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Research Article

Vector valued switching in the products of signed graphs

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Abstract: A signed graph is a graph whose edges are labeled either as positive or negative. The concepts of vector valued switching and balancing dimension of signed graphs were introduced by S. Hameed et al. In this paper, we deal with the balancing dimension of various products of signed graphs, namely the Cartesian product, the lexicographic product, the tensor product, and the strong product.

Keywords: Signed graph, vector valued switching, balancing dimension, product of signed graphs.

AMS Subject classification: 05C22, 05C76

1. Introduction

Throughout this paper, unless otherwise mentioned, we consider only finite, simple, connected, and undirected signed graphs. For the standard notation and terminology in graphs and signed graphs, not given here, the reader may refer to [6] and [10, 11] respectively.
A signed graph $\Sigma = (G, \sigma)$ is a graph G, together with a function σ that assigns a sign $+1$ or -1 to each of its edges. The sign of a cycle in Σ is defined as the product of the signs of its edges, and Σ is balanced if it does not contain any negative cycles. A signed graph $\Sigma = (G, \sigma)$ is said to be "antibalanced" if the signed graph $-\Sigma = (G, -\sigma)$ is balanced. A switching function for Σ is a function $\zeta : V(\Sigma) \to \{-1, 1\}$. For an edge $e = uv$ in Σ, the switched signature σ^ζ is defined as $\sigma^\zeta(e) = \zeta(u)\sigma(e)\zeta(v)$, and the switched signed graph is $\Sigma^\zeta = (G, \sigma^\zeta)$. The signs of cycles are unchanged by switching and every balanced (antibalanced) signed graph can be switched to an

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all-positive (all-negative) signed graph. We call Σ_1 and Σ_2 switching equivalent, and write $\Sigma_1 \sim \Sigma_2$, if there is a switching function ζ such that $\Sigma_2 = \Sigma_1^{\zeta}$ (see [10, Section 3]).
The notions of vector valued switching and balancing dimension of signed graphs were defined by Hameed *et al.* in [5]. In this paper, we focus on computing the balancing dimensions of various products of signed graphs such as the Cartesian product, the lexicographic product, the tensor product, and the strong product.
To begin with, we recall some notations, definitions and fundamental results from [5]. In what follows, $\Omega = \{-1, 0, 1\}$ and the inner product used is the same as that on $\mathbb{R}^k$ restricted to Ω^k.

Definition 1. (Vector Valued Switching or k-switching) [5] Let $\Sigma = (G, \sigma)$ be a given signed graph where $G = (V, E)$. A vector valued switching function is a function $\zeta : V \to \Omega^k \subset \mathbb{R}^k$ such that $\langle \zeta(u), \zeta(v) \rangle \neq 0$ for all edges $uv \in E$. The switched signed graph $\Sigma^{\zeta} = (G, \sigma^{\zeta})$ has the signing

$$\sigma^{\zeta}(uv) = \sigma(uv)\,\mathrm{sgn}(\langle \zeta(u), \zeta(v) \rangle).$$

Note that the switching that has been discussed so far in literature [10] can be considered as 1-switching. Using vector valued switching, the balancing dimension for a signed graph is defined as follows.

Definition 2. (Balancing Dimension) [5] Let $\Sigma = (G, \sigma)$ be a given signed graph where $G = (V, E)$. We say that the balancing dimension of Σ is k, and write it as $\mathrm{bdim}(\Sigma)$, if $k \geq 1$ is the least integer such that a vector-valued switching function $\zeta : V \to \Omega^k \subset \mathbb{R}^k$ switches Σ to an all positive signed graph.
Such a k-switching function ζ is called a positive k-switching function (briefly a k-positive function) for Σ.

One may note that $\mathrm{bdim}(\Sigma) = 1$ if and only if Σ is balanced. Also, the balancing dimension of a subgraph of Σ cannot exceed the balancing dimension of Σ. We will also make use of the fact that the balancing dimension is 1-switching invariant (see [5]).

2. Balancing dimension of the product of signed graphs

In this section, we establish some results regarding the balancing dimension of the Cartesian product, the lexicographic product, the tensor product, and the strong product of signed graphs.

2.1. Balancing dimension of the Cartesian product

The Cartesian product of two signed graphs is defined by Germina et al. in [3].

Definition 3. [3] The Cartesian product $\Sigma_1 \square \Sigma_2$ of two signed graphs $\Sigma_1 = (G_1, \sigma_1)$ and $\Sigma_2 = (G_2, \sigma_2)$ is defined as the Cartesian product of the underlying unsigned graphs with the signature function σ for the labeling of the edges defined by

$$\sigma((u_i, v_j)(u_k, v_l)) = \begin{cases} \sigma_1(u_i u_k), & \text{if } j = l, \\ \sigma_2(v_j v_l), & \text{if } i = k. \end{cases}$$

If Σ_1 and Σ_2 are balanced, then their Cartesian product $\Sigma_1 \square \Sigma_2$ is also balanced (see [3]) and hence $\operatorname{bdim}(\Sigma_1 \square \Sigma_2) = 1$. We now consider the case where one of the factors is balanced.

Theorem 1. *Let $\Sigma_1 = (G_1, \sigma_1)$ and $\Sigma_2 = (G_2, \sigma_2)$ be two signed graphs and let $\Sigma_1 \square \Sigma_2$ be their Cartesian product. Then*

$$\operatorname{bdim}(\Sigma_1 \square \Sigma_2) = \begin{cases} \operatorname{bdim}(\Sigma_1), & \textit{if } \Sigma_2 \textit{ is balanced} \\ \operatorname{bdim}(\Sigma_2), & \textit{if } \Sigma_1 \textit{ is balanced.} \end{cases}$$

Proof. Suppose $\operatorname{bdim}(\Sigma_1) = k$ and Σ_2 is balanced. Let $\zeta_1 : V(\Sigma_1) \to \Omega^k$ and $\zeta_2 : V(\Sigma_2) \to \{-1, 1\}$ be the corresponding switching functions. We now define $\zeta : V(\Sigma_1 \times \Sigma_2) \to \Omega^k$ by $\zeta((u_i, v_j)) = \zeta_1(u_i)\zeta_2(v_j)$ for $1 \leq i \leq |V(\Sigma_1)|$ and $1 \leq j \leq |V(\Sigma_2)|$.
Now for any edge $e = (u_i, v_j)(u_k, v_l)$ in $\Sigma_1 \square \Sigma_2$, we have,

$$\sigma^\zeta((u_i, v_j)(u_k, v_l)) = \sigma((u_i, v_j)(u_k, v_l)) \operatorname{sgn}(\langle \zeta((u_i, v_j)), \zeta((u_k, v_l)) \rangle). \tag{2.1}$$

If $j = l$, Equation 2.1 becomes

$$\begin{aligned} \sigma^\zeta((u_i, v_l)(u_k, v_j)) &= \sigma_1(u_i u_k) \operatorname{sgn}(\langle \zeta((u_i, v_j)), \zeta((u_k, v_l)) \rangle) \\ &= \sigma_1(u_i u_k) \operatorname{sgn}(\langle \zeta_1(u_i)\zeta_2(v_l), \zeta_1(u_k)\zeta_2(v_l) \rangle) \\ &= (\zeta_2(v_l))^2 \sigma_1(u_i u_k) \operatorname{sgn}(\langle \zeta_1(u_i), \zeta_1(u_k) \rangle) \\ &= (\zeta_2(v_l))^2 \sigma_1^{\zeta_1}(u_i u_k) \\ &= +1. \end{aligned}$$

Similarly, if $i = k$, Equation 2.1 becomes

$$\begin{aligned} \sigma^\zeta((u_i, v_j)(u_k, v_l)) &= \sigma_2(v_j v_l) \operatorname{sgn}(\langle \zeta((u_k, v_j)), \zeta((u_k, v_l)) \rangle) \\ &= \sigma_2(v_j v_l) \operatorname{sgn}(\langle \zeta_1(u_k)\zeta_2(v_j), \zeta_1(u_k)\zeta_2(v_l) \rangle) \\ &= \sigma_2(v_j v_l)\zeta_2(v_j)\zeta_2(v_l) \operatorname{sgn}(\langle \zeta_1(u_k), \zeta_1(u_k) \rangle) \\ &= \sigma_2^{\zeta_2}(v_j v_l) \operatorname{sgn}(\|\zeta_1(u_k)\|^2) \\ &= +1. \end{aligned}$$

Thus, ζ switches $\Sigma_1 \square \Sigma_2$ to all-positive, and hence $\operatorname{bdim}(\Sigma_1 \square \Sigma_2) \leq k$. However, since Σ_1 is a subgraph of $\Sigma_1 \square \Sigma_2$, we must have $\operatorname{bdim}(\Sigma_1 \square \Sigma_2) = k = \operatorname{bdim}(\Sigma_1)$.
Similar is the proof of the next part. □

Theorem 2. *[7] Let Σ_1 and Σ_2 be two signed graphs and let $\Sigma_1 \Box \Sigma_2$ be their Cartesian product. If $\Sigma_1 \sim \Sigma_1'$ and $\Sigma_2 \sim \Sigma_2'$, then $\Sigma_1 \Box \Sigma_2 \sim \Sigma_1' \Box \Sigma_2'$.*

Theorem 3. *[5] If Σ contains a negative triangle, then* $\mathrm{bdim}(\Sigma) \geq 3$.

We now compute the balancing dimension of the Cartesian product of two unbalanced signed graphs. To begin with, we consider the Cartesian product of two unbalanced cycles.

Proposition 1. *Let C_m^- and C_n^-, $m, n \geq 3$ be two unbalanced cycles, and let $C_m^- \Box C_n^-$ be their Cartesian product. Then,*

$$\mathrm{bdim}(C_m^- \Box C_n^-) = \begin{cases} 2, & \text{if } m, n > 3 \\ 3, & \text{otherwise.} \end{cases}$$

Proof. Since balancing dimension is 1 - switching invariant, by using Theorem 2, we can consider $C_m^- = u_1 u_2 \cdots u_m$ and $C_n^- = v_1 v_2 \cdots v_n$, where u_1u_2 and v_1v_2 are the only negative edges of C_m^- and C_n^- respectively. Since the Cartesian product is commutative, it suffices to consider three cases: $m, n > 3$, $m = 3$ and $n > 3$, and $n = m = 3$.
In the first case, since $C_m^- \Box C_n^-$ is unbalanced, we have $\mathrm{bdim}(C_m^- \Box C_n^-) \geq 2$. Now, the function $\zeta_1 : V(C_m^- \Box C_n^-) \to \Omega^2$ given in Table 1 switches $C_m^- \Box C_n^-$ to all-positive. Hence, $\mathrm{bdim}(C_m^- \Box C_n^-) = 2$. In the remaining two cases, $C_m^- \Box C_n^-$ contains a negative triangle and hence $\mathrm{bdim}(C_m^- \Box C_n^-) \geq 3$. Now, the functions $\zeta_i : V(C_m^- \Box C_n^-) \to \Omega^3$, where $i \in \{2, 3\}$, given in Tables 2 and 3 respectively, switches $C_m^- \Box C_n^-$ to all-positive. Hence $\mathrm{bdim}(C_m^- \Box C_n^-) = 3$ in each of these cases. $\square$

$\zeta_1((u_i, v_j))$	v_1	v_2	$v_3, v_4, \cdots, v_{n-1}$	v_n
u_1	$(-1, 1)$	$(1, 0)$	$(1, 1)$	$(0, 1)$
u_2	$(1, 0)$	$(-1, -1)$	$(0, -1)$	$(1, -1)$
$u_3, u_4, \cdots, u_{m-1}$	$(1, 1)$	$(0, -1)$	$(1, -1)$	$(1, 0)$
u_m	$(0, 1)$	$(1, -1)$	$(1, 0)$	$(1, 1)$

Table 1. **A 2 - positive function for $C_m^- \Box C_n^-$ for $m, n > 3$.**

$\zeta_2((u_i, v_j))$	v_1	$v_2, v_3, \cdots, v_{n-1}$	v_n
u_1	$(-1, -1, 1)$	$(1, -1, -1)$	$(-1, -1, -1)$
u_2	$(1, -1, -1)$	$(1, 1, 1)$	$(1, 0, 0)$
u_3	$(-1, -1, -1)$	$(1, 0, 0)$	$(1, -1, -1)$

Table 2. **A 3 - positive function for $C_3^- \Box C_n^-$ for $n > 3$.**

$\zeta_3((u_i,v_j))$	v_1	v_2	v_3
u_1	$(-1,-1,1)$	$(1,-1,-1)$	$(-1,-1,-1)$
u_2	$(1,-1,-1)$	$(1,1,1)$	$(1,0,0)$
u_3	$(-1,-1,-1)$	$(1,0,0)$	$(1,-1,-1)$

Table 3. **A** 3 **- positive function for** $C_3^- \square C_3^-$**.**

Next, we consider antibalanced signed complete graphs. We denote the antibalanced signed complete graph on n vertices by K_n^-, and the balancing dimension of K_n^- is defined as $\bar{\nu}(n)$ [5].

Proposition 2. *Let K_m^- and K_n^- be antibalanced signed complete graphs of order m and n respectively, and let $K_m^- \square K_n^-$ be their Cartesian product. Then* $\mathrm{bdim}(K_m^- \square K_n^-) = \bar{\nu}(h)$*, where, $h = \max\{m,n\}$.*

Proof. By adequate 1-switching, we can consider K_m^- and K_n^- as all-negative. Then, the Cartesian product $K_m^- \square K_n^-$ is also all-negative.
Without loss of generality, assume that $m \geq n$. Suppose $\mathrm{bdim}(K_m^-) = k$ and let $\zeta' : V(K_m^-) \to \Omega^k$ be the k - positive function. Since K_m^- is a subgraph of $K_m^- \square K_n^-$, we have $\mathrm{bdim}(K_m^- \square K_n^-) \geq k$. Now, the function $\zeta : V(K_m^- \square K_n^-) \to \Omega^k$ given in Table 4 switches $K_m^- \square K_n^-$ to all-positive. Hence, $\mathrm{bdim}(K_m^- \square K_n^-) = k = \bar{\nu}(m)$.
Similar is the proof of the next part. □

$\zeta((u_i,v_j))$	v_1	v_2	v_3	$\cdots$	v_n
u_1	$\zeta'(u_1)$	$\zeta'(u_2)$	$\zeta'(u_3)$	$\cdots$	$\zeta'(u_n)$
u_2	$\zeta'(u_2)$	$\zeta'(u_3)$	$\zeta'(u_4)$	$\cdots$	$\zeta'(u_{n+1})$
u_3	$\zeta'(u_3)$	$\zeta'(u_4)$	$\zeta'(u_5)$	$\cdots$	$\zeta'(u_{n+2})$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots\vdots\vdots$	$\vdots$
u_m	$\zeta'(u_m)$	$\zeta'(u_1)$	$\zeta'(u_2)$	$\cdots$	$\zeta'(u_{n-1})$

Table 4. **A** k **- positive function for** $K_m^- \square K_n^-$

Corollary 1. *For any antibalanced signed graph Σ on n vertices,* $\mathrm{bdim}(\Sigma \square K_n^-) = \mathrm{bdim}(K_n^-)$.

2.2. Balancing dimension of the lexicographic product

We now focus on the lexicographic product (also called composition) of signed graphs. Two definitions for the lexicographic product of signed graphs are available in the literature. We call the definition given by Hameed et al. [4] the *HG-lexicographic product* and the definition given by Brunetti et al. [2] the *BCD-lexicographic product.*

Definition 4. [4] The *HG-lexicographic product* of two signed graphs $\Sigma_1 = (G_1, \sigma_1)$ and $\Sigma_2 = (G_2, \sigma_2)$ is the signed graph whose underlying graph is the lexicographic product of the underlying unsigned graphs and whose signature function σ for the labeling of the edges is defined by

$$\sigma((u_i,v_j)(u_k,v_l)) = \begin{cases} \sigma_1(u_iu_k) & \text{if } i \neq k, \\ \sigma_2(v_jv_l) & \text{if } i = k. \end{cases}$$

We denote the *HG-lexicographic product* of Σ_1 and Σ_2 by $\Sigma_1[\Sigma_2]$.

Definition 5. [2] The *BCD-lexicographic product* of two signed graphs $\Sigma_1 = (G_1, \sigma_1)$ and $\Sigma_2 = (G_2, \sigma_2)$ as the signed graph whose underlying graph is the lexicographic product of the underlying unsigned graphs and whose signature function σ for the labeling of the edges is defined by

$$\sigma((u_i,v_j)(u_k,v_l)) = \begin{cases} \sigma_1(u_iu_k) & \text{if } u_i \sim u_k \text{ and } v_j \nsim v_l, \\ \sigma_1(u_iu_k)\sigma_2(v_jv_l) & \text{if } u_i \sim u_k \text{ and } v_j \sim v_l, \\ \sigma_2(v_jv_l) & \text{if } u_i = u_k \text{ and } v_j \sim v_l. \end{cases}$$

We denote the *BCD-lexicographic product* of Σ_1 and Σ_2 by $\Sigma_1 * \Sigma_2$.

The *HG-lexicographic product* and *BCD-lexicographic product* of two balanced signed graphs need not be balanced. However, a criterion for the balance of *HG-lexicographic product* of two signed graphs is proved in [4] .

Theorem 4. *[4] If Σ_1 and Σ_2 are two signed graphs with at least one edge for each, then their HG-lexicographic product is balanced if and only if Σ_1 is balanced and Σ_2 is all-positive.*

Theorem 5. *Let Σ_1 and Σ_2 be two signed graphs, with Σ_2 having at least one negative edge. Then* $\mathrm{bdim}(\Sigma_1[\Sigma_2]) \geq 3$.

Proof. Let v_jv_{j+1} be a negative edge of Σ_2. Then for any edge u_iu_{i+1} of Σ_1, $(u_i,v_j)(u_i,v_{j+1})(u_{i+1},v_j)(u_i,v_j)$ forms a negative triangle in $\Sigma_1[\Sigma_2]$. Hence, by Theorem 3, $\mathrm{bdim}(\Sigma_1[\Sigma_2]) \geq 3$. □

Proposition 3. *For any signed graph Σ,* $\mathrm{bdim}(N_k[\Sigma]) = \mathrm{bdim}(\Sigma[N_k]) = \mathrm{bdim}(\Sigma)$*, where N_k is the graph on k vertices without edges.*

Proof. Suppose $\mathrm{bdim}(\Sigma) = k$ and $\zeta : V(\Sigma) \to \Omega^k$ be the k-positive function. Then, $\zeta' : V(N_k[\Sigma]) \to \Omega^k$, defined by $\zeta'((w_i,u_j)) = \zeta(u_j)$ for $1 \leq i \leq k$ and $1 \leq j \leq |V(\Sigma)|$, switches $N_k[\Sigma]$ to all-positive. Hence, $\mathrm{bdim}(N_k[\Sigma]) \leq k$. However, since Σ is a subgraph of $N_k[\Sigma]$, we must have $\mathrm{bdim}(N_k[\Sigma]) = k = \mathrm{bdim}(\Sigma)$.
Similarly, $\zeta'' : V(\Sigma[N_k]) \to \Omega^k$, defined by $\zeta''((u_i,w_j)) = \zeta(u_i)$ for $1 \leq i \leq |V(\Sigma)|$ and $1 \leq j \leq k$, switches $\Sigma[N_k]$ to all-positive, and hence $\mathrm{bdim}(\Sigma[N_k]) = k$. □

Remark 1. The above results show that, even though the lexicographic product is not commutative, there exist signed graphs satisfying $\mathrm{bdim}(\Sigma_1[\Sigma_2]) = \mathrm{bdim}(\Sigma_2[\Sigma_1])$. However, in general, $\mathrm{bdim}(\Sigma_1[\Sigma_2]) \neq \mathrm{bdim}(\Sigma_2[\Sigma_1])$. As an example, consider Σ_1 as the balanced triangle having two negative edges and Σ_2 as the all-positive K_2. Then, Theorem 4 and Theorem 5 respectively shows that $\mathrm{bdim}(\Sigma_1[\Sigma_2]) = 1$ and $\mathrm{bdim}(\Sigma_2[\Sigma_1]) \geq 3$.

Theorem 6. *Let Σ_1 and Σ_2 be two signed graphs and let $\Sigma_1[\Sigma_2]$ be their HG - lexicographic product. If $\Sigma_1 \sim \Sigma_1'$, then $\Sigma_1[\Sigma_2] \sim \Sigma_1'[\Sigma_2]$.*

Proof. Let σ_1, σ_1', σ_2, σ and σ' denote the signatures of Σ_1, Σ_1', Σ_2, $\Sigma_1[\Sigma_2]$ and $\Sigma_1'[\Sigma_2]$ respectively. Since $\Sigma_1 \sim \Sigma_1'$, there exists a switching function $\eta : V(\Sigma_1) \to \{-1, 1\}$ such that $\Sigma_1^\eta = \Sigma_1'$. Define the map $\eta' : V(\Sigma_1[\Sigma_2]) \to \{-1, 1\}$ as $\eta'((u_i, v_j)) = \eta(u_i)$ for $1 \leq i \leq |V(\Sigma_1)|$ and $1 \leq j \leq |V(\Sigma_2)|$. Then, for any edge $(u_i, v_j)(u_k, v_l)$ in $\Sigma_1[\Sigma_2]$, we have,

$$\begin{aligned}
\sigma^{\eta'}((u_i, v_j)(u_k, v_l)) &= \eta'((u_i, v_j))\sigma((u_i, v_j)(u_k, v_l))\eta'((u_k, v_l)) \\
&= \eta(u_i)\sigma((u_i, v_j)(u_k, v_l))\eta(u_k) \\
&= \begin{cases} \eta(u_i)\sigma_1(u_iu_k)\eta(u_k) & \text{if } i \neq k, \\ \eta(u_k)\sigma_2(v_jv_l)\eta(u_k) & \text{if } i = k. \end{cases} \\
&= \begin{cases} \sigma_1^\eta(u_iu_k) & \text{if } i \neq k, \\ \sigma_2(v_jv_l) & \text{if } i = k. \end{cases} \\
&= \begin{cases} \sigma_1'(u_iu_k) & \text{if } i \neq k, \\ \sigma_2(v_jv_l) & \text{if } i = k. \end{cases} \\
&= \sigma'((u_i, v_j)(u_k, v_l)).
\end{aligned}$$

Thus,$(\Sigma_1[\Sigma_2])^{\eta'} = \Sigma_1'[\Sigma_2]$ and hence, $\Sigma_1[\Sigma_2] \sim \Sigma_1'[\Sigma_2]$. □

Since the balancing dimension is 1-switching invariant, we have the following result.

Corollary 2. *If Σ_1 and Σ_2 are any two signed graphs and if $\Sigma_1 \sim \Sigma_1'$, then $\mathrm{bdim}(\Sigma_1[\Sigma_2]) = \mathrm{bdim}(\Sigma_1'[\Sigma_2])$.*

Corollary 3. *If Σ_1 is antibalanced and Σ_2 is all-negative, then $\Sigma_1[\Sigma_2]$ is antibalanced.*

Proof. Since Σ_1 is antibalanced, we have $\Sigma_1 \sim \Sigma_1'$, where Σ_1' is all- negative. Now, since Σ_2 is all-negative, $\Sigma_1'[\Sigma_2]$ is all-negative, and hence antibalanced. Thus, by Theorem 6, $\Sigma_1[\Sigma_2]$ is antibalanced . □

We now consider complete graphs. To begin with, observe that the lexicographic product of two complete graphs, say K_m and K_n is the complete graph K_{mn}. To see this, consider any two vertices $u = (u_i, v_j)$ and $v = (u_k, v_l)$ in $K_m[K_n]$. Then u_i, u_k are adjacent in K_m and v_j, v_l are adjacent in K_n. Therefore, if $u_i \neq u_k$, then since u_i

and u_k are adjacent in K_m, u and v are adjacent in $K_m[K_n]$. On the other hand, if $u_i = u_k$, then since v_j, v_l are adjacent in K_n, u and v are adjacent in $K_m[K_n]$. Thus, any two of the mn vertices of $K_m[K_n]$ are adjacent.
The next result follows immediately from Corollary 3.

Proposition 4. *If Σ_1 and Σ_2 denote the antibalanced signed complete graph K_m^- and the all-negative signed complete graph $-K_n$ respectively, then* $\mathrm{bdim}(\Sigma_1[\Sigma_2]) = \bar{\nu}(mn)$, *where $\bar{\nu}(mn)$ is the balancing dimension of the antibalanced signed complete graph K_{mn}^-.*

Theorem 7. *Let $\Sigma_1 = (G_1, \sigma_1)$ and $\Sigma_2 = (G_2, \sigma_2)$ be two signed graphs, where Σ_2 is all-positive. Then* $\mathrm{bdim}(\Sigma_1[\Sigma_2]) = \mathrm{bdim}(\Sigma_1)$.

Proof. Suppose Σ_2 is all positive. Let $\mathrm{bdim}(\Sigma_1) = k$ and $\zeta_1 : V(\Sigma_1) \to \Omega^k$ be the k-positive function. Now, the function $\zeta : V(\Sigma_1[\Sigma_2]) \to \Omega^k$, defined by $\zeta((u_i, v_j)) = \zeta_1(u_i)$ for $1 \leq i \leq |V(\Sigma_1)|$ and $1 \leq j \leq |V(\Sigma_2)|$, switches $\Sigma_1[\Sigma_2]$ to all-positive, and hence $\mathrm{bdim}(\Sigma_1[\Sigma_2]) \leq k$. However, since Σ_1 is a subgraph of $\Sigma_1[\Sigma_2]$, we must have $\mathrm{bdim}(\Sigma_1[\Sigma_2]) = k = \mathrm{bdim}(\Sigma_1)$. □

Remark 2. Let $\Sigma_1 = +K_2$ and $\Sigma_2 = -K_2$. Then $\Sigma_1[\Sigma_2]$ is the antibalanced signed complete graph K_4^- and hence $\mathrm{bdim}(\Sigma_1[\Sigma_2]) = \bar{\nu}(4) = 3 \neq \mathrm{bdim}(\Sigma_2)$. Thus, $\mathrm{bdim}(\Sigma_1[\Sigma_2])$ and $\mathrm{bdim}(\Sigma_2)$ need not be equal if Σ_1 is all-positive.

We now focus on the *BCD-lexicographic product* of two signed graphs. To begin with, we restate Theorem 2.3 from [2], by removing the incorrect part (see [1]) and provide an alternate proof for it.

Theorem 8. *Let Σ_1 and Σ_2 be two signed graphs and let $\Sigma_1 * \Sigma_2$ be their BCD - lexicographic product. If $\Sigma_1 \sim \Sigma_1'$, then $\Sigma_1 * \Sigma_2 \sim \Sigma_1' * \Sigma_2$.*

Proof. Let σ_1, σ_1', σ_2, σ and σ' denote the signatures of Σ_1, Σ_1', Σ_2, $\Sigma_1 * \Sigma_2$ and $\Sigma_1' * \Sigma_2$ respectively. Since $\Sigma_1 \sim \Sigma_1'$, there exists a switching function $\eta : V(\Sigma_1) \to \{-1, 1\}$ such that $\Sigma_1^\eta = \Sigma_1'$. Define the map $\eta' : V(\Sigma_1 * \Sigma_2) \to \{-1, 1\}$ as $\eta'((u_i, v_j)) = \eta(u_i)$ for $1 \leq i \leq |V(\Sigma_1)|$ and $1 \leq j \leq |V(\Sigma_2)|$. Then, for any edge $(u_i, v_j)(u_k, v_l)$ in $\Sigma_1 * \Sigma_2$, we have, $\sigma^{\eta'}((u_i, v_j)(u_k, v_l)) = \sigma'((u_i, v_j)(u_k, v_l))$. Thus, $(\Sigma_1 * \Sigma_2)^{\eta'} = \Sigma_1' * \Sigma_2$ and hence $\Sigma_1 * \Sigma_2 \sim \Sigma_1' * \Sigma_2$. □

Since the balancing dimension is 1-switching invariant, we have the following result.

Corollary 4. *If Σ_1 and Σ_2 are any two signed graphs and if $\Sigma_1 \sim \Sigma_1'$, then* $\mathrm{bdim}(\Sigma_1 * \Sigma_2) = \mathrm{bdim}(\Sigma_1' * \Sigma_2)$.

Remark 3. Note that Corollary 3 does not hold in the case of *BCD-lexicographic product*. To illustrate this consider $\Sigma_1 = (P_3, \sigma)$ and $\Sigma_2 = -P_2$ depicted in Figure 1. Then,

$(u_1, v_1)(u_2, v_1)(u_2, v_2)(u_1, v_1)$ forms a negative triangle in $-(\Sigma_1 * \Sigma_2)$, making it unbalanced. Thus, $\Sigma_1 * \Sigma_2$ is not antibalanced, though Σ_1 is antibalanced and Σ_2 is all-negative.

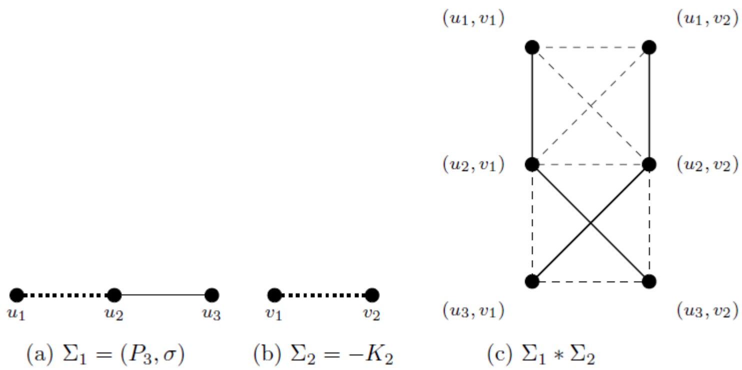

Figure 1. **The BCD-lexicographic product $\Sigma_1 * \Sigma_2$ is not antibalanced**

Theorem 9. *Let $\Sigma_1 = (G_1, \sigma_1)$ and $\Sigma_2 = (G_2, \sigma_2)$ be two signed graphs, where Σ_2 is all-positive. Then* $\text{bdim}(\Sigma_1 * \Sigma_2) = \text{bdim}(\Sigma_1)$.

Proof. Suppose Σ_2 is all positive and let $\text{bdim}(\Sigma_1) = k$. Then the function $\zeta : V(\Sigma_1 * \Sigma_2) \to \Omega^k$, defined by $\zeta((u_i, v_j)) = \zeta_1(u_i)$ for $1 \leq i \leq |V(\Sigma_1)|$ and $1 \leq j \leq |V(\Sigma_2)|$, where $\zeta_1 : V(\Sigma_1) \to \Omega^k$ is the k-positive function for Σ_1, switches $\Sigma_1 * \Sigma_2$ to all-positive, and hence $\text{bdim}(\Sigma_1 * \Sigma_2) \leq k$. However, since Σ_1 is a subgraph of $\Sigma_1 * \Sigma_2$, we must have $\text{bdim}(\Sigma_1 * \Sigma_2) = k = \text{bdim}(\Sigma_1)$. □

Theorem 10. *Let $\Sigma_1 = (G_1, \sigma_1)$ be a balanced signed graph and $\Sigma_2 = (K_n, \sigma_2)$ be a signed complete graph. Then* $\text{bdim}(\Sigma_1 * \Sigma_2) = \text{bdim}(\Sigma_2)$.

Proof. Since, Σ_1 is balanced, by Theorem 8, we can consider it as all-positive. Let $\text{bdim}(\Sigma_2) = k$ and let $\zeta_2 : V(\Sigma_2) \to \Omega^k$ be the corresponding k-positive function. Then the vector valued switching function $\zeta : V(\Sigma_1 * \Sigma_2) \to \Omega^k$, defined by $\zeta((u_i, v_j)) = \zeta_2(v_j)$ for $1 \leq i \leq |V(\Sigma_1)|$ and $1 \leq j \leq n$, switches $\Sigma_1 * \Sigma_2$ to all-positive, and hence $\text{bdim}(\Sigma_1 * \Sigma_2) \leq k$. However, since Σ_2 is a subgraph of $\Sigma_1 * \Sigma_2$, we must have $\text{bdim}(\Sigma_1 * \Sigma_2) = k = \text{bdim}(\Sigma_2)$. □

Remark 4. Theorem 10 does not hold for the *HG-lexicographic product* of signed graphs. As an example, let $\Sigma_1 = +K_2$ and $\Sigma_2 = -K_2$. Then the *HG-lexicographic product* $\Sigma_1[\Sigma_2]$ is the antibalanced signed complete graph (K_4, σ) and hence $\text{bdim}(\Sigma_1[\Sigma_2]) = \bar{\nu}(4) = 3 \neq \text{bdim}(\Sigma_2)$. Thus, $\text{bdim}(\Sigma_1[\Sigma_2])$ and $\text{bdim}(\Sigma_2)$ need not be equal if Σ_1 is balanced and Σ_2 is a signed complete graph.

2.3. Balancing dimension of the tensor product

We now focus on the tensor product of signed graphs. The tensor product of two signed graphs is given in [8] as follows.

Definition 6. The *tensor product* of two signed graphs $\Sigma_1 = (G_1, \sigma_1)$ and $\Sigma_2 = (G_2, \sigma_2)$ is the signed graph $\Sigma = \Sigma_1 \times \Sigma_2$ whose underlying graph is $G = G_1 \times G_2$ and with the sign of an edge $(u_i, v_j)(u_k, v_l)$ of G given by

$$\sigma(((u_i, v_j)(u_k, v_l)) = \sigma_1(u_1 u_k)\sigma_2(v_j v_l).$$

Theorem 11. *[9] Let Σ_1 and Σ_2 be two connected signed graphs of order at least 2. Then, the tensor product $\Sigma_1 \times \Sigma_2$ is balanced if and only if Σ_1 and Σ_2 are both balanced or both antibalanced.*

Theorem 12. *Let $\Sigma_1 = (G_1, \sigma_1)$ and $\Sigma_2 = (G_2, \sigma_2)$ be two signed graphs and $\Sigma_1 \times \Sigma_2$ be their tensor product. Then*

(i) $\mathrm{bdim}(\Sigma_1 \times \Sigma_2) \leq \mathrm{bdim}(\Sigma_1)$, *if Σ_2 is balanced.*

(ii) $\mathrm{bdim}(\Sigma_1 \times \Sigma_2) \leq \mathrm{bdim}(\Sigma_2)$, *if Σ_1 is balanced*

Proof. Suppose $\mathrm{bdim}(\Sigma_1) = k$ and Σ_2 is balanced. Let $\zeta_1 : V(\Sigma_1) \to \Omega^k$ and $\zeta_2 : V(\Sigma_2) \to \{-1, +1\}$ be the corresponding switching functions. Then $\zeta : V(\Sigma_1 \times \Sigma_2) \to \Omega^k$, defined by $\zeta((u_i, v_j)) = \zeta_1(u_i)\zeta_2(v_j)$ for $1 \leq i \leq |V(\Sigma_1)|$ and $1 \leq j \leq |V(\Sigma_2)|$, switches $\Sigma_1 \times \Sigma_2$ to all-positive, and hence $\mathrm{bdim}(\Sigma_1 \times \Sigma_2) \leq k$.
Similar is the proof of the next part. □

Remark 5. Let $\Sigma_1 = -K_3$ and $\Sigma_2 = -K_2$ denote the all-negative signed complete graphs. Then by Theorem 11, $\mathrm{bdim}(\Sigma_1 \times \Sigma_2) = 1$. However, $\mathrm{bdim}(\Sigma_1) = \bar{\nu}(3) = 3$. Hence, unlike the Cartesian product and the lexicographic products, there exist cases in which the balancing dimension of the tensor product is strictly less than the balancing dimension of its factor(s).
As an example for the case where equality holds, consider $\Sigma_3 = u_1u_2u_3$ and $\Sigma_4 = v_1v_2v_3$ as the all-negative and all-positive signed complete graphs on three vertices respectively. Then $(u_1, v_1)(u_2, v_2)(u_3, v_3)$ forms a negative triangle in $\Sigma_3 \times \Sigma_4$. Thus, $\mathrm{bdim}(\Sigma_3 \times \Sigma_4) = \mathrm{bdim}(\Sigma_4)$.

2.4. Balancing dimension of the strong product

Finally, we consider the strong product of signed graphs.

Definition 7. [2] The *strong product* of two signed graphs $\Sigma_1 = (G_1, \sigma_1)$ and $\Sigma_2 = (G_2, \sigma_2)$ is the signed graph $\Sigma = \Sigma_1 \boxtimes \Sigma_2$ whose underlying graph is $G = G_1 \boxtimes G_2$ and with the sign of an edge $(u_i, v_j)(u_k, v_l)$ of G given by

$$\sigma((u_i, v_j)(u_k, v_l)) = \begin{cases} \sigma_1(u_i u_k) & \text{if } u_i \sim u_k \text{ and } v_j = v_l, \\ \sigma_2(v_j v_l) & \text{if } u_i = u_k \text{ and } v_j \sim v_l, \\ \sigma_1(u_i u_k)\sigma_2(v_j v_l) & \text{if } u_i \sim u_k \text{ and } v_j \sim v_l. \end{cases}$$

Lemma 1. *Let Σ_1 and Σ_2 be two signed graphs and let $\Sigma_1 \boxtimes \Sigma_2$ be their strong product.*

(i) *If $\Sigma_1 \sim \Sigma_1'$, then $\Sigma_1 \boxtimes \Sigma_2 \sim \Sigma_1' \boxtimes \Sigma_2$.*

(ii) *If $\Sigma_2 \sim \Sigma_2'$, then $\Sigma_1 \boxtimes \Sigma_2 \sim \Sigma_1 \boxtimes \Sigma_2'$.*

Proof. Let σ_1, σ_1', σ_2, σ_2', σ, σ' and σ'' denote the signatures of Σ_1, Σ_1', Σ_2, Σ_2', $\Sigma_1 \boxtimes \Sigma_2$, $\Sigma_1' \boxtimes \Sigma_2$, and $\Sigma_1 \boxtimes \Sigma_2'$ respectively.
Since $\Sigma_1 \sim \Sigma_1'$, there exists a switching function $\eta : V(\Sigma_1) \to \{-1, 1\}$ such that $\Sigma_1^\eta = \Sigma_1'$. Define the map $\eta' : V(\Sigma_1 \boxtimes \Sigma_2) \to \{-1, 1\}$ as $\eta'((u_i, v_j)) = \eta(u_i)$ for $1 \leq i \leq |V(\Sigma_1)|$ and $1 \leq j \leq |V(\Sigma_2)|$. Then, for any edge $(u_i, v_j)(u_k, v_l)$ in $\Sigma_1 \boxtimes \Sigma_2$, we have, $\sigma^{\eta'}((u_i, v_j)(u_k, v_l)) = \sigma'((u_i, v_j)(u_k, v_l))$. Thus, $(\Sigma_1 \boxtimes \Sigma_2)^{\eta'} = \Sigma_1' \boxtimes \Sigma_2$ and hence $\Sigma_1 \boxtimes \Sigma_2 \sim \Sigma_1' \boxtimes \Sigma_2$.
To prove (ii), consider the map $\mu' : V(\Sigma_1 \boxtimes \Sigma_2) \to \{-1, 1\}$ defined by $\mu'((u_i, v_j)) = \mu(v_j)$ for $1 \leq i \leq |V(\Sigma_1)|$ and $1 \leq j \leq |V(\Sigma_2)|$, where μ is the switching function that switches Σ_2 to Σ_2'. □

Using Lemma 1 we arrive at the following theorem.

Theorem 13. *Let Σ_1 and Σ_2 be two signed graphs and let $\Sigma_1 \boxtimes \Sigma_2$ be their strong product. If $\Sigma_1 \sim \Sigma_1'$ and $\Sigma_2 \sim \Sigma_2'$, then $\Sigma_1 \boxtimes \Sigma_2 \sim \Sigma_1' \boxtimes \Sigma_2'$.*

Corollary 5. *If Σ_1 and Σ_2 are balanced, then so is $\Sigma_1 \boxtimes \Sigma_2$.*

Remark 6. If Σ_1 and Σ_2 are antibalanced, it need not imply that their strong product $\Sigma_1 \boxtimes \Sigma_2$ is antibalanced. As an example, consider $\Sigma_1 = \Sigma_2 = -K_2$. Then their strong product is the balanced signed graph $\Sigma_1 \boxtimes \Sigma_2 = (K_4, \sigma)$. Hence the signed graph $-(\Sigma_1 \boxtimes \Sigma_2) = (K_4, -\sigma)$ contains an unbalanced triangle, making it unbalanced. Thus, $\Sigma_1 \boxtimes \Sigma_2$ is not antibalanced.

Theorem 14. *Let $\Sigma_1 = (G_1, \sigma_1)$ and $\Sigma_2 = (G_2, \sigma_2)$ be two signed graphs and $\Sigma_1 \boxtimes \Sigma_2$ be their strong product. Then*

$$\operatorname{bdim}(\Sigma_1 \boxtimes \Sigma_2) = \begin{cases} \operatorname{bdim}(\Sigma_1), & \text{if } \Sigma_2 \text{ is balanced} \\ \operatorname{bdim}(\Sigma_2), & \text{if } \Sigma_1 \text{ is balanced.} \end{cases}$$

Proof. Suppose $\operatorname{bdim}(\Sigma_1) = k$ and Σ_2 is balanced. Let $\zeta_1 : V(\Sigma_1) \to \Omega^k$ and $\zeta_2 : V(\Sigma_2) \to \{-1, 1\}$ be the corresponding switching functions. Now, the function $\zeta : V(\Sigma_1 \boxtimes \Sigma_2) \to \Omega^k$, defined by $\zeta((u_i, v_j)) = \zeta_1(u_i)\zeta_2(v_j)$, switches $\Sigma_1 \boxtimes \Sigma_2$ to all-positive, and hence $\operatorname{bdim}(\Sigma_1 \boxtimes \Sigma_2) \leq k$. However, since Σ_1 is a subgraph of $\Sigma_1 \boxtimes \Sigma_2$, we must have $\operatorname{bdim}(\Sigma_1 \boxtimes \Sigma_2) = k = \operatorname{bdim}(\Sigma_1)$.
Similar is the proof of the next part. □

3. Conclusion and Scope

In this paper, we have studied the properties of balancing dimension of various products of signed graphs, namely, the Cartesian product, the lexicographic product, the tensor product, and the strong product. We found the relationship between the balancing dimensions of various signed graph products and their factors, provided one of them is balanced or all-positive. We also computed the balancing dimensions of the Cartesian product of unbalanced cycles, the Cartesian product of antibalanced signed complete graphs, and the lexicographic product of antibalanced signed complete graphs. We also proved some results on switching equivalence in the case of the lexicographic products and the strong product. Finding the balancing dimensions of products of general unbalanced signed graphs, finding the relation between balancing dimensions of signed graph products and their factors, and studying properties of balancing dimensions of other existing products of signed graphs are some exciting areas for further investigation.

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