Communications in Combinatorics and Optimization
Vol. ??, No. ?? (????), pp. 1-13
DOI:10.22049/cco.2023.28238.1486

CCO
Commun. Comb. Optim.

Research Article

On Zero-Divisor Graph of the ring $\mathbb{F}_p + u\mathbb{F}_p + u^2\mathbb{F}_p$

N. Annamalai*

Department of Basic Engineering, Lecturer in Mathematics, Government Polytechnic College, Sankarapuram, Kallakurichi-606401, Tamil Nadu, India
algebra.annamalai@gmail.com

Received: 10 January 2023; Accepted: 5 October 2023
Published Online: 15 October 2023

Abstract: In this article, we discussed the zero-divisor graph of a commutative ring with identity $\mathbb{F}_p + u\mathbb{F}_p + u^2\mathbb{F}_p$ where $u^3 = 0$ and p is an odd prime. We find the clique number, chromatic number, vertex connectivity, edge connectivity, diameter and girth of a zero-divisor graph associated with the ring. We find some of topological indices and the main parameters of the code derived from the incidence matrix of the zero-divisor graph $\Gamma(R)$. Also, we find the eigenvalues, energy and spectral radius of both adjacency and Laplacian matrices of $\Gamma(R)$.

Keywords: Zero-divisor graph, Laplacian matrix, Spectral radius

AMS Subject classification: 05C09, 05C40, 05C50

1. Introduction

The zero-divisor graph has attracted a lot of attention in the last few years. In 1988, Beck [6] introduced the zero-divisor graph. He included the additive identity of a ring R in the definition and was mainly interested in the coloring of commutative rings. Let Γ be a simple graph whose vertices are the set of zero-divisors of the ring R, and two distinct vertices are adjacent if the product is zero. Later it was modified by Anderson and Livingston [1]. They redefined the definition as a simple graph that only considers the non-zero zero-divisors of a commutative ring R.
Let R be a commutative ring with identity and $Z(R)$ be the set of zero-divisors of R. The zero-divisor graph $\Gamma(R)$ of a ring R is an undirected graph whose vertices are the non-zero zero-divisors of R with two distinct vertices x and y are adjacent if and only if $xy = 0$. In this article, we consider the zero-divisor graph $\Gamma(R)$ as a graph with

* *Corresponding Author*

vertex set $Z^*(R)$ the set of non-zero zero-divisors of the ring R. Many researchers are doing research in this area [9, 11, 13, 14].

Let $\Gamma = (V, E)$ be a simple undirected graph with vertex set V, edge set E. The incidence matrix of a graph Γ is a $|V| \times |E|$ matrix $Q(\Gamma)$ whose rows are labelled by the vertices and columns by the edges and entries $q_{ij} = 1$ if the vertex labelled by row i is incident with the edge labelled by column j and $q_{ij} = 0$ otherwise.

The adjacency matrix $A(\Gamma)$ of the graph Γ, is the $|V| \times |V|$ matrix defined as follows. The rows and the columns of $A(\Gamma)$ are indexed by V. If $i \neq j$ then the (i, j)-entry of $A(\Gamma)$ is 0 for vertices i and j which are nonadjacent, and the (i, j)-entry is 1 for i and j which are adjacent. The (i, i)-entry of $A(\Gamma)$ is 0 for $i = 1, \ldots, |V|$. For any graph Γ, the energy of the graph is defined as

$$\varepsilon(\Gamma) = \sum_{i=1}^{|V|} |\lambda_i|,$$

where $\lambda_1, \ldots, \lambda_{|V|}$ are the eigenvalues of $A(\Gamma)$ of Γ.

The Laplacian matrix $L(\Gamma)$ of Γ is the $|V| \times |V|$ matrix defined as follows. The rows and columns of $L(\Gamma)$ are indexed by V. If $i \neq j$ then the (i, j)-entry of $L(\Gamma)$ is 0 if vertex i and j are not adjacent, and it is -1 if i and j are adjacent. The (i, i)-entry of $L(\Gamma)$ is d_i, the degree of the vertex i, $i = 1, 2, \ldots, |V|$. Let $D(\Gamma)$ be the diagonal matrix of vertex degrees. If $A(\Gamma)$ is the adjacency matrix of Γ, then note that $L(\Gamma) = D(\Gamma) - A(\Gamma)$. Let $\mu_1, \mu_2, \ldots, \mu_{|V|}$ are eigenvalues of $L(\Gamma)$. Then the Laplacian energy $LE(\Gamma)$ is given by

$$LE(\Gamma) = \sum_{i=1}^{|V|} \left| \mu_i - \frac{2|E|}{|V|} \right|.$$

Lemma 1. *[5] Let $\Gamma = (V, E)$ be a graph, and let $0 = \mu_1 \leq \mu_2 \leq \cdots \leq \mu_{|V|}$ be the eigenvalues of its Laplacian matrix $L(\Gamma)$. Then, $\mu_2 > 0$ if and only if Γ is connected.*

The Wiener index of a connected graph Γ is defined as the sum of distances between each pair of vertices, i.e.,

$$W(\Gamma) = \sum_{\substack{a,b \in V \\ a \neq b}} d(a, b),$$

where $d(a, b)$ is the length of shortest path joining a and b.

The degree of $v \in V$, denoted by d_v, is the number of vertices adjacent to v.

The Randić index (also known under the name connectivity index) is a much investigated degree-based topological index. It was invented in 1976 by Milan Randić [12] and is defined as

$$R(\Gamma) = \sum_{(a,b) \in E} \frac{1}{\sqrt{d_a d_b}}$$

with summation going over all pairs of adjacent vertices of the graph.

The Zagreb indices were introduced more than 50 years ago by Gutman and Trinajestić [8]. For a graph Γ, the first Zagreb index $M_1(\Gamma)$ and the second Zagreb index $M_2(\Gamma)$ are, respectively, defined as follows:

$$M_1(\Gamma) = \sum_{a \in V} d_a^2$$

$$M_2(\Gamma) = \sum_{(a,b) \in E} d_a d_b.$$

An edge-cut of a connected graph Γ is the set $S \subseteq E$ such that $\Gamma - S = (V, E - S)$ is disconnected. The edge-connectivity $\lambda(\Gamma)$ is the minimum cardinality of an edge-cut. The minimum k for which there exists a k-vertex cut is called the vertex connectivity or simply the connectivity of Γ it is denoted by $\kappa(\Gamma)$.
For any connected graph Γ, we have $\lambda(\Gamma) \leq \delta(\Gamma)$ where $\delta(\Gamma)$ is minimum degree of the graph Γ.
The chromatic number of a graph Γ is the minimum number of colors needed to color the vertices of Γ so that adjacent vertices of Γ receive distinct colors and is denoted by $\chi(\Gamma)$. A clique of a graph Γ is a complete subgraph of Γ. The clique number $\omega(\Gamma)$ of a graph Γ is the number of vertices in a maximum clique of Γ. Note that for any graph Γ, $\omega(\Gamma) \leq \chi(\Gamma)$. The girth of an undirected graph is the length of a shortest cycle contained in the graph.
Beck[6] conjectured that if R is a finite chromatic ring, then $\omega(\Gamma(R)) = \chi(\Gamma(R))$ where $\omega(\Gamma(R)), \chi(\Gamma(R))$ are the clique number and the chromatic number of $\Gamma(R)$, respectively. He also verified that the conjecture is true for several examples of rings. Anderson and Naseer, in [1], disproved the above conjecture with a counterexample. $\omega(\Gamma(R))$ and $\chi(\Gamma(R))$ of the zero-divisor graph associated to the ring $\mathbb{F}_p + u\mathbb{F}_p + u^2\mathbb{F}_p$ are same. For basic graph theory, one can refer [4, 5].
Let $\mathbb{F}_q$ be a finite field with q elements. Let $x = (x_1, \ldots, x_n) \in \mathbb{F}_q^n$, then the Hamming weight $w_H(x)$ of x is defined by the number of non-zero coordinates in x. Let $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{F}_q^n$, the Hamming distance $d_H(x, y)$ between x and y is defined by the number of coordinates in which they differ.
A q-ary code of length n is a non-empty subset C of $\mathbb{F}_q^n$. If C is a subspace of $\mathbb{F}_q^n$, then C is called a q-ary linear code of length n. An element of C is called a *codeword*. The minimum Hamming distance of a code C is defined by

$$d_H(C) = \min\{d_H(c_1, c_2) \mid c_1 \neq c_2, c_1, c_2 \in C\}.$$

The minimum weight $w_H(C)$ of a code C is the smallest among all weights of the non-zero codewords of C. For q-ary linear code, we have $d_H(C) = w_H(C)$. For basic coding theory, we refer [10].
A linear code of length n, dimension k and minimum distance d is denoted by $[n, k, d]_q$. The code generated by the rows of the incidence matrix $Q(\Gamma)$ of the graph Γ is denoted by $C_p(\Gamma)$ over the finite field $\mathbb{F}_p$.

Theorem 1. *[7]*

1. *Let $\Gamma = (V, E)$ be a connected graph and let G be a $|V| \times |E|$ incidence matrix for Γ. Then, the main parameters of the code $C_2(G)$ is $[|E|, |V| - 1, \lambda(\Gamma)]_2$.*

2. *Let $\Gamma = (V, E)$ be a connected bipartite graph and let G be a $|V| \times |E|$ incidence matrix for Γ. Then the incidence matrix generates $[|E|, |V| - 1, \lambda(\Gamma)]_p$ code for odd prime p.*

Codes from the row span of incidence matrix or adjacency matrix of various graphs are studied in [2, 3, 7, 15, 16].

Let p be an odd prime. The ring $\mathbb{F}_p + u\mathbb{F}_p + u^2\mathbb{F}_p$ is defined as a characteristic p ring subject to restrictions $u^3 = 0$. The ring isomorphism $\mathbb{F}_p + u\mathbb{F}_p + u^2\mathbb{F}_p \cong \frac{\mathbb{F}_p[x]}{\langle x^3 \rangle}$ is obvious to see. An element $a + ub + u^2c \in R$ is unit if and only if $a \neq 0$.

Throughout this article, we denote the ring $\mathbb{F}_p + u\mathbb{F}_p + u^2\mathbb{F}_p$ by R. In this article, we discussed the zero-divisor graph of a commutative ring with identity $\mathbb{F}_p + u\mathbb{F}_p + u^2\mathbb{F}_p$ where $u^3 = 0$ and we find the clique number, chromatic number, vertex connectivity, edge connectivity, diameter, and girth of the graph $\Gamma(R)$, in Section 2. In Section 3, we find some of topological indices of $\Gamma(R)$. In Section 4, we find the main parameters of the code derived from incidence matrix of the zero-divisor graph $\Gamma(R)$. Finally, We find the eigenvalues, energy and spectral radius of both adjacency and Laplacian matrices in Section 5.

2. Zero-divisor graph $\Gamma(R)$ of the ring R

In this section, we discuss the zero-divisor graph $\Gamma(R)$ of the ring R and we find the clique number, chromatic number, vertex connectivity, edge connectivity, diameter, and girth of the graph $\Gamma(R)$.

Let $A_u = \{xu \mid x \in \mathbb{F}_p^*\}$, $A_{u^2} = \{xu^2 \mid x \in \mathbb{F}_p^*\}$ and $A_{u+u^2} = \{xu + yu^2 \mid x, y \in \mathbb{F}_p^*\}$. Then $|A_u| = (p-1)$, $|A_{u^2}| = (p-1)$ and $|A_{u+u^2}| = (p-1)^2$. Therefore, $Z^*(R) = A_u \cup A_{u^2} \cup A_{u+u^2}$ and $|Z^*(R)| = |A_u| + |A_{u^2}| + |A_{u+u^2}| = (p-1) + (p-1) + (p-1)^2 = p^2 - 1$. As $u^3 = 0$, every vertices of A_u is adjacent with every vertices of A_{u^2}, every vertices

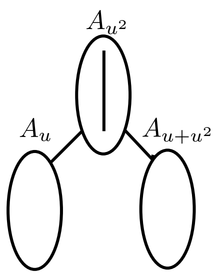

Figure 1. **Zero-divisor graph of** $R = \mathbb{F}_p + u\mathbb{F}_p + u^2\mathbb{F}_p$

of A_{u^2} is adjacent with every vertices of A_{u+u^2} and any two distinct vertices of A_{u^2}

are adjacent. From the diagram, the graph $\Gamma(R)$ is connected with p^2-1 vertices and $(p-1)^2+(p-1)^3+\frac{(p-1)(p-2)}{2}=\frac{1}{2}(2p^3-3p^2-p+2)$ edges.

Example 1. For $p=3$, $R=\mathbb{F}_3+u\mathbb{F}_3+u^2\mathbb{F}_3$. Then $A_u=\{u,2u\}$, $A_{u^2}=\{u^2,2u^2\}$, $A_{u+u^2}=\{u+u^2,2u+2u^2,u+2u^2,2u+u^2\}$. The number of vertices is 8 and the number

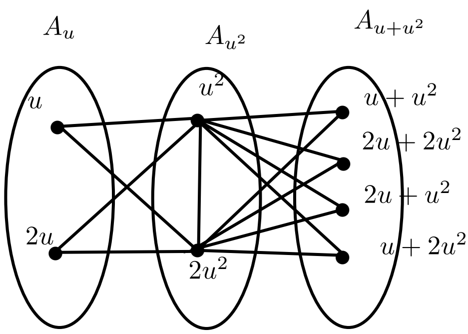

Figure 2. **Zero-divisor graph of** $R=\mathbb{F}_3+u\mathbb{F}_3+u^2\mathbb{F}_3$

of edges is 13.

Theorem 2. *The diameter of the zero-divisor graph* $diam(\Gamma(R))=2$.

Proof. From the Figure 1, we can see that the distance between any two distinct vertices are either 1 or 2. Therefore, the maximum of distance between any two distinct vertices is 2. Hence, $diam(\Gamma(R))=2$. □

Theorem 3. *The clique number* $\omega(\Gamma(R))$ *of* $\Gamma(R)$ *is* p.

Proof. From the Figure 1, A_{u^2} is a complete subgraph(clique) in $\Gamma(R)$. If we add exactly one vertex v from either A_u or A_{u+u^2}, then resulting subgraph form a complete subgraph(clique). Then $A_{u^2}\cup\{v\}$ forms a complete subgraph with maximum vertices. Therefore, the clique number of $\Gamma(R)$ is $\omega(\Gamma(R))=|A_{u^2}\cup\{v\}|=p-1+1=p$. □

Theorem 4. *The chromatic number* $\chi(\Gamma(R))$ *of* $\Gamma(R)$ *is* p.

Proof. Since A_{u^2} is a complete subgraph with $p-1$ vertices in $\Gamma(R)$, then at least $p-1$ different colors needed to color the vertices of A_{u^2}. And no two vertices in A_u are adjacent then one color different from previous $p-1$ colors is enough to color all vertices in A_u. We take the same color in A_u to color vertices of A_{u+u^2} as there is no direct edge between A_u and A_{u+u^2}. Therefore, minimum p different colors required for proper coloring. Hence, the chromatic number $\chi(\Gamma(R))$ is p. □

The above two theorems show that the clique number and the chromatic number of our graph are same.

Theorem 5. *The girth of the graph* $\Gamma(R)$ *is 3.*

Proof. Since $p \geq 3$, we have $\Gamma(R)$ contains a cycle of length 3. Hence, the result follows from the definition of girth. □

Theorem 6. *The vertex connectivity* $\kappa(\Gamma(R))$ *of* $\Gamma(R)$ *is* $p-1$.

Proof. As the minimum degree $\delta(\Gamma(R))$ of $\Gamma(R)$ is $p-1$, $\kappa(\Gamma(R)) \leq \delta(\Gamma(R)) = p-1$. Note that, every vertex of $A_u \cup A_{u+u^2}$ is adjacent to every vertex of A_{u^2}. Hence there is no vertex cut of cardinality $p-2$ and therefore the result follows. □

Theorem 7. *The edge connectivity* $\lambda(\Gamma(R))$ *of* $\Gamma(R)$ *is* $p-1$.

Proof. As $\Gamma(R)$ connected graph, $\kappa(\Gamma(R)) \leq \lambda(\Gamma(R)) \leq \delta(\Gamma(R))$. Since $\kappa(\Gamma(R)) = p-1$ and $\delta(\Gamma(R)) = p-1$, then $\lambda(\Gamma(R)) = p-1$. □

3. Some Topological Indices of $\Gamma(R)$

In this section, we find the Wiener index, first Zagreb index, second Zagreb index and Randić index of the zero divisor graph $\Gamma(R)$.

Theorem 8. *The Wiener index of the zero-divisor graph* $\Gamma(R)$ *of* R *is* $W(\Gamma(R)) = \frac{p(2p^3-2p^2-7p+5)}{2}$.

Proof. Consider,

$$\begin{aligned}
W(\Gamma(R)) &= \sum_{\substack{x,y \in Z^*(R) \\ x \neq y}} d(x,y) \\
&= \sum_{\substack{x,y \in A_u \\ x \neq y}} d(x,y) + \sum_{\substack{x,y \in A_{u^2} \\ x \neq y}} d(x,y) + \sum_{\substack{x,y \in A_{u+u^2} \\ x \neq y}} d(x,y) \\
&\quad + \sum_{\substack{x \in A_u \\ y \in A_{u^2}}} d(x,y) + \sum_{\substack{x \in A_u \\ y \in A_{u+u^2}}} d(x,y) + \sum_{\substack{x \in A_{u^2} \\ y \in A_{u+u^2}}} d(x,y) \\
&= (p-1)(p-2) + \frac{(p-1)(p-2)}{2} + p(p-2)(p-1)^2 \\
&\quad + (p-1)^2 + 2(p-1)^3 + (p-1)^3 \\
&= (p-1)^2 + 3(p-1)^3 + \frac{(p-1)(p-2)}{2} + (p-1)(p-2)(p^2-p+1) \\
&= \frac{p(2p^3-2p^2-7p+5)}{2}.
\end{aligned}$$

□

Denote $[A,B]$ be the set of edges between the subset A and B of V. For any $a \in A_u$, $d_a = p-1$, for any $a \in A_{u^2}, d_a = p^2-2$ and any $a \in A_{u+u^2}, d_a = p-1$.

Theorem 9. *The Randić index of the zero-divisor graph $\Gamma(R)$ of R is*

$$R(\Gamma(R)) = \frac{(p-1)}{2(p^2-2)}\left[2p\sqrt{(p-1)(p^2-2)} + (p-2)\right].$$

Proof. Consider,

$$\begin{aligned}
R(\Gamma(R)) &= \sum_{(a,b)\in E} \frac{1}{\sqrt{d_a d_b}} \\
&= \sum_{(a,b)\in[A_u,A_{u^2}]} \frac{1}{\sqrt{d_a d_b}} + \sum_{(a,b)\in[A_{u^2},A_{u^2}]} \frac{1}{\sqrt{d_a d_b}} + \sum_{(a,b)\in[A_{u^2},A_{u+u^2}]} \frac{1}{\sqrt{d_a d_b}} \\
&= (p-1)^2 \frac{1}{\sqrt{(p-1)(p^2-2)}} + \frac{(p-1)(p-2)}{2}\frac{1}{\sqrt{(p^2-2)(p^2-2)}} \\
&\quad + (p-1)^3 \frac{1}{\sqrt{(p^2-2)(p-1)}} \\
&= \frac{(p-1)^2}{\sqrt{(p-1)(p-2)}}[p(p-1)] + \frac{(p-1)(p-2)}{2(p^2-2)} \\
&= \frac{p(p-1)^2}{\sqrt{(p-1)(p^2-2)}} + \frac{(p-1)(p-2)}{2(p^2-2)} \\
&= \frac{(p-1)}{2(p^2-2)}\left[2p\sqrt{(p-1)(p^2-2)} + (p-2)\right]
\end{aligned}$$

□

Theorem 10. *The first Zagreb index of the zero-divisor graph $\Gamma(R)$ of R is $M_1(\Gamma(R)) = (p-1)[p^4+p^3-4p^2+p+4]$.*

Proof. Consider,

$$\begin{aligned}
M_1(\Gamma(R)) &= \sum_{a\in Z^*(R)} d_a^2 \\
&= \sum_{a\in A_u} d_a^2 + \sum_{a\in A_{u^2}} d_a^2 + \sum_{a\in A_{u+u^2}} d_a^2 \\
&= (p-1)(p-1)^2 + (p-1)(p^2-2)^2 + (p-1)^2(p-1)^2 \\
&= (p-1)^3 + (p-1)^4 + (p^2-2)^2(p-1) \\
&= p(p-1)^3 + (p-1)(p^2-2) \\
&= (p-1)[p^4+p^3-4p^2+p+4].
\end{aligned}$$

□

Theorem 11. *The second Zagreb index of the zero-divisor graph $\Gamma(R)$ of R is*

$$M_2(\Gamma(R)) = \frac{1}{2}[3p^6 - 9p^5 + 22p^3 - 16p^2 - 8p + 8].$$

Proof. Consider,

$$\begin{aligned}
M_2(\Gamma(R)) &= \sum_{(a,b)\in E} d_a d_b \\
&= \sum_{(a,b)\in[A_u, A_{u^2}]} d_a d_b + \sum_{(a,b)\in[A_{u^2}, A_{u^2}]} d_a d_b + \sum_{(a,b)\in[A_{u^2}, A_{u+u^2}]} d_a d_b \\
&= (p-1)^2(p-1)(p^2-2) + \frac{(p-1)(p-2)}{2}(p^2-2)(p^2-2) \\
&\qquad + (p-1)^3(p^2-2)(p-1) \\
&= \frac{(p-1)(p^2-2)}{2}[3p^3 - 6p^2 + 4] \\
&= \frac{1}{2}[3p^6 - 9p^5 + 22p^3 - 16p^2 - 8p + 8].
\end{aligned}$$

□

4. Codes from Incidence Matrix of $\Gamma(R)$

In this section, we find the incidence matrix of the graph $\Gamma(R)$ and we find the parameters of the linear code generated by the rows of incidence matrix $Q(\Gamma(R))$. The incidence matrix $Q(\Gamma(R))$ is given below

$$Q(\Gamma(R)) = \begin{matrix} \\ A_u \\ A_{u^2} \\ A_{u+u^2} \end{matrix} \begin{matrix} [A_u, A_{u^2}] & [A_{u^2}, A_{u^2}] & [A_{u^2}, A_{u+u^2}] \\ \left(\begin{matrix} D^{(p-1)}_{(p-1)\times(p-1)^2} \\ J_{(p-1)\times(p-1)^2} \\ \mathbf{0}_{(p-1)^2\times(p-1)^2} \end{matrix}\right. & \begin{matrix} \mathbf{0}_{(p-1)\times\frac{(p-1)(p-2)}{2}} \\ J_{(p-1)\times\frac{(p-1)(p-2)}{2}} \\ \mathbf{0}_{(p-1)^2\times\frac{(p-1)(p-2)}{2}} \end{matrix} & \left.\begin{matrix} \mathbf{0}_{(p-1)\times(p-1)^3} \\ J_{(p-1)\times(p-1)^3} \\ D^{(p-1)}_{(p-1)^2\times(p-1)^3} \end{matrix}\right) \end{matrix},$$

where J is a all one matrix, $\mathbf{0}$ is a zero matrix with appropriate order, $\mathbf{1}_{(p-1)}$ is a all one $1\times(p-1)$ row vector and $D^{(p-1)}_{k\times l} = \begin{pmatrix} \mathbf{1}_{(p-1)} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{(p-1)} & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{1}_{(p-1)} \end{pmatrix}_{k\times l}$.

Example 2. The incidence matrix of the zero-divisor graph $\Gamma(R)$ given in the Example 1 is

$$Q(\Gamma(R)) = \begin{matrix} u \\ 2u \\ u^2 \\ 2u^2 \\ u+u^2 \\ 2u+2u^2 \\ 2u+u^2 \\ u+2u^2 \end{matrix} \left(\begin{array}{cccc|c|cccccccc}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{array}\right)_{8\times 13}.$$

The number of linearly independent rows is 7 and hence the rank of the matrix $Q(\Gamma(R))$ is 7. The rows of the incidence matrix $Q(\Gamma(R))$ is generate a $[n = 13, k = 7, d = 2]_2$ code over $\mathbb{F}_2$.

The edge connectivity of the zero-divisor graph $\Gamma(R)$ is $p-1$, then we have the following theorem:

Theorem 12. *The linear code generated by the incidence matrix $Q(\Gamma(R))$ of the zero-divisor graph $\Gamma(R)$ is a $C_2(\Gamma(R)) = [\frac{1}{2}(2p^3 - 3p^2 - p + 2), p^2 - 2, p - 1]_2$ linear code over the finite field $\mathbb{F}_2$.*

5. Adjacency and Laplacian Matrices of $\Gamma(R)$

In this section, we find the eigenvalues, energy and spectral radius of both adjacency and Laplacian matrices of $\Gamma(R)$.
If μ is an eigenvalue of matrix A then $\mu^{(k)}$ means that μ is an eigenvalue with multiplicity k.
The vertex set partition into A_u, A_{u^2} and A_{u+u^2} of cardinality $p-1, p-1$ and $(p-1)^2$, respectively. Then the adjacency matrix of $\Gamma(R)$ is

$$A(\Gamma(R)) = \begin{array}{c} \\ A_u \\ A_{u^2} \\ A_{u+u^2} \end{array} \begin{array}{c} \begin{array}{ccc} A_u & A_{u^2} & A_{u+u^2} \end{array} \\ \begin{pmatrix} \mathbf{0}_{p-1} & J_{p-1} & \mathbf{0}_{(p-1)\times(p-1)^2} \\ J_{p-1} & J_{p-1} - I_{p-1} & J_{(p-1)\times(p-1)^2} \\ \mathbf{0}_{(p-1)^2\times(p-1)} & J_{(p-1)^2\times(p-1)} & \mathbf{0}_{(p-1)^2} \end{pmatrix} \end{array},$$

where J_k is an $k \times k$ all one matrix, $J_{n\times m}$ is an $n \times m$ all matrix, $\mathbf{0}_k$ is an $k \times k$ zero matrix, $\mathbf{0}_{n\times m}$ is an $n \times m$ zero matrix and I_k is an $k \times k$ identity matrix.
All the rows in A_{u^2} are linearly independent and all the rows in A_u and A_{u+u^2} are linearly dependent. Therefore, $p - 1 + 1 = p$ rows are linearly independent. So, the rank of $A(\Gamma(R))$ is p. By Rank-Nullity theorem, nullity of $A(\Gamma(R)) = p^2 - p - 1$. Hence, zero is an eigenvalue with multiplicity $p^2 - p - 1$.
For $p = 3$, the adjacency matrix of $\Gamma(R)$ is

$$A(\Gamma(R)) = \left(\begin{array}{cc|cc|cccc} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{array}\right)_{8\times 8}.$$

The eigenvalues of $A(\Gamma(R))$ are $0^{(5)}, 4^{(1)}, (-1)^{(1)}$ and $(-3)^{(1)}$. For $p = 5$, the eigenvalues of $A(\Gamma(R))$ are $0^{(19)}, 10^{(1)}, (-1)^{(3)}$ and $(-7)^{(1)}$.

Theorem 13. *The energy of the adjacency matrix $A(\Gamma(R))$ is $\varepsilon(\Gamma(R)) = 6p - 10$.*

Proof. For any odd prime p, the eigenvalues of $A(\Gamma(R))$ are $0^{(p^2-p-1)}$, $(3p-5)^{(1)}$, $(-1)^{(p-2)}$, $(3-2p)^{(1)}$. The energy of adjacency matrix $A(\Gamma(R))$ is the sum of the absolute values of all eigenvalues of $A(\Gamma(R))$. That is,

$$\begin{aligned}\varepsilon(\Gamma(R)) &= \sum_{i=1}^{p^2-1} |\lambda_i| \qquad \text{where } \lambda_i\text{'s are eigenvalues of } A(\Gamma(R))\\ &= |3p-5| + (p-2)|-1| + |3-2p|\\ &= 3p-5+p-2+2p-3 \qquad \text{since } p>2\\ &= 6p-10.\end{aligned}$$

□

The degree matrix of the graph $\Gamma(R)$ is

$$D(\Gamma(R)) = \begin{matrix} & A_u & A_{u^2} & A_{u+u^2} \\ A_u \\ A_{u^2} \\ A_{u+u^2} \end{matrix} \begin{pmatrix} (p-1)I_{p-1} & \mathbf{0}_{p-1} & \mathbf{0}_{(p-1)\times(p-1)^2} \\ \mathbf{0}_{p-1} & (p^2-2)I_{p-1} & \mathbf{0}_{(p-1)\times(p-1)^2} \\ \mathbf{0}_{(p-1)^2\times(p-1)} & \mathbf{0}_{(p-1)^2\times(p-1)} & (p-1)I_{(p-1)^2} \end{pmatrix}.$$

The Laplacian matrix $L(\Gamma(R))$ of $\Gamma(R)$ is defined by $L(\Gamma(R)) = D(\Gamma(R)) - A(\Gamma(R))$. Therefore,

$$L(\Gamma(R)) = \begin{matrix} & A_u & A_{u^2} & A_{u+u^2} \\ A_u \\ A_{u^2} \\ A_{u+u^2} \end{matrix} \begin{pmatrix} (p-1)I_{p-1} & -J_{p-1} & \mathbf{0}_{(p-1)\times(p-1)^2} \\ -J_{p-1} & (p^2-1)I_{p-1} - J_{p-1} & -J_{(p-1)\times(p-1)^2} \\ \mathbf{0}_{(p-1)^2\times(p-1)} & -J_{(p-1)^2\times(p-1)} & (p-1)I_{(p-1)^2} \end{pmatrix}.$$

Since each row sum is zero, zero is one of the eigenvalues of $L(\Gamma(R))$. By Lemma 1, the second smallest eigenvalue of $L(\Gamma(R))$ is positive as $\Gamma(R)$ is connected. Hence zero is an eigenvalue with multiplicity one, and all other eigenvalues are positive.
For $p = 3$, the Laplacian matrix is

$$L(\Gamma(R)) = \begin{pmatrix} 2 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 7 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 7 & -1 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 2 \end{pmatrix}_{8\times 8}.$$

The eigenvalues of $L(\Gamma(R))$ are $0^{(1)}, 8^{(2)}, 2^{(5)}$.
For $p = 5$, the eigenvalues of $L(\Gamma(R))$ are $0^{(1)}, 24^{(4)}, 4^{(19)}$.
For any prime p, the eigenvalues of $L(\Gamma(R))$ are $0^{(1)}, (p^2-1)^{(p-1)}, (p-1)^{(p^2-p-1)}$.

Theorem 14. *The Laplacian energy of* $\Gamma(R)$ *is* $LE(\Gamma(R)) = \dfrac{2p^5 - 6p^4 + 6p^3 - 4p + 1}{p^2-1}$.

Proof. Let $|V| = n$ and $|E| = m$. Let $\mu_1, \mu_2, \ldots, \mu_n$ are eigenvalues of $L(\Gamma(R))$. Then the Laplacian energy $LE(\Gamma(R))$ is given by

$$LE(\Gamma(R)) = \sum_{i=1}^{n} \Big|\mu_i - \frac{2m}{n}\Big|.$$

We know that the eigenvalues of $L(\Gamma(R))$ are $0^{(1)}, (p^2-1)^{(p-1)}, (p-1)^{(p^2-p-1)}$. Then

$$\begin{aligned}
LE(\Gamma(R)) &= \sum_{i=1}^{n} \Big|\mu_i - \frac{2m}{n}\Big| \\
&= \sum_{i=1}^{n} \Big|\mu_i - \frac{2p^3-3p^2-p+2}{p^2-1}\Big| \\
&= \Big|0 - \frac{2p^3-3p^2-p+2}{p^2-1}\Big| + (p-1)\Big|(p^2-1) - \frac{2p^3-3p^2-p+2}{p^2-1}\Big| \\
&\qquad + (p^2-p-1)\Big|(p-1) - \frac{2p^3-3p^2-p+2}{p^2-1}\Big| \\
&= \frac{2p^5-6p^4+6p^3-4p+1}{p^2-1} \qquad \text{since } p \geq 2.
\end{aligned}$$

□

We denote by $\rho(\Gamma(R))$ the largest eigenvalue in absolute of $A(\Gamma(R))$ and call it the spectral radius of $\Gamma(R)$; we denote by $\mu(\Gamma(R))$ the largest eigenvalue in absolute of $L(\Gamma(R))$ and call it the Laplacian spectral radius of $\Gamma(R)$.

Theorem 15. *For any odd prime p, $\rho(\Gamma(R)) = 3p-5$ and $\mu(\Gamma(R)) = p^2-1$.*

Proof. The eigenvalues of the adjacency matrix $A(\Gamma(R))$ are $0^{(p^2-p-1)}$, $(3p-5)^{(1)}$, $(-1)^{(p-2)}$ and $(3-2p)^{(1)}$. Then the largest eigenvalue in absolute is $3p-5$ as $p > 2$. That is, $\rho(\Gamma(R)) = 3p-5$.
The eigenvalues of the Laplacian matrix $L(\Gamma(R))$ are $0^{(1)}$, $(p^2-1)^{(p-1)}$ and $(p-1)^{(p^2-p-1)}$. Then the largest eigenvalue in absolute is p^2-1. That is, $\mu(\Gamma(R)) = p^2-1$. □

Conclusion

In this article, we discussed the zero-divisor graph of a commutative ring with identity $\mathbb{F}_p + u\mathbb{F}_p + u^2\mathbb{F}_p$ where $u^3 = 0$ and p is an odd prime. We find the clique number, chromatic number, vertex connectivity, edge connectivity, diameter and girth of a zero-divisor graph associated with the ring. We find some of topological indices and the main parameters of the code derived from the incidence matrix of the zero-divisor graph $\Gamma(R)$. Also, we find the eigenvalues, energy and spectral radius of both adjacency and Laplacian matrices of $\Gamma(R)$.

Acknowledgements. The author thanks Satya Bagchi and C. Durairajan for their help to improving the presentation of the paper.

Conflict of interest. The authors declare that they have no conflict of interest.

Data Availability. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

References

[1] D. F. Anderson, A. Frazier, A. Lauve, and P. S. Livingston, *The zero-divisor graph of a commutative ring, II*, Ideal theoretic methods in commutative algebra, CRC Press, 2019, pp. 61–72
http://dx.doi.org/10.1201/9780429187902–5.

[2] N. Annamalai and C. Durairajan, *Linear codes from incidence matrices of unit graphs*, J. Inf. Optim. Sci. **42** (2021), no. 8, 1943–1950
https://doi.org/10.1080/02522667.2021.1972617.

[3] ______, *Codes from the incidence matrices of a zero-divisor graphs*, J. Discrete Math. Sci. Cryptogr. (In press), https://doi.org/10.1080/09720529.2021.1939955.

[4] R. Balakrishnan and K. Ranganathan, *A Textbook of Graph Theory*, Springer, New York, 2012.

[5] R.B. Bapat, *Graphs and Matrices*, Springer, London, 2014.

[6] I. Beck, *Coloring of commutative rings*, J. Algebra **116** (1988), no. 1, 208–226
https://doi.org/10.1016/0021–8693(88)90202–5.

[7] P. Dankelmann, J.D. Key, and B.G. Rodrigues, *Codes from incidence matrices of graphs*, Designs, Codes and Cryptography **68** (2013), no. 1, 373–393
https://doi.org/10.1007/s10623–011–9594–x.

[8] I. Gutman and N. Trinajstić, *Graph theory and molecular orbitals. total φ-electron energy of alternant hydrocarbons*, Chem. Phy. Lett. **17** (1972), no. 4, 535–538
https://doi.org/10.1016/0009–2614(72)85099–1.

[9] T. Kavaskar, *Beck's coloring of finite product of commutative ring with unity*, Graphs Combin. **38** (2022), no. 2, 1–9
https://doi.org/10.1007/s00373–021–02401–x.

[10] S. Ling and C. Xing, *Coding Theory: A First Course*, Cambridge University Press, Cambridge, 2004.

[11] A. Mukhtar, R. Murtaza, S.U. Rehman, S. Usman, and A.Q. Baig, *Computing the size of zero divisor graphs*, J. Inf. Optim. Sci. **41** (2020), no. 4, 855–864
https://doi.org/10.1080/02522667.2020.1745378.

[12] M. Randić, *On chracterization of molecular branching*, J. Am. Chem. Soc. **97**, no. 23, 6609–6615
https://doi.org/10.1021/ja00856a001.

[13] B.S. Reddy, R.S. Jain, and N. Laxmikanth, *Vertex and edge connectivity of the zero divisor graph* $\gamma[\mathbb{Z}_n]$, Comm. Math. Appl. **11** (2020), no. 2, 253–258 https://doi.org/10.26713/cma.v11i2.1319.

[14] S.P. Redmond, *The zero-divisor graph of a non-commutative ring*, Internat. J. Commut. Rings **1** (2002), no. 4, 203–211.

[15] R. Saranya and C. Durairajan, *Codes from incidence matrices of some regular graphs*, Discrete Math. Algorithms Appl. **13** (2021), no. 4, Article ID: 2150035 https://doi.org/10.1142/S179383092150035X.

[16] ———, *Codes from incidence matrices of* $(n, 1)$*-arrangement graphs and* $(n, 2)$*-arrangement graphs*, J. Discrete Math. Sci. Cryptogr. **25** (2022), no. 2, 373–393 https://doi.org/10.1080/09720529.2019.1681674.