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Research Article

Finite abelian groups with isomorphic inclusion graphs of cyclic subgroups

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Abstract: Let G be a finite group. The directed inclusion graph of cyclic subgroups of G, $\overrightarrow{\mathcal{I}_c}(G)$, is the digraph with vertices of all cyclic subgroups of G, and for two distinct cyclic subgroups $\langle a\rangle$ and $\langle b\rangle$, there is an arc from $\langle a\rangle$ to $\langle b\rangle$ if and only if $\langle b\rangle \subset \langle a\rangle$. The (undirected) inclusion graph of cyclic subgroups of G, $\mathcal{I}_c(G)$, is the underlying graph of $\overrightarrow{\mathcal{I}_c}(G)$, that is, the vertex set is the set of all cyclic subgroups of G and two distinct cyclic subgroups $\langle a\rangle$ and $\langle b\rangle$ are adjacent if and only if $\langle a\rangle \subset \langle b\rangle$ or $\langle b\rangle \subset \langle a\rangle$. In this paper, we first show that, if G and H are finite groups such that $\mathcal{I}_c(G) \cong \mathcal{I}_c(H)$ and G is cyclic, then H is cyclic. We show that for two cyclic groups G and H of orders $p_1^{\alpha_1}\dots p_t^{\alpha_t}$ and $q_1^{\beta_1}\dots q_s^{\beta_s}$, respectively, $\mathcal{I}_c(G) \cong \mathcal{I}_c(H)$ if and only if $t = s$ and by a suitable σ, $\alpha_i = \beta_{\sigma(i)}$. Also for any cyclic groups G, H, if $\mathcal{I}_c(G) \cong \mathcal{I}_c(H)$, then $\overrightarrow{\mathcal{I}_c}(G) \cong \overrightarrow{\mathcal{I}_c}(H)$. We also show that for two finite abelian groups G and H, $\mathcal{I}_c(G) \cong \mathcal{I}_c(H)$ if and only if $|\pi(G)| = |\pi(H)|$ and by a convenient permutation the graph of their sylow subgroups are isomorphic. In this case, their directed inclusion graphs are isomorphic too.

Keywords: Inclusion graph, power graph, cyclic subgroup, abelian group

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1. Introduction

Graphs associated to groups have a long history. Cayley graphs are first such notion introduced by Cayley. Power graph of a group was introduced by Kelarev and Quinn in [10]. In [2, 3], Cameron and Ghosh obtained interesting results about power graphs of finite groups. In recent years, the study of power graphs has been growing, see, for

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example, [4, 5, 9, 11–13]. Also, see [1] for a survey of results and open problems on power graphs. On the other hand, the concept of a power graph can be generalized or modified in various ways, such as reduced power graphs [14] and quotient power graphs [17].

Rajkumar and Anitha [14, 15] defined the *reduced power graph of G*, $\mathcal{RP}(G)$, is a graph with vertex set G, and two vertices u and v are adjacent if and only if $u \neq v$ and $\langle v\rangle \subset \langle u\rangle$ or $\langle u\rangle \subset \langle v\rangle$, and they study some interplay between the algebraic properties of a group and the graph theoretical properties of its (directed and undirected) reduced power graphs.

Shaker and Iranmanesh [17] defined the *quotient power graph* of a group G as follows: if $\mathcal{P} = (G, E)$ is a power graph and $\sim$ is a relation on G, in which for any $x, y \in G$, $x \sim y$ if and only if $\langle x\rangle = \langle y\rangle$, then the quotient power graph of G, $\mathcal{P}(G)/\sim$, denoted by $\overset{\sim}{\mathcal{P}}(G)$, is a graph with vertex set $[G] = G/\sim$ and edge set $[E]$, in which two distinct vertices $[x]$ and $[y]$ are adjacent if and only if there exists $x' \in [x]$ and $y' \in [y]$ such that $\{x', y'\} \in E$. In this paper they investigated some relationships between the power and quotient power graphs of a finite group and fined some graph theoretical properties of the quotient and proper quotient power graphs of a finite group G. Also they classify those groups whose quotient (proper quotient) power graphs are isomorphic to trees or paths.

Inspired by ideas from Shaker and Iranmanesh in [17], we study in [8] the *inclusion graph of cyclic subgroups* of a group G as follows: if G is a finite group, the inclusion graph of cyclic subgroups, $\mathcal{I}_c(G)$, is the (undirected) graph with vertices of all cyclic subgroups of G, and two distinct cyclic subgroups $\langle a\rangle$ and $\langle b\rangle$, are adjacent if and only if $\langle a\rangle \subset \langle b\rangle$ or $\langle b\rangle \subset \langle a\rangle$, and we classified all abelian groups whose inclusion graph is planar. Also we studied planarity of this graph for finite groups G, where $|\pi(Z(G))| \geq 2$. We denote $\mathcal{I}_c^*(G) = \mathcal{I}_c(G)\backslash\{\langle e\rangle\}$. Since $\mathcal{I}_c(G) \cong \mathcal{I}_c(H)$ if and only if $\mathcal{I}_c^*(G) \cong \mathcal{I}_c^*(H)$, so throughout of this paper we use $\mathcal{I}_c(G)$ instead of $\mathcal{I}_c^*(G)$. The *directed inclusion graph of cyclic subgroups* of G, $\overrightarrow{\mathcal{I}_c}(G)$, is the digraph with vertex set of all non-trivial cyclic subgroups of G, and for two distinct cyclic subgroups $\langle a\rangle$ and $\langle b\rangle$, there is an arc from $\langle a\rangle$ to $\langle b\rangle$ if and only if $\langle b\rangle \subset \langle a\rangle$.

In the first part of this paper, we provided some definitions and preliminaries which are required. In the next part we show that, if G and H are finite groups such that $\mathcal{I}_c(G) \cong \mathcal{I}_c(H)$ and G is cyclic, then H is cyclic too. We also show that for two cyclic groups G, H of orders $p_1^{\alpha_1} \cdots p_t^{\alpha_t}$ and $q_1^{\beta_1} \cdots q_s^{\beta_s}$, respectively, $\mathcal{I}_c(G) \cong \mathcal{I}_c(H)$ if and only if $t = s$ and by a suitable σ, $\alpha_i = \beta_{\sigma(i)}$. Also for any cyclic groups G and H, if $\mathcal{I}_c(G) \cong \mathcal{I}_c(H)$, then $\overrightarrow{\mathcal{I}_c}(G) \cong \overrightarrow{\mathcal{I}_c}(H)$. In the last part, we show that for two finite abelian groups G and H, $\mathcal{I}_c(G) \cong \mathcal{I}_c(H)$ if and only if $|\pi(G)| = |\pi(H)|$ and by a convenient permutation the graph of their sylow subgroups are isomorphic. Moreover, for two abelian groups if their inclusion graph of cyclic subgroups are isomorphic, then their directed inclusion graphs are isomorphic too.

2. Definition and preliminaries

All groups and graphs in this paper are assumed to be finite. Throughout the paper by a graph we mean a simple graph which has no multiple edges or loops. The following notations are used in the rest of this paper.

Let $\Gamma = (V, E)$ be any graph, and let $X \subseteq V$ be any subset of vertices of Γ. The *induced subgraph* $\Gamma[X]$, is the graph whose vertex set is X and whose edge set consists of all of the edges in E that have both endpoints in X. The non-empty graph Γ is *connected* if it has a x, y-path whenever $x, y \in V$, otherwise Γ is disconnected. Also we mean the *null graph* is a graph that has no edges. In other words, the graph Γ is null, if $|E| = 0$. The *Union of graphs* Γ_1 and Γ_2, written $\Gamma_1 \cup \Gamma_2$, is the graph with vertex set $V(\Gamma_1) \cup V(\Gamma_2)$ and edge set $E(\Gamma_1) \cup E(\Gamma_2)$. The distance $d(x, y)$ between vertices x and y of a connected graph Γ is the length of a shortest path connecting them, and we define $d(x, x) = 0$. Also, suppose A and B are two subsets of V, then we define:

$$d(A, B) = min\{d(x, y) \mid x \in A\ ,\ y \in B\}.$$

A non-empty subset $X \subseteq V$ is called a *clique*, if the induced subgraph on X is a complete graph. The maximum size of a clique in Γ is called the *clique number* of Γ and denoted by $\omega(\Gamma)$. The degree of a vertex x in a graph Γ, $deg_\Gamma(x)$, is the number of Γ incident to x. The maximum degree is $\Delta(\Gamma)$, the minimum degree is $\delta(\Gamma)$ and the set of all neighbors of x in Γ is denoted by $N_\Gamma(x)$, or briefly by $N(x)$. Furthermore, $N[x] = N(x) \cup \{x\}$. If Γ is a directed graph, we define, the set of input edges to x, $i(x) = \{y \in V \mid y \to x\}$ and the set of output edges of x, $O(x) = \{y \in V \mid x \to y\}$.

Given a group G and a subset X of cyclic subgroups of G, we denote $[X(G)]$ instead of $\mathcal{I}_c(G)[X]$ and we shall write $[X(G)]'$ instead of $\mathcal{I}'_c(G)[X]$. Also for vertex set and edge set of $\mathcal{I}_c(G)$, we denote $V(G)$ and $E(G)$ instead of $V(\mathcal{I}_c(G))$ and $E(\mathcal{I}_c(G))$, respectively. We also denote $\omega(G)$, $\Delta(G)$ and $\delta(G)$ instead of $\omega(\mathcal{I}_c(G))$, $\Delta(\mathcal{I}_c(G))$ and $\delta(\mathcal{I}_c(G))$, respectively. For $\langle x\rangle \in V(G)$, $i(\langle x\rangle) = \{\langle y\rangle \in V(G) | \langle x\rangle \subset \langle y\rangle\}$ and $O(\langle x\rangle) = \{\langle y\rangle \in V(G) | \langle y\rangle \subset \langle x\rangle\}$.

The cyclic group of order n is denoted by $\mathbb{Z}_n$. A group G is called *homocyclic* if G isomorphic to the direct product of cyclic groups, each of the same order. We also denote by $\pi(n)$ the set of the prime divisors of a positive integer n, and given a group G, we shall write $\pi(G)$ instead of $\pi(|G|)$.

We begin with the following theorem, which will be used frequently in the sequel.

Theorem 1. *Let $\overrightarrow{\mathcal{I}_c}(G_1) \cong \overrightarrow{\mathcal{I}_c}(G_2)$, $\overrightarrow{\mathcal{I}_c}(H_1) \cong \overrightarrow{\mathcal{I}_c}(H_2)$ and $(|G_i|, |H_i|) = 1$, $i \in \{1, 2\}$. Then $\overrightarrow{\mathcal{I}_c}(G_1 \times H_1) \cong \overrightarrow{\mathcal{I}_c}(G_2 \times H_2)$, and moreover $\mathcal{I}_c(G_1 \times H_1) \cong \mathcal{I}_c(G_2 \times H_2)$.*

Proof. Let $\overrightarrow{\mathcal{I}_c}(G_1) \cong_{\varphi_1} \overrightarrow{\mathcal{I}_c}(G_2)$ and $\overrightarrow{\mathcal{I}_c}(H_1) \cong_{\varphi_2} \overrightarrow{\mathcal{I}_c}(H_2)$. Since $(|G_1|, |H_1|) = 1$, any cyclic subgroups of $G_1 \times H_1$ are in the form $\langle a\rangle \times \langle b\rangle$ where $\langle a\rangle \in V(G_1) \cup \{\langle e_{G_1}\rangle\}$ and $\langle b\rangle \in V(H_1) \cup \{\langle e_{H_1}\rangle\}$. Also $\langle a_1\rangle \times \langle b_1\rangle \subseteq \langle a\rangle \times \langle b\rangle$ if and only if $\langle a_1\rangle \subseteq \langle a\rangle$ and $\langle b_1\rangle \subseteq \langle b\rangle$. Then the map $\varphi : \overrightarrow{\mathcal{I}_c}(G_1 \times H_1) \longrightarrow \overrightarrow{\mathcal{I}_c}(G_2 \times H_2)$ defined by $\varphi(\langle g\rangle \times \langle h\rangle) = \varphi_1(\langle g\rangle) \times \varphi_2(\langle h\rangle)$, $\varphi(\langle e_{G_1}\rangle \times \langle h\rangle) = \langle e_{G_2}\rangle \times \varphi_2(\langle h\rangle)$ and $\varphi(\langle g\rangle \times \langle e_{H_1}\rangle) =$

$\varphi_1(\langle g\rangle)\times\langle e_{H_2}\rangle$, for any non-trivial cyclic subgroups $\langle g\rangle \subseteq G_1$ and $\langle h\rangle \subseteq H_1$, is a graph isomorphism, and then $\overrightarrow{\mathcal{I}_c}(G_1 \times H_1) \cong \overrightarrow{\mathcal{I}_c}(G_2 \times H_2)$. $\square$

Example 1. Let $G_1 = A_1 \times H$ and $G_2 = A_2 \times H$, where $A_1 = \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$, $A_2 = \mathbb{Z}_p \ltimes (\mathbb{Z}_p \times \mathbb{Z}_p)$ and H be any finite group where $(|H|, p) = 1$ and $p \geq 3$. Then $\mathcal{I}_c(G_1) \cong \mathcal{I}_c(G_2)$.

Proof. Clearly $\overrightarrow{\mathcal{I}_c}(A_1) \cong \overrightarrow{\mathcal{I}_c}(A_2) \cong K'_{p^2+p+1}$. Since $\overrightarrow{\mathcal{I}_c}(H) \cong \overrightarrow{\mathcal{I}_c}(H)$, by Theorem 1, $\mathcal{I}_c(G_1) \cong \mathcal{I}_c(G_2)$. $\square$

3. Cyclic Groups

Obviously if $\overrightarrow{\mathcal{I}_c}(G) \cong \overrightarrow{\mathcal{I}_c}(H)$, then $\mathcal{I}_c(G) \cong \mathcal{I}_c(H)$. In this section, we show that for a cyclic group G, if $\mathcal{I}_c(G) \cong \mathcal{I}_c(H)$, then H is cyclic too and $\overrightarrow{\mathcal{I}_c}(G) \cong \overrightarrow{\mathcal{I}_c}(H)$.

Lemma 1. *Let G be a group and $|\pi(G)| \geq 3$. Then for any non-trivial subgroup $\langle x\rangle$ of G, $O(\langle x\rangle)'$ is connected if and only if $|\pi(\langle x\rangle)| \geq 2$ or $|\langle x\rangle| = p^2$, where p is a prime.*

Proof. Assume $|\langle x\rangle| = p_1^{r_1} \cdots p_t^{r_t}$, $t \geq 2$. One can see that, all subgroups of distinct prime power orders are adjacent in $O(\langle x\rangle)'$. Suppose $\langle a\rangle \in O(\langle x\rangle)$ is not prime powers and $|\langle a\rangle| = p_1^{\gamma_1} \cdots p_t^{\gamma_t}$, where $\gamma_j < r_j$, for some j. Then $\langle a\rangle$ and $\langle b\rangle$ are adjacent in $O(\langle x\rangle)'$, in which $|\langle b\rangle| = p_j^{r_j}$. Thus $O(\langle x\rangle)'$ is connected.
Now suppose that $|\langle x\rangle| = p^k$, $k \geq 2$. Then $O(\langle x\rangle)$ is complete graph of order $k-1$. Hence $O(\langle x\rangle)'$ is connected if and only if $O(\langle x\rangle)' = K_1$. Also if $|\langle x\rangle| = p$, then $O(\langle x\rangle)' = \emptyset$ and completes the proof. $\square$

Let $\mathcal{I}_1(G) = \mathcal{I}_c(G) \setminus \{G\}$. Clearly, $O_{\mathcal{I}_c(G)}(\langle x\rangle) = O_{\mathcal{I}_1(G)}(\langle x\rangle)$ for any proper cyclic subgroup $\langle x\rangle$ of G. Also if G is not cyclic, then $\mathcal{I}_c(G) = \mathcal{I}_1(G)$.
Since induced subgraph on $O(\langle x\rangle)$ and $\mathcal{I}_1(\langle x\rangle)$ are isomorphic and induced subgraph on $i(\langle x\rangle)$ and $\mathcal{I}_c(\frac{G}{\langle x\rangle})$ are isomorphic, we have the following.

Lemma 2. *Let G be a cyclic group and $|\pi(G)| \geq 3$. Then for any proper non-trivial subgroup $\langle x\rangle$ of G, $i_{\mathcal{I}_1(G)}(\langle x\rangle)'$ is connected if and only if $|\pi(\frac{G}{\langle x\rangle})| \geq 2$ or $|\frac{G}{\langle x\rangle}| = p^2$, where p is a prime.*

Let G be a finite group and $\langle x\rangle \subseteq G$. Then $\langle x\rangle$ is a *prime-element*, if the order of $\langle x\rangle$ is prime. Also, $\langle x\rangle$ is a *maximal cyclic subgroup* of G, if it is a maximal in the set of all proper cyclic subgroups of G.

Remark 1. Let G be a cyclic group and $\langle x\rangle \subseteq G$. Then $O_{\mathcal{I}_c(G)}(\langle x\rangle) = \emptyset$ or $i_{\mathcal{I}_c(G)}(\langle x\rangle) = \emptyset$ if and only if $\langle x\rangle$ is a prime-element or $\langle x\rangle = G$.

Corollary 1. *Let G be a cyclic group, $|\pi(G)| \geq 3$ and $\langle x \rangle \subset G$. Then $N_{\mathcal{I}_1(G)}(\langle x \rangle)'$ $= A \cup B$, where A is a connected component with at least one edge and B is null, if and only if $|\langle x \rangle|$ or $|\frac{G}{\langle x \rangle}|$ is a positive prime power.*

Proof. Let $\langle x \rangle$ be a subgroup of G and $N_{\mathcal{I}_1(G)}(\langle x \rangle)'$ is union of a connected component with at least one edge and some points. Since there is no edge between vertex set of $O(\langle x \rangle)'$ and $i_{\mathcal{I}_1(G)}(\langle x \rangle)'$ in $N_{\mathcal{I}_1(G)}(\langle x \rangle)'$, $N_{\mathcal{I}_1(G)}(\langle x \rangle)' = O(\langle x \rangle)' \cup i_{\mathcal{I}_1(G)}(\langle x \rangle)'$. If $|\pi(\langle x \rangle)| \geq 2$ and $|\pi(\frac{G}{\langle x \rangle})| \geq 2$, then $O(\langle x \rangle)'$ and $i_{\mathcal{I}_1(G)}(\langle x \rangle)'$ are connected components with at least one edge, and so $N_{\mathcal{I}_1(G)}(\langle x \rangle)'$ have two connected components with at least one edge, a contradiction. Since $|\pi(G)| \geq 3$, $|\langle x \rangle| = p^\alpha$ and $|\frac{G}{\langle x \rangle}| = p^\alpha$, $\alpha \geq 1$, don't happen together. Now, suppose $|\langle x \rangle| = p^\alpha$, $\alpha \geq 1$. Hence $O(\langle x \rangle)' = K'_{\alpha-1}$ and by Lemma 2, $i_{\mathcal{I}_1(G)}(\langle x \rangle)'$ is a connected component with at least one edge. Also, if $|\frac{G}{\langle x \rangle}| = p^\alpha$, $\alpha \geq 1$, then $i_{\mathcal{I}_1(G)}(\langle x \rangle)'$ is null and by Lemma 1, $O(\langle x \rangle)'$ is a connected component with at least one edge, as required. □

Utilizing the above corollary we have the following result.

Corollary 2. *Let G be a cyclic group and $|\pi(G)| \geq 3$. Then for any subgroup $\langle x \rangle$ of G, $N_{\mathcal{I}_1(G)}(\langle x \rangle)'$ is connected if and only if $\langle x \rangle$ is a prime-element or $\langle x \rangle$ is a maximal cyclic subgroup of G.*

Lemma 3. *Let G be a cyclic group. Then $\mathcal{I}_c(G)$ has a vertex of degree 1 if and only if $|G| = p^2$ or pq, for some distinct prime numbers p and q.*

Proof. Let G be a cyclic group and $\langle x \rangle \subset G$ such that $deg(\langle x \rangle) = 1$. Then we have $|i(\langle x \rangle)| = 1$ and $|O(\langle x \rangle)| = 0$ or $|i(\langle x \rangle)| = 0$ and $|O(\langle x \rangle)| = 1$. Suppose $|i(\langle x \rangle)| = 1$ and $|O(\langle x \rangle)| = 0$, then $\langle x \rangle$ is a p-element and maximal subgroup. Hence $|G| = p^2$ or pq. If $|i(\langle x \rangle)| = 0$ and $|O(\langle x \rangle)| = 1$, then $\langle x \rangle$ is a generator, which has exactly one proper non-trivial subgroup. So $\langle x \rangle$ is a p-group of order p^2. Hence $|G| = p^2$.
The converse is trivial. □

In the next theorem, we use the following famous theorem.

Theorem 2. *[16, Theorem 5.3.6] A finite p-group has exactly one subgroup of order p if and only if it is cyclic or generalized quaternion group, where the generalized quaternion group $Q_{2^n} = \langle a,\ b | a^{2^{n-1}} = b^2,\ b^{-1}ab = a^{-1} \rangle$.*

Theorem 3. *Let G and H be finite groups and G be a cyclic group such that $\mathcal{I}_c(G) \cong \mathcal{I}_c(H)$. Then H is cyclic too.*

Proof. Since G is a cyclic group, there exists $\langle a \rangle \subseteq G$ such that $\langle a \rangle = G$ and $N[\langle a \rangle] = \mathcal{I}_c(G)$. Also since $\mathcal{I}_c(G) \cong \mathcal{I}_c(H)$, H has a non-trivial cyclic subgroup $\langle h \rangle$ such that $N[\langle h \rangle] = \mathcal{I}_c(H)$.

Firstly assume that $|\langle h\rangle| = p^\alpha$. Then there exists $\langle y\rangle \subset \langle h\rangle$ such that $|\langle y\rangle| = p$. Since $N[\langle h\rangle] \subseteq N[\langle y\rangle]$, $N[\langle y\rangle] = \mathcal{I}_c(H)$. Also since prime-elements are not adjacent, H is a p-group and has exactly one subgroup of order p. By Theorem 2, H is a cyclic or a generalized quaternion 2-group. If H is generalized quaternion group, then H has a cyclic subgroup $\langle x\rangle$ of order 4 such that $deg(\langle x\rangle) = 1$. Thus G has a subgroup a with $deg(\langle a\rangle) = 1$ and by Lemma 3, $|G| = p^2$ or $|G| = pq$. Then $|V(G)| < 4$ but $|V(H)| \geq 4$, a contradiction. Thus H is a cyclic p-group.
Now suppose that $|\pi(\langle h\rangle)| \geq 2$. It is clear that $\pi(\langle h\rangle) = \pi(H)$. If $\langle h\rangle \neq H$ then there is a sylow subgroup P of H such that $P \nsubseteq h$. If $p \in P \setminus \langle h\rangle$, then $\langle h\rangle \subseteq \langle p\rangle$. Hence $\langle h\rangle$ is a p-group, a contradiction. □

Lemma 4. *Let G be a cyclic group and $\langle x\rangle \subset G$. Then the following statements hold:*

i. If $|\pi(G)| \geq 3$, then for each subgroup $\langle x\rangle$ of G, $N(\langle x\rangle)'$ is not null.

ii. $N(\langle x\rangle)$ is a complete graph if and only if $|G| = p^m$ or $|G| = p^m q^n$ where $m, n \geq 1$ and $|\langle x\rangle| = p^m$ or q^n.

Proof. Let $|\pi(G)| \geq 3$. If $|\pi(\langle x\rangle)| \geq 2$, then by Lemma 1, $O(\langle x\rangle)'$ is connected with at least one edge and if $|\langle x\rangle| = p^\alpha$, $\alpha \geq 1$, then $|\pi(\frac{G}{\langle x\rangle})| \geq 2$ and by Lemma 2, $i_{\mathcal{I}_1(G)}(\langle x\rangle)'$ is connected with at least one edge, as required.
Now suppose that $N(\langle x\rangle)$ is a complete graph. By the first part $|\pi(G)| \leq 2$. Let $|G| = p^m q^n$ and $|\langle x\rangle| = p^\alpha$, $\alpha < m$ or $|\langle x\rangle| = p^\alpha q^\beta$, $\alpha, \beta \geq 1$. Then by Lemmas 2 and 1, $i_{\mathcal{I}_1(G)}(\langle x\rangle)'$ or $O(\langle x\rangle)'$ is connected, a contradiction. Also if $|G| = p^m q^n$ and $|\langle x\rangle| = p^m$ or q^n, then $N(\langle x\rangle)$ is complete, as required. □

Lemma 5. *Let G and H be cyclic groups of orders $p_1^{\alpha_1} \cdots p_t^{\alpha_t}$ and $q_1^{\alpha_1} \cdots q_t^{\alpha_t}$, respectively. Then $\mathcal{I}_c(G) \cong \mathcal{I}_c(H)$.*

Proof. By hypothesis $|V(G)| = |V(H)| = \prod_{i=1}^{t}(\alpha_i + 1) - 1$. Suppose $\pi_e(G)$ and $\pi_e(H)$ are the set of all numbers that divide the order of G and H, respectively. Since G and H are cyclic, for any $n \in \pi_e(G)$ and $m \in \pi_e(H)$, G and H has exactly one subgroup of order n and m, respectively. Hence the map $f : V(G) \to V(H)$ defined by $f(\langle a\rangle) = \langle b\rangle$ where $|\langle a\rangle| = p_1^{\beta_1} \cdots p_t^{\beta_t}$ and $|\langle b\rangle| = q_1^{\beta_1} \cdots q_t^{\beta_t}$, is a graph isomorphism. □

The following theorem provides a necessary and sufficient condition for cyclic groups to have isomorphic graphs.

Theorem 4. *Let G and H be cyclic groups of orders $p_1^{\alpha_1} \cdots p_t^{\alpha_t}$, $t \geq 3$, and $q_1^{\beta_1} \cdots q_s^{\beta_s}$, respectively. Then $\mathcal{I}_c(G) \cong \mathcal{I}_c(H)$ if and only if $t = s$ and by a convenient permutation σ, $\alpha_i = \beta_{\sigma(i)}$.*

Proof. Let $\mathcal{I}_c(G) \cong_\varphi \mathcal{I}_c(H)$. Since G and H are cyclic, $\mathcal{I}_1(G) \cong_\varphi \mathcal{I}_1(H)$. Also since $\mathcal{I}_c(G)$ is not complete, $s \geq 2$. If $|H| = p^m q^n$, then H has a subgroup $\langle x\rangle$ of order

p^m such that $N_{\mathcal{I}_c(H)}(\langle x\rangle)'$ is null, which contradicts Lemma 4. Hence $|\pi(H)| \geq 3$. Let $S(G)$ be the set of all non-trivial subgroups $\langle a\rangle$ of G such that $N_{\mathcal{I}_1(G)}(\langle a\rangle)' = A \cup B$ where A is a connected component with at least one edge and B is null. By Corollary 1, $\langle x\rangle \in S(G)$ if and only if $|\langle x\rangle| = p^\alpha$ or $|\frac{G}{\langle x\rangle}| = p^\alpha$, $\alpha \geq 1$. Let $A_i = \{\langle x\rangle \in S(G) \big| \ |\langle x\rangle| \big| p_i^{\alpha_i}\}$ and $B_i = \{\langle y\rangle \in S(G) \big| \ |\frac{G}{\langle y\rangle}| \big| p_i^{\alpha_i}\}$, $1 \leq i \leq t$. One can see that $[A_i \cup A_j]' \cong K_{|A_i|,|A_j|}$, $[B_i \cup B_j]' \cong K_{|B_i|,|B_j|}$ and $[A_i \cup B_j]'$ is a null graph, for any $i \neq j$. We assume that $\langle x_i\rangle$ and $\langle y_i\rangle$, $1 \leq i \leq t$, are subgroups of orders $p_i^{\alpha_i}$ and $\frac{|G|}{p_i^{\alpha_i}}$, respectively. If $|\langle a_{ij}\rangle| = p_i^{\alpha_i - j+1}$ and $|\langle b_{ij}\rangle| = \frac{|G|}{p_i^{\alpha_i - j+1}}$, $1 \leq j \leq \alpha_i$, then $|\langle x_i\rangle| \nmid |\langle b_{ij}\rangle|$ and $|\langle b_{ij}\rangle| \nmid |\langle x_i\rangle|$ (similarly $|\langle y_i\rangle| \nmid |\langle a_{ij}\rangle|$ and $|\langle a_{ij}\rangle| \nmid |\langle y_i\rangle|$), so $\langle x_i\rangle \sim \langle b_{ij}\rangle$, $\langle y_i\rangle \sim \langle a_{ij}\rangle \in [S(G)]'$ and $[A_i \cup B_i]'$ is a connected graph.

Claim 1. $\omega([S(G)]') = t$.

Since $[A_1 \cup A_2 \cup \cdots \cup A_t]'$ is complete t-partite graph, $\omega([S(G)]') \geq t$. By a contrary, let $\omega([S(G)]') \geq t+1$. Let K be a complete subgraph of $[S(G)]'$ with $t+1$ vertices. Since A_i' and B_i' are null, each pair of vertices of K don't lie in one set of A_i or B_i. Thus there exist A_i and B_j such that $|V(K) \cap A_i| = |V(K) \cap B_j| = 1$. Since $[A_i \cup B_j]'$ is null, for $i \neq j$, $V(K) \subseteq A_i \cup B_i$. Also since $[A_i \cup B_i]'$ is bipartite graph, $|V(K)| \leq 2$, which contradicts to $t \geq 3$, and the claim is proved.

Claim 2. $\Delta([S(G)]') = \alpha_1 + \cdots + \alpha_t$.

Let $\langle x_{ij}\rangle \in A_i$ be a subgroup of order p_i^j and $\langle y_{ij}\rangle \in B_i$ be a subgroup of order $\frac{|G|}{p_i^{\alpha_i - j+1}}$, for $1 \leq j \leq \alpha_i$ and $1 \leq i \leq t$. Since any vertex $\langle x_{ij}\rangle$ is joined with all subgroups $\langle y_{il}\rangle$ in B_i when $(|\langle x_{ij}\rangle|, |\langle y_{il}\rangle|) = p_i^k$, $0 \leq k \leq j-1$, $deg_{[S(G)]'}(\langle x_{ij}\rangle) = j + \Sigma_{k\neq i}\alpha_k$. Similarly $deg_{[S(G)]'}(\langle y_{ij}\rangle) = (\alpha_i - j + 1) + \Sigma_{k\neq i}\alpha_k$. Therefore $\Delta([S(G)]') = \alpha_1 + \alpha_2 + \cdots + \alpha_t = deg_{[S(G)]'}(\langle x_{i\alpha_i}\rangle) = deg_{[S(G)]'}(\langle y_{i1}\rangle)$.

Similarly $\omega([S(H)]') = s$. Since $\mathcal{I}_1(G) \cong \mathcal{I}_1(H)$, $[S(G)]' \cong [S(H)]'$. Hence $\omega([S(G)]') = \omega([S(H)]')$ and $t = s$. Let $S(H) = C_1 \cup \cdots \cup C_t \cup D_1 \cup \cdots \cup D_t$, in which $C_i = \{\langle x\rangle \in S(H) \ \big| \ |\langle x\rangle| \big| q_i^{\beta_i}\}$ and $D_i = \{\langle y\rangle \in S(H) \ \big| \ |\frac{G}{\langle y\rangle}| \big| q_i^{\beta_i}\}$, $1 \leq i \leq t$. By the proof of claim 2, if $\langle a\rangle \in A_i$ has the maximum degree in $[S(G)]'$, then $N_{[S(G)]'}(\langle a\rangle) = \left(\bigcup_{k\neq i} A_k\right) \cup B_i$. Also if $\langle b\rangle \in B_i$ has the maximum degree in $[S(G)]'$, then $N_{[S(G)]'}(\langle b\rangle) = \left(\bigcup_{k\neq i} B_k\right) \cup A_i$. Now we assume that $\langle a_i\rangle \in A_i$ has the maximum degree. Then $\{\langle a_1\rangle, \cdots, \langle a_t\rangle\}$ is a clique. Then $\varphi(\langle a_i\rangle)$ has the maximum degree and $\{\varphi(\langle a_1\rangle), \cdots, \varphi(\langle a_t\rangle)\}$ is a clique too. Hence $\{\varphi(\langle a_1\rangle), \cdots, \varphi(\langle a_t\rangle)\} \subseteq \bigcup_{i=1}^t C_i$ or $\{\varphi(\langle a_1\rangle), \cdots, \varphi(\langle a_t\rangle)\} \subseteq \bigcup_{i=1}^t D_i$. Let $\{\varphi(\langle a_1\rangle), \cdots, \varphi(\langle a_t\rangle)\} \subseteq \bigcup_{i=1}^t C_i$ and $\varphi(\langle a_i\rangle) \in C_{\sigma(i)}$, for some permutation σ. Now one can see that $\bigcap_{i\neq j} N_{[S(G)]'}(\langle a_i\rangle) = A_j$ and $\bigcap_{i\neq j} N_{[S(H)]')}(\varphi(\langle a_i\rangle)) = C_{\sigma(j)}$, and then $\varphi(A_i) = C_{\sigma(i)}$. Consequently, $\alpha_i = \beta_{\sigma(i)}$, as required.

The converse holds by Lemma 5. □

Now we focus on the group G with $|\pi(G)| = 2$. In what follows,

$$S_1(G) = \{\langle x\rangle \subset G \ \big| \ N_{\mathcal{I}_1(G)}(\langle x\rangle)' \text{ is connected } \}.$$

In the following lemma all cyclic groups G are characterized , in which $[S_1(G)]$ is completed.

Lemma 6. *Let G be a cyclic group. Then $[S_1(G)]$ is a complete graph if and only if $|G| = p^m q$, $m > 1$, or $|G| = p^3$.*

Proof. Let $[S_1(G)]$ be complete graph and $G = P_1 \times \cdots \times P_t$, $t \geq 3$. Then by Corollary 2, $S_1(G) = \{\langle x\rangle \subset G \mid \langle x\rangle$ is a prime-element or $\langle x\rangle$ is maximal $\}$. Since prime-elements are not adjacent, $[S_1(G)]$ is not complete, a contradiction. Let $t = 1$ and $|G| = p^m$, $m \neq 3$. Then $S_1(G) = \emptyset$. Now if $|G| = p^3$, then $[S_1(G)] \cong K_2$, as required. Suppose $t = 2$ and $|G| = p^m q^n$, m, $n > 1$. Let $\langle a\rangle$, $\langle b\rangle subset G$ of orders p and q, respectively. Since $|\pi(\frac{G}{\langle a\rangle})| = |\pi(\frac{G}{\langle b\rangle})| = 2$, by Lemma 2, $\langle a\rangle$, $\langle b\rangle \in S_1(G)$, a contradiction. Also if $m = n = 1$, then $S_1(G) = \emptyset$. Now assume that $m > 1$ and $n = 1$. Then $S_1(G) = \{\langle x\rangle \subset G \big| |\langle x\rangle| = p$ or $|\langle x\rangle| = p^{m-1}q\}$ and $[S_1(G)] \cong K_2$. □

Theorem 5. *Let G be a cyclic group of order $p^m q^n$, $m, n \geq 1$ and $\mathcal{I}_c(G) \cong \mathcal{I}_c(H)$. Then $|H| = p_1^m q_1^n$.*

Proof. By Theorems 3 and 4, H is cyclic and $\pi(H) < 3$. If $|H| = p^m$, then $\mathcal{I}_c(H)$ is a complete graph, a contradiction. Hence $|H| = p_1^{m_1} q_1^{n_1}$.
Let $n = 1$. By Lemma 6, $m_1 = 1$ or $n_1 = 1$. Also $|V(G)| = 2m + 1$. Assume $n_1 = 1$. Then $|V(H)| = 2m_1 + 1$ and $m = m_1$.
Now assume that $m, n > 1$. Similar to the argument of the previous theorem, set

$$S(G) = \{\langle x\rangle \subset G \mid N_{\mathcal{I}_1(G)}(\langle x\rangle)' = A \cup B,\ A \text{ is connected and } B \text{ is null }\}.$$

It is obvious that $\langle a\rangle \in S(G)$ if and only if $|\langle a\rangle| = p^i q^n$ or p^i for $1 \leq i < m$, or $|\langle a\rangle| = p^m q^j$ or q^j for $1 \leq j < n$. Let $A_1 = \{\langle a\rangle \in S(G) \big| |\langle a\rangle| = p^i, i < m\}$, $B_1 = \{\langle a\rangle \in S(G) \big| |\langle a\rangle| = p^i q^n, i < m\}$, $A_2 = \{\langle a\rangle \in S(G) \big| |\langle a\rangle| = q^j, j < n\}$ and $B_2 = \{\langle a\rangle \in S(G) \big| |\langle a\rangle| = p^m q^j, j < n\}$. Let $\langle a_i\rangle \in A_1$ of order p^i, $\langle b_i\rangle \in A_2$ of order q^i, $\langle c_i\rangle \in B_1$ of order $p^i q^n$ and $\langle d_i\rangle \in B_2$ of order $p^m q^i$. Then $deg_{[S(G)]'}(\langle a_i\rangle) = n + i - 2$, $deg_{[S(G)]'}(\langle b_i\rangle) = m + i - 2$ and $deg_{[S(G)]'}(\langle c_i\rangle) = deg_{[S(G)]'}(\langle d_i\rangle) = m + n - (i+2)$. Hence $\delta([S(G)]') = min\{m, n\} - 1$ and $\Delta([S(G)]') = m + n - 3$. Assume that $m \leq n$ and $m_1 \leq n_1$. Then $m + n - 3 = m_1 + n_1 - 3$ and $m - 1 = m_1 - 1$. Therefore $m = m_1$ and $n = n_1$, as required. □

Since for cyclic p-groups G of order p^n, $\mathcal{I}_c(G) \cong K_n$, by Theorems 4 and 5, we have the following result.

Corollary 3. *Let G be a cyclic group of order $p_1^{\alpha_1} \cdots p_t^{\alpha_t}$ and H be a group of order $q_1^{\beta_1} \cdots q_s^{\beta_s}$. Then $\mathcal{I}_c(G) \cong \mathcal{I}_c(H)$ if and only if H is cyclic, $t = s$ and by a convenient permutation σ, $\alpha_i = \beta_{\sigma(i)}$.*

Also by Corollary 3 and Theorem 1, one can see that the following result holds.

Corollary 4. *If G is a cyclic group and $\mathcal{I}_c(G) \cong \mathcal{I}_c(H)$, then $\overrightarrow{\mathcal{I}_c}(G) \cong \overrightarrow{\mathcal{I}_c}(H)$.*

4. Abelian Groups

In this section we focus on the abelian groups. Let G be a p-group. We denote the number of cyclic subgroups of order p^i of G with $\mathfrak{c}_{p^i}(G)$.

Theorem 6. *Let G and H be abelian p-groups such that for any positive integer k, they have the same number of cyclic subgroups of order p^k. Then $G \cong H$.*

Proof. Let $G \cong \mathbb{Z}_{p^{\alpha_1}} \times \cdots \times \mathbb{Z}_{p^{\alpha_t}}$ and $H \cong \mathbb{Z}_{p^{\beta_1}} \times \cdots \times \mathbb{Z}_{p^{\beta_s}}$. Since $\mathfrak{c}_p(G) = \mathfrak{c}_p(H)$, $\frac{p^t-1}{p-1} = \frac{p^s-1}{p-1}$, and then $t = s$.
If G is an elementary abelian group, the proof is obvious. Now assume that $G = G_1 \times G_2$ where G_2 is homocyclic of exponent $p^\alpha = exp(G)$, $\alpha > 1$, and $G_2 \cong \underbrace{\mathbb{Z}_{p^\alpha} \times \cdots \times \mathbb{Z}_{p^\alpha}}_{t_1-\text{times}}$. Also $H = H_1 \times H_2$ where H_2 is homocyclic of exponent $p^\beta = exp(H)$ and $H_2 \cong \underbrace{\mathbb{Z}_{p^\beta} \times \cdots \times \mathbb{Z}_{p^\beta}}_{s_1-\text{times}}$. By hypothesis $exp(G) = exp(H)$, so $\alpha = \beta$, and then $\mathfrak{c}_{p^\alpha}(G) = \mathfrak{c}_{p^\alpha}(H)$. Therefore

$$\begin{aligned} \frac{(p^\alpha)^{t_1} - (p^{\alpha-1})^{t_1}}{p^\alpha - p^{\alpha-1}}|G_1| &= \frac{(p^\alpha)^{s_1} - (p^{\alpha-1})^{s_1}}{p^\alpha - p^{\alpha-1}}|H_1| \\ \Rightarrow \quad (p^{\alpha-1})^{t_1}(p^{t_1} - 1)|G_1| &= (p^{\alpha-1})^{s_1}(p^{s_1} - 1)|H_1| \end{aligned}$$

Since $G_1,\ H_1$ are p-groups, $p^{t_1} - 1 = p^{s_1} - 1$, and then $t_1 = s_1$.
On the other hand since the number of cyclic subgroups of order p^k, $k < \alpha$, in $G_1 \times G_2^p$ and G are equal and the number of cyclic subgroups of order p^k, $k < \alpha$, in $H_1 \times H_2^p$ and H are equal too, $G_1 \times G_2^p$ and $H_1 \times H_2^p$ have equal number of cyclic subgroups. By induction $G_1 \times G_2^p \cong H_1 \times H_2^p$. Consequently $G_1 \cong H_1$ and $G_2 \cong H_2$. $\square$

In the following, the distance between two distinct prime-elements and two distinct maximal cyclic subgroups is calculated.

Lemma 7. *Let G be an abelian group and $|\pi(G)| \geq 2$. Then for each two distinct prime-elements or two maximal cyclic subgroups $\langle x\rangle$ and $\langle y\rangle$, $d(\langle x\rangle, \langle y\rangle) \in \{2, 4\}$.*

Proof. Firstly assume that $\langle x\rangle$, $\langle y\rangle$ are prime-elements and $|\langle x\rangle| = |\langle y\rangle| = p$. By hypothesis there exist $\langle a\rangle \subset G$ such that $|\langle a\rangle| = q$ where $q \neq p$ is a prime. Since G is abelian, $\langle x\rangle \sim \langle ax\rangle \sim \langle a\rangle \sim \langle ay\rangle \sim \langle y\rangle$ is a path of length 4 from $\langle x\rangle$ to $\langle y\rangle$ and $d(\langle x\rangle, \langle y\rangle) \leq 4$. If $d(\langle x\rangle, \langle y\rangle) = 2$ and $\langle x\rangle \sim \langle c\rangle \sim \langle y\rangle$, then $\langle x\rangle, \langle y\rangle \subset \langle c\rangle$, a contradiction. Now assume that $\langle x\rangle \sim \langle a_1\rangle \sim \langle a_2\rangle \sim \langle y\rangle$ be a path of length 3 from $\langle x\rangle$ to $\langle y\rangle$. Since $\langle x\rangle$ and $\langle y\rangle$ are prime-elements, $\langle x\rangle \subset \langle a_1\rangle$ and $\langle y\rangle \subset \langle a_2\rangle$. Also $\langle a_1\rangle \subset \langle a_2\rangle$ or $\langle a_2\rangle \subset \langle a_1\rangle$. We can assume that $\langle a_2\rangle \subset \langle a_1\rangle$. Hence $\langle y\rangle \subset \langle a_1\rangle$, a contradiction. Thus

$d(\langle x\rangle, \langle y\rangle)$=4. Also if $(|\langle x\rangle|, |\langle y\rangle|)$=1, then $\langle x\rangle \sim \langle xy\rangle \sim \langle y\rangle$ is a path of length 2 and since $\langle x\rangle$, $\langle y\rangle$ are not adjacent, $d(\langle x\rangle, \langle y\rangle)$=2.
Now suppose that $\langle x\rangle$, $\langle y\rangle$ are distinct maximal cyclic subgroups. If $\langle x\rangle \cap \langle y\rangle = \langle e\rangle$, then there are prime-elements $\langle a\rangle \subset \langle x\rangle$ and $\langle b\rangle \subset \langle y\rangle$ such that $|\langle a\rangle| \neq |\langle b\rangle|$ and $\langle x\rangle \sim \langle a\rangle \sim \langle ab\rangle \sim \langle b\rangle \sim \langle y\rangle$. Hence $d(\langle x\rangle, \langle y\rangle) \leq 4$. If $d(\langle x\rangle, \langle y\rangle) = 2$ and $\langle x\rangle \sim \langle a\rangle \sim \langle y\rangle$ for some cyclic subgroup $\langle a\rangle$, then $\langle a\rangle \subset \langle x\rangle \cap \langle y\rangle$, a contradiction. Let $d(\langle x\rangle, \langle y\rangle)$=3 and $\langle x\rangle \sim \langle a_1\rangle \sim \langle a_2\rangle \sim \langle y\rangle$ be a path of length 3 from $\langle x\rangle$ to $\langle y\rangle$. Then $\langle a_1\rangle \subset \langle x\rangle$ and $\langle a_2\rangle \subset \langle y\rangle$. Since $\langle a_1\rangle \sim \langle a_2\rangle$, we can assume that $\langle a_1\rangle \subset \langle a_2\rangle$, and then $\langle a_1\rangle \subset \langle y\rangle$, a contradiction. Thus $d(\langle x\rangle, \langle y\rangle)$=4. Also if $\langle x\rangle \cap \langle y\rangle \neq \langle e\rangle$, then $\langle x\rangle \sim \langle x\rangle \cap \langle y\rangle \sim \langle y\rangle$ is a path of length 2 from $\langle x\rangle$ to $\langle y\rangle$ and $d(\langle x\rangle, \langle y\rangle)$=2. □

Let $n_c(G)$ be the number of connected components of the inclusion graph of a p-group G. The following theorem shows that the isomorphic inclusion graphs of cyclic subgroups of some non-cyclic abelian groups implies that their corresponding groups are isomorphic.

Theorem 7. *Let G and H be non-cyclic abelian p-group and q-group, respectively, and G is an not elementary abelian group such that $\mathcal{I}_c(G) \cong \mathcal{I}_c(H)$. Then $G \cong H$.*

Proof. Let $G = G_1 \times \cdots \times G_t$ and $H = H_1 \times \cdots \times H_s$. Suppose that $G_1 \cong \cdots \cong G_{r_1-1} \cong \mathbb{Z}_{p^{\beta_0}}$, $G_{r_1} \cong \cdots \cong G_{r_2-1} \cong \mathbb{Z}_{p^{\beta_1}}$, $\cdots$, $G_{r_m} \cong \cdots \cong G_t \cong \mathbb{Z}_{p^{\beta_m}}$, $\beta_0 < \beta_1 < \cdots < \beta_m$ and $H_1 \cong \cdots \cong H_{r'_1-1} \cong \mathbb{Z}_{q^{\beta'_0}}$, $H_{r'_1} \cong \cdots \cong H_{r'_2-1} \cong \mathbb{Z}_{q^{\beta'_1}}$, $\cdots$, $H_{r'_n} \cong \cdots \cong H_s \cong \mathbb{Z}_{q^{\beta'_n}}$, $\beta'_0 < \beta'_1 < \cdots < \beta'_n$. Let $B_1 \subset G$ be a cyclic subgroup. Then $N[B_1]$ is a connected component of $\mathcal{I}_c(G)$ if and only if $|B_1| = p$. Let $B_2 \in N[B_1] \setminus \{B_1\}$. $N[B_2] \setminus \{B_1\}$ is a connected component of $N[B_1] \setminus \{B_1\}$ if and only if $|B_2| = p^2$. By continuing this process we can find the order of any cyclic subgroup of G, and also all chains $B_1 \subset B_2 \subset \cdots \subset B_k$ where $|B_i| = p^i$ and B_k is a maximal cyclic subgroup. For any element B of order p, let $L(B)$ be the maximum length of chains with started to B. One can see that $L(B) \in \{\beta_0, \cdots, \beta_m\}$. Let $B = \langle(a_1, \cdots, a_t)\rangle$. Then $L(B) = \beta_j$ if and only if $a_i = e_i$ for $i < r_j$ and there exists k, $r_j \leq k \leq r_{j+1} - 1$ such that $a_k \neq e_k$. Hence the number of cyclic subgroups B of order p where $L(B) = \beta_j$, is equal to:

$$\frac{p^{r_{j+1}-r_j} - 1}{p - 1} \times p^{t - r_{j+1} + 1}$$

where $r_0 = 1$ and $r_{m+1} - 1 = t$.
Similarly for any cyclic subgroup B' of order q of H, $L(B') \in \{\beta'_0, \cdots, \beta'_n\}$, and then $n = m$ and $\{\beta_0, \cdots, \beta_m\} = \{\beta'_0, \cdots, \beta'_m\}$ which shows that $\beta_i = \beta'_i$. Let $L(B) = \beta_m$. The number of connected component of $N[B] \setminus \{B\}$ is equal to the number of cyclic subgroups of order p^2 contains B. Thus

$$n_c(N[B] \setminus \{B\}) = \frac{p^{t-1}(p^2 - p)}{p^2 - p} = p^{t-1}.$$

Similarly for H, $n_c(N[B'] \setminus \{B'\}) = \frac{q^{s-1}(q^2-q)}{q^2-q} = q^{s-1}$. Therefore $p = q$ and $t = s$. Now by comparing the number of cyclic subgroups B where $L(B) = \beta_j$ for G and H, we can find the number of all factors of H and the orders of each factor. Hence H is completely specified and $G \cong H$. □

A group G is said to be an EPO-group, if all non-trivial element orders of G are prime. A complete classification of EPO-group has been given in [6, 7]. It is clear that for a finite group G, $\mathcal{I}_c(G)$ is null if and only if G is an EPO-group.
The following remark shows that the Lemma 7, is not true for elementary abelian groups and non-abelian groups.

Remark 2. If G and H are abelian groups and G is an elementary abelian group such that $\mathcal{I}_c(G) \cong \mathcal{I}_c(H)$, then H is elementary abelian too. But we cannot in general conclude that $G \cong H$. For example if $G = \underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{5-times}$ and $H = \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5$, $|V(\mathcal{I}_c(G))| = |V(\mathcal{I}_c(H))| = 31$, then $\mathcal{I}_c(G) \cong \mathcal{I}_c(H)$ but $G \ncong H$.
Also if G is an elementary abelian group and H be a group such that $\mathcal{I}_c(G) \cong \mathcal{I}_c(H)$, then H is not in general abelian. For example let $G_1 = \mathbb{Z}_3 \times \mathbb{Z}_3$ and $H_1 = S_3$ or $G_2 = \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5$ and $H_2 = A_5$. By [7, Main Theorem], $G_i,\ H_i$, $i \in \{1,\ 2\}$ are EPO-groups and one can see that $\mathcal{I}_c(G_i) \cong \mathcal{I}_c(H_i)$ but $G_i \ncong H_i$.

Next lemma is important for further investigations.

Lemma 8. *Let $G = P_1 \times \cdots \times P_t$, $t \geq 2$, be a non-cyclic abelian group and P_i be sylow subgroups of G. Then the following statements hold.*

- **i.** *If $\langle x \rangle \subset G$ and $N(\langle x \rangle)'$ is connected, then $\langle x \rangle$ is a prime-element or $\langle x \rangle$ is a maximal cyclic subgroup.*
- **ii.** *If $\langle x \rangle \subset G$ is a maximal cyclic subgroup, then $N(\langle x \rangle)'$ is connected.*
- **iii.** *If P_1 is non-cyclic and $\langle x \rangle \subset P_i$, $i \geq 2$, is a prime-element, then $N(\langle x \rangle)'$ is connected.*

Proof. Since $N(\langle x \rangle)' = O(\langle x \rangle)' \cup i(\langle x \rangle)'$ and $N(\langle x \rangle)'$ is connected, we have $O(\langle x \rangle)' = \emptyset$ or $i(\langle x \rangle)' = \emptyset$. Hence $|\langle x \rangle|$ is a prime or $\langle x \rangle$ is a maximal cyclic subgroup, respectively.
Suppose that $\langle x \rangle = \langle (x_1, \cdots, x_t) \rangle$ is a maximal cyclic subgroup. Then $i(\langle x \rangle) = \emptyset$. Also since G is non-cyclic, $|\pi(\langle x \rangle)| = t$. Now by Lemma 1, $O(\langle x \rangle)'$ and consequently $N(\langle x \rangle)'$ is connected.
Now let $|\langle x \rangle| = p_2$. Then

$$N(\langle x \rangle) = \{\langle a \rangle = \langle (a_1, \cdots, a_t) \rangle \mid a_2^{p_2^i} = x, \text{for some } i \in \mathbb{Z} \text{ and } a_j \in P_j,\ j \neq 2\}.$$

Let $\langle b \rangle$, $\langle c \rangle$ be two distinct prime-elements of P_1. For any $\langle a \rangle \in N(\langle x \rangle)$, either $\langle xb \rangle \notin N(\langle a \rangle)$ or $\langle xc \rangle \notin N(\langle a \rangle)$. Also $\langle xb \rangle$ is not adjacent to $\langle xc \rangle$, as desired. □

The following key lemma will be used frequently in the rest of the paper.

Lemma 9. *Let $G = P_1 \times \cdots \times P_t$, $t \geq 2$, be an abelian group and P_1 is not cyclic. If $\langle x \rangle$ is a maximal cyclic subgroup of G, then there is a prime-element $\langle y \rangle$ of P_2 such that $deg(\langle y \rangle) > deg(\langle x \rangle)$.*

Proof. Let $\langle x \rangle \subset G$ be a maximal cyclic subgroup of order $p_1^{r_1} \cdots p_t^{r_t}$. Then

$$deg(\langle x \rangle) = (r_1 + 1)(r_2 + 1) \cdots (r_t + 1) - 2.$$

Suppose that $\langle y \rangle \subset \langle x \rangle$ and $|\langle y \rangle| = p_2$. Then

$$\begin{aligned} deg(\langle y \rangle) &\geq |V(\mathbb{Z}_{p_1^{r_1}} \times \mathbb{Z}_{p_1})| \times r_2 \times (r_3 + 1) \cdots (r_t + 1) - 1 \\ &> (r_1 p_1 + 2)(r_2)(r_3 + 1) \cdots (r_t + 1) - 2. \end{aligned}$$

Since $(r_1 p_1 + 2) r_2 \geq (r_1 + 1)(r_2 + 1)$, $deg(\langle y \rangle) > deg(\langle x \rangle)$. □

Let G be a non-cyclic group. Similar to the last section, we consider $S_1(G) = \{\langle x \rangle \subset G \mid N(\langle x \rangle)' \text{ is connected }\}$. By Lemma 8, one can see that

$$S_1(G) \subseteq \{\langle x \rangle \subset G \mid \langle x \rangle \text{ is a prime-element or maximal cyclic subgroup}\}.$$

Theorem 8. *Let G, H be abelian groups and $G = P_1 \times \cdots \times P_t$, $t \geq 2$, where P_i are sylow subgroups of G and P_1, P_2 are not cyclic. Also let $H = Q_1 \times \cdots \times Q_s$. Then $\mathcal{I}_c(G) \cong_\varphi \mathcal{I}_c(H)$ if and only if $t = s$ and by a convenient permutation σ, $\mathcal{I}_c(P_i) \cong \mathcal{I}_c(Q_{\sigma(i)})$.*

Proof. By Lemma 8,

$$S_1(G) = \{\langle x \rangle \subset G \mid \langle x \rangle \text{ is a prime-element or maximal cyclic subgroup}\}.$$

Also by Lemma 9, any element of maximum degree in $S_1(G)$ has prime order. Let $\langle a \rangle$ has maximum degree and $|\langle a \rangle| = p$. If $\langle b \rangle \in S_1(G)$ and $|\langle b \rangle|$ is not prime, then there is $q \neq p$ such that $q \big| |\langle b \rangle|$. Assume that $\langle y \rangle \subset \langle b \rangle$ and $|\langle y \rangle| = q$. Hence $d(\langle a \rangle, \langle y \rangle) = 2$ and $d(\langle a \rangle, \langle b \rangle) \leq 3$. If $d(\langle a \rangle, \langle b \rangle) = 2$ and $\langle a \rangle \sim \langle d \rangle \sim \langle b \rangle$, then $\langle a \rangle \subset \langle d \rangle$ and $\langle d \rangle \subset \langle b \rangle$, a contradiction. Thus $d(\langle a \rangle, \langle b \rangle) = 1$ or 3. Set $S_2(G) = \{\langle x \rangle \in S_1(G) \mid d(\langle a \rangle, \langle x \rangle) = 0 \text{ or } 2 \text{ or } 4\}$. By Lemma 7,

$$S_2(G) = \{\langle b \rangle \in S_1(G) \mid |\langle b \rangle| \text{ is prime}\}.$$

By Theorem 3, H is not cyclic. Also since $\mathcal{I}_c(G)$ is connected, $s \geq 2$ and by Lemma 9, $\varphi(\langle a \rangle)$ is a prime-element, and then

$$\varphi(S_2(G)) = \{\langle x \rangle \in S_1(H) \mid \langle x \rangle \text{ is a prime-element}\}.$$

By a contrary, let $\langle h\rangle \in V(H)\setminus S_1(H)$ be a prime-element. Then $d(\langle h\rangle, \varphi(S_2(G))) = 2$ but $d(\varphi^{-1}(\langle h\rangle), S_2(G)) \in \{0,1\}$, a contradiction. Hence $S_1(H)$ contains all prime-elements of H and $\varphi(S_2(G))$ is the set of all prime-elements of H. Thus all elements of prime order in G and H are specified. Set

$$T(G) = \{\langle x\rangle \subset G \;\big|\; |N(\langle x\rangle) \cap S_2(G)| = 1\} \cup S_2(G).$$

It can be seen that $\langle x\rangle \in T(G)$ if and only if $|\langle x\rangle|$ is prime power. Thus $T(G) = V(P_1)\cup\cdots\cup V(P_t)$. Let A_1, A_2 be two connected components of $[T(G)]$. Then A_1, A_2 are in the same sylow subgroup if and only if $d_{\mathcal{I}_c(G)}(A_1, A_2) = 4$ and A_1, A_2 are in the distinct sylow subgroups if and only if $d_{\mathcal{I}_c(G)}(A_1, A_2) = 2$. Thus any sylow subgroup of G is specified by the graph. Set

$$T(H) = \{\ \langle y\rangle \subset H \;\big|\; |N(\langle y\rangle) \cap \varphi(S_2(G))| = 1\ \} \cup \varphi(S_2(G)).$$

Similarly $T(H) = V(Q_1) \cup \cdots \cup V(Q_s)$. By the hypothesis, $\varphi(T(G)) = T(H)$. Hence $s = t$ and for a convenient permutation σ, $\mathcal{I}_c(P_i) \cong_\varphi \mathcal{I}_c(Q_{\sigma(i)})$.
Conversely, let $s = t$ and σ be a permutation, where $\mathcal{I}_c(P_i) \cong \mathcal{I}_c(Q_{\sigma(i)})$. By Corollary 3 and Theorem 6, $\overrightarrow{\mathcal{I}_c}(P_i) \cong \overrightarrow{\mathcal{I}_c}(Q_{\sigma(i)})$. Now Theorem 1, completes the proof. $\square$

Now we prove the previous theorem for abelian groups with only one non-cyclic sylow subgroup.

Theorem 9. *Let G, H be abelian groups and $G = P_1 \times \cdots \times P_t$, $t \geq 2$, and $H = Q_1 \times \cdots \times Q_s$, where P_1 is not cyclic and P_i are cyclic for $i > 1$. Then $\mathcal{I}_c(G) \cong_\varphi \mathcal{I}_c(H)$ if and only if $s = t$ and for a convenient permutation σ, $\mathcal{I}_c(P_i) \cong_\varphi \mathcal{I}_c(Q_{\sigma(i)})$.*

Proof. By Theorem 3, H is not cyclic. Also since $\mathcal{I}_c(G)$ is connected, $s \geq 2$. We consider the following two cases:

Case 1. P_1 is not an elementary abelian group. Hence by the hypothesis there are distinct cyclic subgroups $\langle b\rangle$, $\langle b_1\rangle$ and $\langle b_2\rangle$ of P_1 such that $\langle b\rangle$ is a prime-element and $\langle b\rangle \subset \langle b_1\rangle \cap \langle b_2\rangle$ and $|\langle b_1\rangle| = |\langle b_2\rangle| = p_1^2$. Since for any $\langle x\rangle \in N(\langle b\rangle)$, $\langle x\rangle \notin N(\langle b_1\rangle)$ or $\langle x\rangle \notin N(\langle b_2\rangle)$ and $\langle b_1\rangle$ is not adjacent to $\langle b_2\rangle$, $N(\langle b\rangle)'$ is connected and by Lemma 8, for any i, $S_1(G) \cap P_i \neq \emptyset$.
Let $deg(\langle a\rangle) = Max\{deg(\langle x\rangle) \;\big|\; \langle x\rangle \in S_1(G)\}$. Since P_1 is not cyclic, by Lemma 9, $\langle a\rangle$ is prime-element. Let $\langle c\rangle \in S_1(G)$. $d(\langle a\rangle, \langle c\rangle) = 2$ or 4 if and only if $\langle c\rangle$ is a prime-element. Also for $\langle x\rangle \in S_1(G)$, $\langle x\rangle \subset P_1$ if and only if there is $\langle y\rangle \subset G$ such that $d(\langle x\rangle, \langle y\rangle) = 4$. Assume that $\langle x_1\rangle \in S_1(G) \cap V(P_1)$ and $A = \{\langle y\rangle \subset G \;\big|\; d(\langle y\rangle, \langle x_1\rangle) = 4\}$. One can see that $\langle y\rangle \in A$ if and only if $\langle y\rangle \subset P_1$ and $\langle y\rangle \notin N[\langle x_1\rangle]$. Let $\langle y_1\rangle \in A$ and $B = \{\langle y\rangle \subset G \;\big|\; d(\langle y\rangle, \langle y_1\rangle) = 4\ \}$. Similarly $B \subseteq V(P_1)$ and $N_{\mathcal{I}_c(P_1)}[\langle x_1\rangle] \subseteq B$. One can see that $V(P_1) = A \cup B$. Now for $\langle z\rangle \in S_1(G)$, $d(\langle z\rangle, V(P_1)) = 2$ if and only if

$\langle z\rangle$ is a prime-element and $|\langle z\rangle| \neq p_1$. Let $\langle a_2\rangle \subset P_2, \cdots, \langle a_t\rangle \subset P_t$ be prime-elements. Then for $i \geq 2$,

$$V(P_i)=\{\langle x\rangle\subset G \big| \langle x\rangle\in N[\langle a_i\rangle] \text{ and } N(\langle x\rangle) \cap (V(P_1) \cup (\{\langle a_2\rangle, \cdots, \langle a_t\rangle\} \setminus \{\langle a_i\rangle\}))=\emptyset \ \}.$$

Hence all vertices of all sylow subgroups of G are specified. By Theorem 8, one of $Q_1, \cdots, Q_s$ is not cyclic and any others are cyclic. We can assume Q_1 is not cyclic and $H = Q_1 \times \cdots \times Q_t$. consequently $\varphi(\langle a\rangle)$ is a prime-element and by an elementary argument $\mathcal{I}_c(P_i) \cong_\varphi \mathcal{I}_c(Q_i)$.

Case 2. P_1 is an elementary abelian group. Let $\langle x\rangle \subset P_1$ be a cyclic subgroup. Then $|\langle x\rangle| = p_1$ and

$$N(\langle x\rangle) = \{ \ \langle y\rangle \subset G \ \big| \ \langle x\rangle \subset \langle y\rangle \text{ and } |\langle y\rangle| = p_1 p_2^{r_2} \cdots p_t^{r_t} \ , \ r_j \geq 1 \text{ for some } \ j \ \}.$$

Let $P_i = \langle b_i\rangle$, $i \geq 2$, and $\langle y\rangle = \langle x b_2\rangle \cdots \langle b_t\rangle$. Then $\langle y\rangle \in N(\langle x\rangle)$ and for any $\langle z\rangle \in N(\langle x\rangle)$, $\langle z\rangle \subset \langle y\rangle$. Hence $N(\langle x\rangle)'$ is not connected, and then $V(P_1) \cap S_1(G) = \emptyset$. Let $\langle a\rangle \in S_1(G)$ has the maximum degree in $S_1(G)$. Then $\langle a\rangle$ is prime-element and similar to last case all prime-elements of $P_2, \cdots, P_t$ are specified. Assume that $\langle a_2\rangle \subset P_2, \cdots, \langle a_t\rangle \subset P_t$ are prime-elements. Set

$$A_1 = \{\langle x\rangle \subset G \ \big| \ d(\langle x\rangle, \{\langle a_2\rangle, \cdots, \langle a_t\rangle\}) = 2 \ \},$$

and for $i \geq 2$,

$$A_i = \{\langle x\rangle \subset G \ \big| \ d(\langle x\rangle, A_1 \cup \{\langle a_2\rangle, \cdots, \langle a_{i-1}\rangle, \langle a_{i+1}\rangle, \cdots \langle a_t\rangle\}) = 2\}.$$

Then $V(P_j) = A_j$, for any j, $1 \leq j \leq t$. Hence all vertices of all sylow subgroups of G are specified. By Theorem 8, and last case $H = Q_1 \times \cdots \times Q_t$ where Q_1 is an elementary abelian group and $Q_2, \cdots, Q_t$ are cyclic. By Theorem 4 and Remark 2, $\mathcal{I}_c(P_i) \cong_\varphi \mathcal{I}_c(Q_i)$, as required.
The converse of theorem by Corollary 3, Theorem 6 and Theorem 1, is clear. □

According to the above theorems and Theorem 1, the following results can be easily observed.

Corollary 5. *Let $G = P_1 \times \cdots \times P_t \times \mathbb{Z}_{p_{t+1}^{\alpha_{t+1}}} \times \cdots \times \mathbb{Z}_{p_n^{\alpha_n}}$ be an abelian group such that P_i are non-cyclic and $exp(P_i) \geq p_i^2$ and H be an abelian group. Then $\mathcal{I}_c(G) \cong \mathcal{I}_c(H)$ if and only if $H = P_1 \times \cdots \times P_t \times \mathbb{Z}_{q_{t+1}^{\alpha_{t+1}}} \times \cdots \times \mathbb{Z}_{q_n^{\alpha_n}}$ where $q_{t+1}, \cdots, q_n$ are distinct primes and $\{p_1, \cdots, p_t\} \cap \{q_{t+1}, \cdots, q_n\} = \emptyset$.*

Corollary 6. *Let G and H be abelian groups. Then $\mathcal{I}_c(G) \cong \mathcal{I}_c(H)$ if and only if $\overrightarrow{\mathcal{I}_c}(G) \cong \overrightarrow{\mathcal{I}_c}(H)$.*

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References

[1] J. Abawajy, A. Kelarev, and M. Chowdhury, *Power graphs: A survey*, Electron. J. Graph Theory Appl. **1** (2013), no. 2, 125–147
https://dx.doi.org/10.5614/ejgta.2013.1.2.6.

[2] P.J. Cameron, *The power graph of a finite group, II*, J. Group Theory **13** (2010), no. 779–783
https://doi.org/10.1515/jgt.2010.023.

[3] P.J. Cameron and S. Ghosh, *The power graph of a finite group*, Discrete Math. **311** (2011), no. 13, 1220–1222
https://doi.org/10.1016/j.disc.2010.02.011.

[4] P.J. Cameron, H. Guerra, and Š. Jurina, *The power graph of a torsion-free group*, J. Algebraic Combin. **49** (2019), no. 1, 83–98
https://doi.org/10.1007/s10801–018–0819–1.

[5] P.J. Cameron and S.H. Jafari, *On the connectivity and independence number of power graphs of groups*, Graphs Combin. **36** (2020), no. 3, 895–904
https://doi.org/10.1007/s00373–020–02162–z.

[6] K.N. Cheng, M. Deaconescu, M.L. Lang, and W.J. Shi, *Corrigendum and addendum to: "classification of finite groups with all elements of prime order"*, Proc. Amer. Math. Soc. **117** (1993), no. 4, 1205–1207
https://doi.org/10.2307/2159554.

[7] M. Deaconescu, *Classification of finite groups with all elements of prime order*, Proc. Amer. Math. Soc. **106** (1989), no. 3, 625–629
https://doi.org/10.2307/2047414.

[8] Z. Garibbolooki and S.H. Jafari, *Planarity of inclusion graph of cyclic subgroups of finite group*, Math. Interdisc. Res. **5** (2020), no. 4, 303–314
https://doi.org/10.22052/mir.2020.209251.1183.

[9] S.H. Jafari and S. Chattopadhyay, *Spectrum of proper power graphs of the direct product of certain finite groups*, Linear Multilinear Algebra **70** (2022), no. 20, 5460–5481
https://doi.org/10.1080/03081087.2021.1918051.

[10] A.V. Kelarev and S.J. Quinn, *A combinatorial property and power graphs of groups*, Contrib. General Algebra **12** (2000), 229–235.

[11] X. Ma, *Proper connection of power graphs of finite groups*, J. Algebra . Appl. **20** (2021), no. 3, Article ID: 2150033
https://doi.org/10.1142/S021949882150033X.

[12] X. Ma and L. Zhai, *Strong metric dimensions for power graphs of finite groups*, Comm. Algebra **49** (2021), no. 11, 4577–4587 https://doi.org/10.1080/00927872.2021.1924764.

[13] G.R. Pourgholi, H. Yousefi-Azari, and A.R. Ashrafi, *The undirected power graph of a finite group*, Bull. Malays. Math. Sci. **38** (2015), no. 4, 1517–1525 https://doi.org/10.1007/s40840–015–0114–4.

[14] R. Rajkumar and T. Anitha, *Reduced power graph of a group*, Electron. Notes Discrete Math. **63** (2017), 69–76 https://doi.org/10.1016/j.endm.2017.10.063.

[15] ———, *Some results on the reduced power graph of a group*, Southeast Asian Bull. Math. **45** (2021), 241–262.

[16] D.J.S. Robinson, *A Course in the Theory of Groups*, vol. 80, Springer Science & Business Media, 2012.

[17] M. Shaker and M.A. Iranmanesh, *On groups with specified quotient power graphs*, Int. J. Group Theory **5** (2016), no. 3, 49–60 https://doi.org/10.22108/ijgt.2016.8542.