1}(\{b_1, b_2\})| \leq |N[\{b_2, b_3\}] \cap S| = 5$. Thus no defense is successful against every attack. Therefore S is not an ultra secure set.

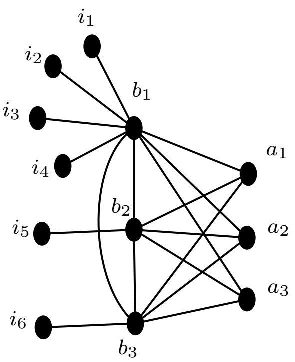

Figure 1. **Graph** G

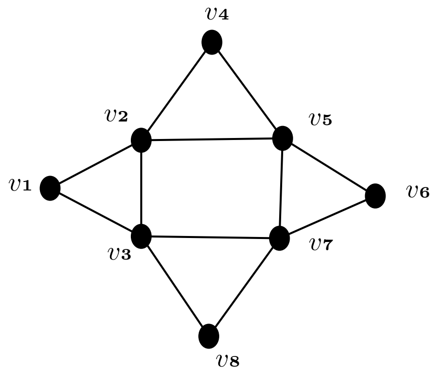

Figure 2. **Graph** H

Example 2. In the graph H of Figure 2, Consider the set $\{v_1, v_2, v_3, v_4\}$. Consider the defense $D : \{v_1, v_2, v_3, v_4\} \to \{v_2, v_3, v_4\}$ given by

$$D(v_1) = D(v_3) = v_3, D(v_2) = v_2, D(v_4) = v_4.$$

Note that D is successful against any attack. Thus S is an ultra secure set. But since $|N[\{v_2, v_3, v_4\}] \cap S| - |N[\{v_2, v_3, v_4\}] - S| = 4 - 3 = 1 < 3 = |\{v_2, v_3, v_4\}|$, S is not a 1-secure set.

The following theorem is a sharpening of Theorem 7.

Theorem 11. *A set S is ultra secure if and only if* $|N[X] \cap S| \geq \sum_{x \in X} |N[x] - S|$ *for every* $X \subseteq Bord(S)$.

Proof. If S is an ultra secure set, then by Theorem 7, clearly $|N[X] \cap S| \geq \sum_{x \in X} |N[x] - S|$ for every $X \subseteq Bord(S)$. Conversely, assume that $|N[X] \cap S| \geq \sum_{x \in X} |N[x] - S|$ for every $X \subseteq Bord(S)$. Then for any $Y \subseteq S$,

$$|N[Y] \cap S| \geq |N[Y \cap Bord(S)] \cap S| \geq \sum_{y \in Y \cap Bord(S)} |N[y] - S| = \sum_{y \in Y} |N[y] - S|.$$

Then by Theorem 7, S is an ultra secure set. □

Theorem 12. *If S is a 1-secure set and $G^{Bord(S)}$ has no cycle, then S is an ultra secure set.*

Proof. Since $G^{Bord(S)}$ has no cycle, every G^X contains fewer edges than vertices and hence $\sum_{x \in X} |N[x] - S| \leq |X| + |N[X] - S| \leq |N[X] \cap S|$ for every $X \subseteq Bord(S)$. Therefore S is ultra secure by Theorem 11. □

4. 1-Security Number of Graphs

In this section, 1-security numbers of certain classes of graphs are obtained.

Theorem 13. *Let $G = (V, E)$ be a graph. Then,*

1. *$s_1(G) = 1$ if and only if $|V| = 1$.*
2. *$s_1(G) = 2$ if and only if there exist two adjacent vertices u and v such that $\deg(u) + \deg(v) \leq 3$.*
3. *$s_1(G) = 3$ if and only if $s_1(G) \neq 2$ and there exists $S = \{u, v, w\} \subseteq V$ such that one of the following holds.*
   - *(a) $\langle S \rangle = P_3$ with $\deg(u) = \deg(v) = 1$ and $3 \leq \deg(w) \leq 4$.*
   - *(b) $\langle S \rangle = P_3$ with $\langle S \cup \partial S \rangle = C_4$.*
   - *(c) $\langle S \rangle = C_3$ with $\deg(u) = \deg(v) = 2$ and $\deg(w) \leq 4$.*

Proof. 1. The proof is trivial.

2. Let $S = \{u, v\}$ be a minimum 1-secure set. Then by Proposition 1, u and v are adjacent. If $\partial(S) = \emptyset$, then $G = P_2$. If $\partial(S) \neq \emptyset$, then $|Bord(S)| = |\partial(S)| = 1$. In both the cases, $\deg(u) + \deg(v) \leq 3$. Conversely, suppose that there exist adjacent vertices u and v with $\deg(u) + \deg(v) \leq 3$. Then exactly one of them is of degree 1 whereas the other is of at most 2. Then $S = \{u, v\}$ is 1-secure.

3. If $s_1(G) \neq 2$ and there exists $S = \{u, v, w\}$ such that the given condition holds, then it is clear that $s_1(G) = 3$. Now suppose that $s_1(G) = 3$ and let $S = \{u, v, w\}$ be a minimum 1-secure set. By Proposition 1, $\langle S \rangle = P_3$ or C_3. Note that $|\partial S| \leq 2$. Assume that $\langle S \rangle = P_3$. Let u, v be the end vertices of $\langle S \rangle = P_3$. Since $s_1(G) \neq 2$, $Bord(S) \neq \emptyset$. If w is a border vertex, then w is the only border vertex and can have at most 2 neighbors outside S. Therefore $\deg(u) = \deg(v) = 1$ and $3 \leq \deg(w) \leq 4$. If u is a border vertex, then it can have exactly one neighbor x outside S. Further $\partial S = \{x\}$ and w must be an interior vertex. If x is not adjacent to v, then $\{v, w\}$ form a 1-secure set which is a contradiction to the fact that $s_1(G) \neq 2$. Therefore the subgraph induced

by $\{u, v, w, x\} = C_4$. Now assume that $\langle S\rangle = C_3$. Note that $|Bord(S)| \leq 1$ and $|\partial S| \leq 2$. Then two vertices of S have degree 2 whereas the remaining vertex has a degree at most 4.

□

Theorem 14. *Let $n \geq 2$ be an integer and $n_1 \leq n_2 \leq ... \leq n_l$ be positive integers. Then,*

1. $s_1(P_n) = 2$.
2. $s_1(C_3) = s_1(C_4) = 3$ *and for* $n \geq 5$, $s_1(C_n) = 4$.
3. $s_1(K_n) = n$.
4. $s_1(K_{n_1,n_2,\dots,n_l}) = n_1 + n_2 + ...n_{l-1} + \lceil \frac{n_l}{2} \rceil$.

Proof. 1. Follows by (2) of Theorem 13.

2. By (3) of Theorem 13, $s_1(C_3) = s_1(C_4) = 3$. Suppose $n \geq 5$. Since any set of four consecutive vertices form a 1-secure set, $s_1(C_n) \leq 4$. Further, no set with lesser cardinality is 1-secure.

3. Suppose S is a 1-secure set in K_n with $|S| = m < n$. Then $|\partial S| = n - m > 0$ and $Bord(S) = S$. Let A be an attack on S. Then there exists a defense D such that $|D^{-1}(z)| \geq |A^{-1}(z)| + 1$ for all $z \in S$. Then $0 < n - m = |\partial S| = |\bigcup\limits_{z\in S} A^{-1}(z)| = \sum\limits_{z\in S} |A^{-1}(z)| \leq \sum\limits_{z\in S} \Big(|D^{-1}(z)| - 1\Big) = \sum\limits_{z\in S} (|D^{-1}(z)|) - |S| \leq 0$; therefore, $n = m$. So, $S = V(K_n)$.

4. Let $X_i = \{u_{i1}, u_{i2}, \dots, u_{in_i}\}$, $1 \leq i \leq l$ be the partition of $K_{n_1,\dots,n_l}$. Then $\{u_{ij} : 1 \leq j \leq n_i, 1 \leq i \leq l-1\} \cup \{u_{lj} : 1 \leq j \leq \lceil \frac{n_l}{2} \rceil\}$ is a 1-secure set. Let S be a minimum 1-secure set. Suppose that $Int(S) = \emptyset$. Note that $|S| \leq n_1 + \cdots + n_{l-1} + \lceil \frac{n_l}{2} \rceil$ and hence $Bord(S) \neq \emptyset$. Thus $S = Bord(S)$ and $\partial S \neq \emptyset$. Let $|S| = m$. Then $|\partial S| = n_1 + n_2 + \cdots + n_l - m > 0$. Note that for any attack A with $\bigcup\limits_{z\in S} A^{-1}(z) = \partial S$, there exists a defense D such that $|D^{-1}(z)| \geq |A^{-1}(z)| + 1$ for all $z \in S$. Then $0 < n_1 + n_2 + \cdots + n_l - m = |\partial S| = |\bigcup\limits_{z\in S} A^{-1}(z)| = \sum\limits_{z\in S} |A^{-1}(z)| \leq \sum\limits_{z\in S} \Big(|D^{-1}(z)| - 1\Big) = \sum\limits_{z\in S} (|D^{-1}(z)|) - |S| \leq 0$ which is a contradiction. Hence $Int(S) \neq \emptyset$. Thus at least $l-1$ X_i's are contained in S and at least $\lceil \frac{n_i}{2} \rceil$ vertices of the remaining one X_i lie in S. Therefore $|S| \geq n_1 + n_2 + ... + n_{l-1} + \lceil \frac{n_l}{2} \rceil$ which completes the proof.

□

The first two rows (columns) in the Cartesian product $P_m \times P_n$ and any four consecutive rows (columns) in $C_m \times C_n$ form 1-secure sets. In $P_m \times C_n$, the first two rows and any four consecutive columns separately form 1-secure sets. This leads to the following result.

Theorem 15. *Let m and n be positive integers.*

1. *If* $2 \leq m \leq n$, *then* $s_1(P_m \times P_n) \leq 2m$.
2. *If* $3 \leq m \leq n$, *then* $s_1(C_m \times C_n) \leq 4m$.
3. *If* $m \geq 2$ *and* $n \geq 3$, *then* $s_1(P_m \times C_n) \leq min\{4m, 2n\}$.

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