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Research Article

Simultaneous coloring of vertices and incidences of hypercubes

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Abstract: An element $i = (v, e)$ of a graph G is called an incidence of G, if $v \in V(G)$, $e \in E(G)$ and $v \in e$. A vi-simultaneous proper coloring of a graph G is a coloring of the vertices and incidences of G properly at the same time such that any two adjacent or incident elements receive distinct colors. In this paper, we investigate the simultaneous coloring of vertices and incidences of hypercubes.

Keywords: Incidence of graph, simultaneous coloring of graph, hypercube

AMS Subject classification: 05C15

1. Introduction

All graphs we consider in this paper are simple, finite and undirected. For a graph G, $V(G)$, $E(G)$ are its vertex set and edge set, respectively. Also, maximum degree of a graph G is denoted by $\Delta(G)$. For vertex $v \in V(G)$, $N_G(v)$ is the set of neighbors of v in G and any vertex of degree k is called a k-vertex. From now on, we use the notation $[n]$ instead of $\{1, \ldots, n\}$.
Apart from vertices and edges of a graph, incidences are other elements of the graph introduced by Brualdi and Massey in 1993 in [1]. Any pair $i = (v, e)$ of a graph G is called an incidence of G, if $v \in V(G)$, $e \in E(G)$ and $v \in e$. Also in this case the elements v and i are called incident. Let $I(G)$ be the set of incidences of a graph G. Two incidences (v, e) and (w, f) are adjacent if (i) $v = w$, or (ii) $e = f$, or (iii) $\{v, w\} = e$ or f. For any edge $e = \{u, v\}$, we call (u, e), the first incidence of u and (v, e), the second incidence of u.

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A mapping c from $V(G)$ to $[k]$ is a proper k-coloring of G, if $c(v) \neq c(u)$ for any two adjacent vertices. A minimum integer k that G has a proper k-coloring is the chromatic number of G and denoted by $\chi(G)$. Incidence coloring of graphs as another type of coloring is a proper coloring of incidences of a graph such that any two adjacent incidences have different colors. The incidence chromatic number of a graph G, denoted by $\chi_i(G)$, is the minimum integer k such that G is incidence k-colorable. Proper coloring of vertices and incidences of a graph G at the same time, is a concept named vi-simultaneous proper coloring of G, which is introduced by Mozafari-Nia et al. in [4].

Definition 1. *[4] Let G be a graph. A vi-simultaneous proper k-coloring of G is a coloring $c : V(G) \cup I(G) \longrightarrow [k]$ in which any two adjacent or incident elements in the set $V(G) \cup I(G)$ receive distinct colors. The vi-simultaneous chromatic number, denoted by $\chi_{vi}(G)$, is the smallest integer k such that G has a vi-simultaneous proper k-coloring.*

The authors in [4] showed that the minimum number of colors for vi-simultaneous proper coloring of G is equal to the chromatic number of $G^{\frac{3}{3}}$, where $G^{\frac{3}{3}}$ is $\frac{3}{3}$-power of the graph G introduced by Iradmusa in [6] and defined in the following.
For a positive integer k, a k-power of G, denoted by G^k, is a graph with $V(G^k) = V(G)$. Two vertices u and v of G^k are adjacent if and only if $1 \leq d_G(u,v) \leq k$. Moreover, the k-subdivision of G, denoted by $G^{\frac{1}{k}}$, is constructed by replacing each edge xy of G with a path of length k, say P_{xy} with vertices $(xy)_0, (xy)_1, \ldots, (xy)_k$, where for $l \in \{0,1,2,\ldots,k\}$, $(xy)_l$ has distance l from the vertex x. Note that $(xy)_l = (yx)_{k-l}$, $x = (xy)_0 = (yx)_k$ and $y = (yx)_0 = (xy)_k$. Any vertex $(xy)_0$ of $G^{\frac{1}{k}}$ is called a *terminal vertex* (or briefly t-vertex) and any of the remaining vertices is called an *internal vertex* (or briefly i-vertex). The fractional power of graphs which is in association with the two concepts above is defined as follows.

Definition 2. *[6]* Let G be a graph and $m, n \in \mathbb{N}$. The graph $G^{\frac{m}{n}}$ is defined to be the *m-power* of the n-subdivision of G. In other words, $G^{\frac{m}{n}} = (G^{\frac{1}{n}})^m$.

According to the case $G^{\frac{3}{3}}$ in Definition 2 and the definition of the adjacency between incidences in the graph G, the internal vertices of $G^{\frac{3}{3}}$ can be considered as the incidences of G and so we denote two internal vertices $(uv)_1$ and $(uv)_2$ on the path P_{uv} with (u,v) and (v,u), respectively. In general, for a vertex $v \in V(G)$, we can define $I_1^G(v) = \{(vu)_1 \mid u \in N_G(v)\}$ and $I_2^G(v) = \{(vu)_2 \mid u \in N_G(v)\}$. In addition, we have $I_G(v) = I_1^G(v) \cup I_2^G(v)$, $I_1^G[v] = \{v\} \cup I_1^G(v)$ and $I_G[v] = \{v\} \cup I_G(v)$. Sometime we remove the index G for simplicity.
So far, the chromatic number of $\frac{3}{3}$-power of some classes of graphs are investigated in [3–5, 7, 10]. Clearly, for any graph G with maximum degree Δ, the value $\Delta + 2$ is a trivial lower bound for $\chi(G^{\frac{3}{3}}) = \chi_{vi}(G)$. Regarding the upper bound, it is conjectured that the chromatic number of $\frac{3}{3}$-power of any graph G with maximum degree Δ is at most $2\Delta + 1$ [3].

Conjecture 1. *[3] For any graph G with $\Delta(G) \geq 2$, $\chi(G^{\frac{3}{3}}) \leq 2\Delta(G) + 1$.*

In [4], the vi-simultaneous coloring of some classes of graphs like k-degenerated graphs, cycles, forests, complete graphs and regular bipartite graphs are investigated and the correctness of Conjecture 1 has shown for such classes of graphs. In this paper, we are going to consider the vi-simultaneous coloring of hypercubes and show that the Conjecture 1 is true for these graphs. The main theorems of this paper are as follows.

Theorem 2. *For $n \in \mathbb{N}$, the following statements are equivalent:*

(i) $\chi(Q_n^{\frac{3}{3}}) = n + 2$,

(ii) $\chi(Q_{n+1}^2) = n + 2$,

(iii) $n = 2^k - 2$ *where* $1 < k \in \mathbb{N}$.

Theorem 3. *If $n \in \{2^k + l \mid k \in \mathbb{N}, l \in \mathbb{Z}, -1 \leq l \leq 2\} \setminus \{2, 6\}$, then $\chi(Q_n^{\frac{3}{3}}) = n + 3$.*

Definition 3. Let G be a nonempty graph with vi-simultanious chromatic number equal to $\Delta(G) + 1 + s$, where $s \in \mathbb{N}$. We say that G is a graph of vi-class s.

According to the main results of the paper, $\chi_{vi}(Q_n) \leq n+3$ for any $n \in \{1, 2, \ldots, 10\}$. So these hypercubes are graphs of vi-class one or two and we conjecture that there is no hypercube of vi-class three.

Conjecture 4. $\chi_{vi}(Q_n) \leq n + 3$ for any $n \in \mathbb{N}$.

2. Proof of the main theorems

In this section, the chromatic number of $\frac{3}{3}$-power of hypercubes are investigated. A Cartesian product of two graphs G and H are a graph denoted by $G \square H$, whose vertex set is $\{(u, v) : u \in V(G), v \in V(H)\}$. Two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if $u_1 = u_2$ and v_1 is adjacent to v_2 or u_1 is adjacent to u_2 and $v_1 = v_2$.
The n-cube or n-dimensional hypercube Q_n is a graph whose set of vertices V, consists of the 2^n n-dimensional boolean vectors where two vertices are adjacent whenever they differ in exactly one coordinate. It can be easily seen that Q_n is defined in terms of the Cartesian product of n copy of K_2 and so $Q_n = Q_r \square Q_t$ where $r + t = n$.
In [8] and [2], it is shown that the incidence chromatic number of Q_n is $n+1$ or $n+2$.

Theorem 5. [2, 8] *For any $n \in \mathbb{N}$,* $\chi_i(Q_n) = \begin{cases} n+1 & n = 2^k - 1, k \in \mathbb{N} \\ n+2 & \textit{otherwise.} \end{cases}$

In [9], using a relation between dominating sets and incidence chromatic number, the author characterize the $(r + 1)$-incidence colorable r-regular graphs.

Theorem 6. [9] *If G is an r-regular graph, then $\chi_i(G) = \chi(G^2) = r+1$ if and only if $V(G)$ is a disjoint union of $r+1$ dominating sets.*

Theorem 7. *For a hypercube graph Q_n with $n \geq 1$, $\chi_{vi}(Q_n) \leq \Delta(Q_n) + 4 = n+4$.*

Proof. Since $\chi_{vi}(Q_n) \leq \chi_i(Q_n) + \chi(Q_n)$ and Q_n is a bipartite graph, by Theorem 5 we have $\chi_{vi}(Q_n) \leq n+2+2 = n+4$. $\square$

In the following, we are going to show that $\chi_{vi}(Q_n) = n+2$ if and only if $n = 2^k - 2$ for $k \geq 2$, using the following lemmas. The set $\{c(a) \mid a \in A\}$ is denoted by $c(A)$, where $c : D \to R$ is a function and $A \subseteq D$.

Lemma 1. [4] Let G be a graph with maximum degree Δ and c is a proper $(\Delta+2)$-coloring of $G^{\frac{3}{3}}$ with colors from $[\Delta+2]$. Then $|c(I_2(v))| \leq \Delta - d_G(v) + 1$ for any t-vertex v. Specially $|c(I_2(v))| = 1$ for any Δ-vertex v of G.

Lemma 2. *Let $n \in \mathbb{N}$ and $\chi(Q_n^{\frac{3}{3}}) = n+2$. Then n is an even number.*

Proof. Suppose that $c : V(Q_n^{\frac{3}{3}}) \longrightarrow [n+2]$ is a proper coloring, $c(v) = k$ and $N_{Q_n}(v) = \{w_1, \ldots, w_n\}$. Therefore, $k \notin c(I_1(v) \cup I_2(v) \cup N_{Q_n}(v))$. So there is exactly one vertex x_1 in $N_{Q_n}(w_1) \setminus \{v\}$ such that $c((w_1, x_1)) = k$.
Since the vertices v and x_1 have a common neighbor w_1 in Q_n, there should be one other common neighbor, named w_2. Since $c((w_1, x_1)) = k$, by Lemma 1 we have $c((w_2, x_1)) = k$. Now, consider the t-vertex w_3. Similarly, since $c(w_3) \neq k$, there is a vertex $x_2 \in N_{Q_n}(w_3) \setminus \{v\}$ such that $c((w_3, x_2)) = k$. Note that, $x_1 \neq x_2$. Otherwise, two vertices v and x_1 have three common neighbors which is a contradiction. Again, since the vertices v and x_2 have the common neighbor w_3, there should be one other common neighbor, named w_4 such that $c((w_4, x_2)) = k$. By continuing this process, $N_{Q_n}(v)$ is partitioned into sets of size two. Therefore n must be even. $\square$

Lemma 3. *Suppose that $c : V(Q_n^{\frac{3}{3}}) \longrightarrow [n+2]$ is a proper coloring. If $c((v, u_0)) = k$, where $(v, u_0) \in I_1(v)$ then there is a vertex $u_i \in N_{Q_n}(v)$ with color k.*

Proof. Suppose that $N_{Q_n}(v) = \{u_0, u_1, \ldots, u_{n-1}\}$ and $k \notin c(N_{Q_n}(v) \setminus \{u_0\})$. Then for any vertex $u_i \in N_{Q_n}(v) \setminus \{u_0\}$, we have $k \in c(I_1(u_i))$. Moreover, since $c((v, u_0)) = k$, there is no color k on i-vertices of the edge $\{v, u_i\}$, $i \in [n-1]$. Hence, there is a vertex $x_1 \in N(u_1) \setminus \{v\}$ such that $c((u_1, x_1)) = k$.
Since the vertices v and x_1 have a common neighbor u_1, so there should be one other common neighbor, named u_2. Since $c((u_1, x_1)) = k$, by Theorem 1 we have $c((u_2, x_1)) = k$. Now, consider the t-vertex u_3. Similarly, since $c(u_3) \neq k$, there is a vertex $x_2 \in N(u_3) \setminus \{v\}$ such that $c((u_3, x_2)) = k$. Note that, $x_1 \neq x_2$. Otherwise, two vertices v and x_1 have three common neighbors in Q_n which is a contradiction. Again, since two vertices v and x_2 have a common neighbor u_3, there should be one

other common neighbor, named u_4 and $c((u_4, x_2)) = k$.
By continuing this process, $\{u_1, \ldots, u_{n-1}\}$ is partitioned into sets of size two. But $n-1$ is an odd integer and so this partition of $\{u_1, \ldots, u_{n-1}\}$ is impossible. $\square$

Corollary 1. *Let $n \in \mathbb{N}$ and $c : V(Q_n^{\frac{3}{3}}) \longrightarrow [n+2]$ be a proper coloring. Then $c(I_1(v)) = c(N_{Q_n}(v))$ for any vertex $v \in V(Q_n)$ and the restriction of c to $V(Q_n)$ is a proper coloring of Q_n^2.*

Proof of Theorem 2. The equivalence of (ii) and (iii) is derived from Theorems 5 and 6. Therefore, we only prove that (i) and (ii) are equivalent.
$(i) \Rightarrow (ii)$. Suppose that $G = Q_n$, $V(G) = \{v_i \mid 1 \leq i \leq 2^n\}$, $\chi(G^{\frac{3}{3}}) = n+2$ and $c : V(G^{\frac{3}{3}}) \longrightarrow [n+2]$ is a proper coloring. We know that $Q_{n+1} = Q_n \Box K_2$ and so we can consider $V(Q_{n+1}) = V_1 \cup V_2$ such that $V_1 = \{v_i \mid 1 \leq i \leq 2^n\}$, $V_2 = \{u_i \mid 1 \leq i \leq 2^n\}$, $Q_{n+1}[V_1] \cong G \cong Q_{n+1}[V_2]$ and $\{v_i, u_i\} \in E(Q_{n+1})$ for each $i \in [2^n]$. Now we prove that the following coloring

$$c' : V(Q_{n+1}) \to [n+2], c'(x) = \begin{cases} c(x) & x \in V_1 \\ c((v_i, v_j)) & x = u_j \in V_2, v_i \in N_G(v_j) \end{cases}$$

is a proper coloring of Q_{n+1}^2.
Suppose that x and y are two adjacent vertices in Q_{n+1}^2. There are six cases.

Case 1. $d_{Q_{n+1}}(x, y) = 1$ and $x, y \in V_1$.
In this case, x and y are adjacent in G and so $c'(x) = c(x) \neq c(y) = c'(y)$.

Case 2. $d_{Q_{n+1}}(x, y) = 1$, $x, y \in V_2$.
In this case $x = u_i$ and $y = u_j$ and so $c'(x) = c((v_j, v_i) \neq c((v_i, v_j)) = c'(y)$.

Case 3. $d_{Q_{n+1}}(x, y) = 1$, $x \in V_1$ and $y \in V_2$.
In this case $x = v_i$ and $y = u_i$ and so $c'(x) = c(v_i) \neq c((v_l, v_j)) = c'(y)$ where $v_l \in N_G(v_j)$.

Case 4.$d_{Q_{n+1}}(x, y) = 2$ and $x, y \in V_1$.
In this case, Corollary 1 implies that $c(x) \neq c(y)$ and so $c'(x) = c(x) \neq c(y) = c'(y)$.

Case 5. $d_{Q_{n+1}}(x, y) = 2$ and $x, y \in V_2$.
In this case, $x = u_i$ and $y = u_j$ such that $d_G(v_i, v_j) = 2$. So $c'(x) = c((v_l, v_i) \neq c((v_l, v_j)) = c'(y)$ where $v_l \in N_G(v_i) \cap N_G(v_j)$.

Case 6. $d_{Q_{n+1}}(x, y) = 2$, $x \in V_1$ and $y \in V_2$.
In this case, $x = v_i$, $y = u_j$ such that $d_G(v_i, v_j) = 1$. So $c'(x) = c(v_i) \neq c((v_i, v_j)) = c'(y)$.
Therefore, c' is a proper coloring and $\chi(Q_{n+1}^2) = n+2$.

$(ii) \Rightarrow (i)$. Suppose that $\chi(Q_{n+1}^2) = n+2$ and $c : V(Q_{n+1}^2) \longrightarrow [n+2]$ is a proper coloring. Again, suppose that $V(Q_{n+1}) = V_1 \cup V_2$ such that $V_1 = \{v_i \mid 1 \leq i \leq 2^n\}$,

$V_2 = \{u_i \mid 1 \le i \le 2^n\}$, $Q_{n+1}[V_1] \cong Q_n \cong Q_{n+1}[V_2]$ and $\{v_i, u_i\} \in E(Q_{n+1})$ for each $i \in [2^n]$. We prove that the following coloring

$$c' : V(H) \to [n+2], c'(x) = \begin{cases} c(v_i) & x = v_i \in V_1 \\ c(u_j) & x = (v_i, v_j), v_i \in N_{Q_n}(v_j) \end{cases}$$

is a proper coloring of $H = Q_n^{\frac{3}{3}}$ where $V(Q_n) = V_1$.
Suppose that x and y are two adjacent vertices in H. There are five cases.

Case 1. $x, y \in V_1$.
In this case, x and y are adjacent in Q_{n+1} and so $c'(x) = c(x) \ne c(y) = c'(y)$.

Case 2. $x \in V_1$ and $y \in I_1(x)$.
In this case $x = v_i$ and $y = (v_i, v_j)$. So $c'(x) = c(v_i) \ne c(u_j) = c'(y)$.

Case 3. $x \in V_1$ and $y \in I_2(x)$.
In this case $x = v_i$ and $y = (v_j, v_i)$. So $c'(x) = c(v_i) \ne c(u_i) = c'(y)$.

Case 4. $x, y \in I_1(v_i)$.
In this case, $x = (v_i, v_j)$ and $y = (v_i, v_l)$ such that $v_j, v_l \in N_G(v_i)$. So $c'(x) = c(u_j) \ne c(u_l) = c'(y)$.

Case 5. $x = (v_i, v_j)$ and $y = (v_l, v_i)$ where $v_j, v_l \in N_{Q_n}(v_i)$.
In this case, $c'(x) = c(u_j) \ne c(u_i) = c'(y)$.
Note that $d_{Q_{n+1}}(x, y) = 1$ in Cases 1, 3 and 5 and $d_{Q_{n+1}}(x, y) = 2$ in other cases. Therefore, c' is a proper coloring and $\chi(H) = n + 2$. ■

As you see, $\chi(Q_n^{\frac{3}{3}})$ takes on three distinct values, $n+2$, $n+3$, or $n+4$. In continue, we find some $n \in \mathbb{N}$ such that $\chi(Q_n^{\frac{3}{3}}) = n + 3$.

Proof of Theorem 3. We divide the proof to four parts:
(i) $l = -1$. By Theorem 2, $\chi(Q_n^{\frac{3}{3}}) \ge n+3$. In addition, by Theorem 5, $\chi_i(Q_n) = n+1$. Now, since Q_n is a bipartite graph, then $\chi(Q_n^{\frac{3}{3}}) \le \chi_i(Q_n) + \chi(Q_n) = n+3$. Therefore, $\chi(Q_n^{\frac{3}{3}}) = n + 3$.
(ii) $l = 0$. By Theorem 2, $\chi(Q_n^{\frac{3}{3}}) \ge n + 3$. It is enough to present a proper $(n+3)$-coloring for $Q_n^{\frac{3}{3}}$. We know that $Q_n = Q_{n-1} \square K_2$ and so we can consider $V(Q_n) = V \cup V'$ such that $V = \{v_i \mid 1 \le i \le 2^{n-1}\}$, $V' = \{v'_i \mid 1 \le i \le 2^{n-1}\}$, $G_1 = Q_n[V] \cong Q_{n-1} \cong Q_n[V'] = G_2$ and $\{v_i, v'_i\} \in E(Q_n)$ for each $i \in [2^{n-1}]$. Also let (A_1, B_1) and (A_2, B_2) be the bipartitions of G_1 and G_2, respectively and $(A_1 \cup A_2, B_1 \cup B_2)$ be the bipartition of Q_n. Since $n - 1 = 2^k - 1$, by Theorem 6, $V(G_1)$ is a disjoint union of n dominating sets $\mathcal{S} = \{S_1, \dots, S_n\}$ and also $V(G_2)$ is a disjoint union of n dominating sets $\mathcal{S}' = \{S'_1, \dots, S'_n\}$. It can be easily seen that $|S_i \cap A_1| = |S_i \cap B_1|$ for any dominating set $S_i \in \mathcal{S}$ in G_1 (and similarly for $\mathcal{S}'$ and G_2).
Suppose that $r \ge 2$. For any t-vertex $v_i \in S_r$, color all vertices of $I_2(v_i) \cap V(G_1^{\frac{3}{3}})$ with color r and for any vertex $v'_i \in S'_r$, color all vertices of $I_2(v'_i) \cap V(G_2^{\frac{3}{3}})$ with color

r. Then for any two t-vertices $v_i \in S_1$ and $v'_i \in S'_1$, if $v_i \in A_1$ and $v'_i \in B_2$, then color all i-vertices of $I_2(v_i) \cap V(G_1^{\frac{3}{3}})$ and $I_2(v'_i) \cap V(G_2^{\frac{3}{3}})$ with color 1 and if $v_i \in B_1$ and $v'_i \in A_2$, then color all i-vertices of $I_2(v_i) \cap V(G_1^{\frac{3}{3}})$ and $I_2(v'_i) \cap V(G_2^{\frac{3}{3}})$ with color $n+1$.
To color t-vertices of Q_n, color all t-vertices of S_1 and S'_1 with colors $n+2$ and $n+3$, respectively. Also for $i \geq 2$, assign color 1 and $n+1$ to all t-vertices of $(S_i \cap A_1) \cup (S'_i \cap A_2)$ and $(S_i \cap B_1) \cup (S'_i \cap B_2)$, respectively.
The only uncolored vertices of Q_n are all i-vertices on the edges between G_1 and G_2. Let F_i be the set of edges between S_i and S'_i ($i \in [n]$). Color all i-vertices on edges of F_i as follows.

- If $e = \{v_i, v'_i\} \in F_1$ such that $v_i \in A_1$ and $v'_i \in B_2$, then color i-vertices (v_i, v'_i) and (v'_i, v_i) with colors $n+1$ and 1, respectively.
- If $e = \{v_i, v'_i\} \in F_1$ such that $v_i \in B_1$ and $v'_i \in A_2$, then color i-vertices (v_i, v'_i) and (v'_i, v_i) with colors 1 and $n+1$, respectively.
- If $e = \{v_i, v'_i\} \in F_j$ and $j > 1$, then color i-vertices (v_i, v'_i) and (v'_i, v_i) with colors $n+2$ and $n+3$, respectively.

With a simple review, we can show that the given coloring is a proper $(n+3)$-coloring for $Q_n^{\frac{3}{3}}$.
(iii) $l = 1$. By Theorem 2, $\chi(Q_n^{\frac{3}{3}}) \geq n+3$. It is enough to show that $\chi(Q_n^{\frac{3}{3}}) \leq n+3$. Consider Q_n as a Cartesian product of two graphs Q_{n-2} and $Q_2 = C_4$. Note that since $n-2 = 2^k - 1$, $\chi_i(Q_{n-2}) = n-1$. Now, we divide the vertices of $Q_n^{\frac{3}{3}}$ to two sets: the first set, V_1, is the union of internal vertices of disjoint copies of Q_{n-2} and the second set, V_2, is the union of terminal and internal vertices of disjoint copies of Q_2. Therefore $\chi(Q_n^{\frac{3}{3}}) \leq \chi(Q_n^{\frac{3}{3}}[V_1]) + \chi(Q_n^{\frac{3}{3}}[V_2])$. But $Q_n^{\frac{3}{3}}[V_2]$ is a subgraph of $Q_{n-2} \Box Q_2^{\frac{3}{3}}$. So $\chi(Q_n^{\frac{3}{3}}) \leq \chi_i(2^2 Q_{n-2}) + \chi(Q_{n-2} \Box Q_2^{\frac{3}{3}}) = \chi_i(Q_{n-2}) + \max\{\chi(Q_{n-2}), \chi(Q_2^{\frac{3}{3}})\} = n-1+4 = n+3$.
(iv) $l = 2$. By Theorem 2, $\chi(Q_n^{\frac{3}{3}}) \geq n+3$. It is enough to present a proper $(n+3)$-coloring c of $Q_n^{\frac{3}{3}}$ using Cartesian product of graphs.
Consider Q_n as the Cartesian product of two graphs Q_{n-2} and $Q_2 = C_4$. So we have 4 copies $G_1, \ldots, G_4$ of Q_{n-2}, in which any t-vertex of G_i is adjacent to its correspond t-vertex in G_{i+1}, in which the indices are in module of 4. According to the proof of Part (ii), for $i \in [4]$, the bipartitie graph G_i with bipatition $((A_{i,1} \cup A_{i,2}), (B_{i,1} \cup B_{i,2}))$ is the union of two copies of Q_{n-3}, named $G_{i,1}$ and $G_{i,2}$ and a perfect matching between them. Color each copy G_i like the given coloring in Part (ii) of the proof with colors in $[n+1]$. The $(n+1)$-coloring of $G_1^{\frac{3}{3}}$ is shown in Figure 1. Now, we are going to recolor some t-vertices and i-vertices of copies G_2, G_3 and G_4 and then color the remain uncolored vertices.

- Color $G_{2,1}^{\frac{3}{3}}$ and $G_{2,2}^{\frac{3}{3}}$ as same as coloring of $G_{1,2}^{\frac{3}{3}}$ and $G_{1,1}^{\frac{3}{3}}$, repectively. Due to this coloring, for $i > 1$ and each edge $e = \{v, v'\} \in F_i \subseteq E(G_2)$ we have

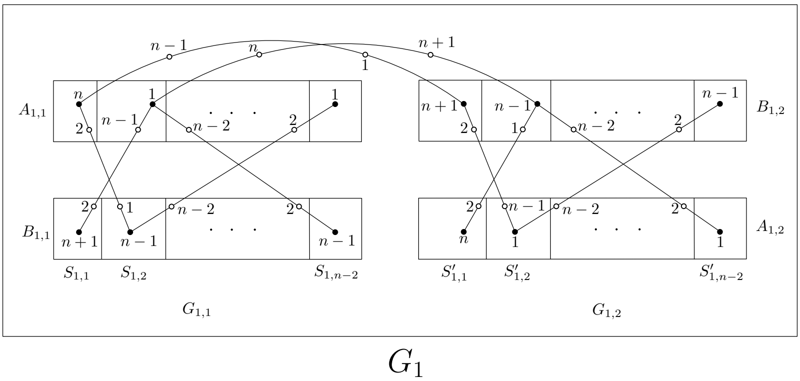


Figure 1. **$(n+1)$-coloring of $G_1^{\frac{3}{3}}$.**

$c((v,v')) = n$ and $c((v',v)) = n+1$, where F_i is the edge set defined in the proof of Part (ii). Also if $v \in (S_{2,1} \cap A_{2,1}) \cup (S'_{2,2} \cap A_{2,2})$, then $c(v) = n$ and if $v \in (S_{2,1} \cap B_{2,1}) \cup (S'_{2,1} \cap B_{2,2})$, then $c(v) = n+1$. Now, for $i > 1$ and an edge $e = \{v, v'\} \in F_i \subseteq E(G_2)$, change the color of two i-vertices (v,v') and (v',v) to $n+2$ and $n+3$. Also recolor t-vertices $v \in (S_{2,1} \cap A_{2,1}) \cup (S'_{2,1} \cap A_{2,2})$ and $v \in (S_{2,1} \cap B_{2,1}) \cup (S'_{2,1} \cap B_{2,2})$ with colors $n+2$ and $n+3$, respectively. Color of some i-vertices and t-vertices of $G_2^{\frac{3}{3}}$ are shown in Figure 2.

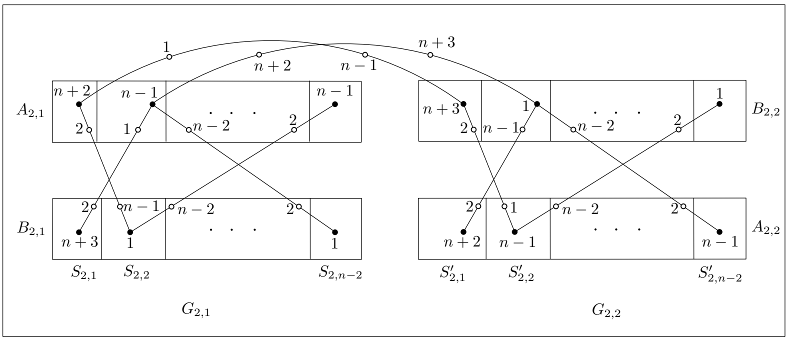
- First, color $G_{3,1}^{\frac{3}{3}}$ and $G_{3,2}^{\frac{3}{3}}$ as same as coloring of $G_{1,2}^{\frac{3}{3}}$ and $G_{1,1}^{\frac{3}{3}}$, repectively. Then, for $e = \{v, v'\} \in F_i$ and $i > 1$, recolor to i-vertices (v,v') and (v',v) with colors $n+1$ and n, respectively. Also, recolor the t-vertices of $(S_{3,1} \cap A_{3,1}) \cup (S'_{3,1} \cap A_{3,2})$ and $(S_{3,1} \cap B_{3,1}) \cup (S'_{3,1} \cap B_{3,2})$ with colors $n+1$ and n, respectively.

- First, color $G_{4,1}^{\frac{3}{3}}$ and $G_{4,2}^{\frac{3}{3}}$ as same as coloring of $G_{1,2}^{\frac{3}{3}}$ and $G_{1,1}^{\frac{3}{3}}$, repectively. Then, for $e = \{v, v'\} \in F_i$ and $i > 1$, recolor to i-vertices (v,v') and (v',v) with colors $n+2$ and $n+3$, respectively. Also recolor the t-vertices of $(S_{4,1} \cap A_{4,1}) \cup (S'_{4,1} \cap A_{4,2})$ and $(S_{4,1} \cap B_{4,1}) \cup (S'_{4,1} \cap B_{4,2})$ with colors $n+3$ and $n+2$, respectively.

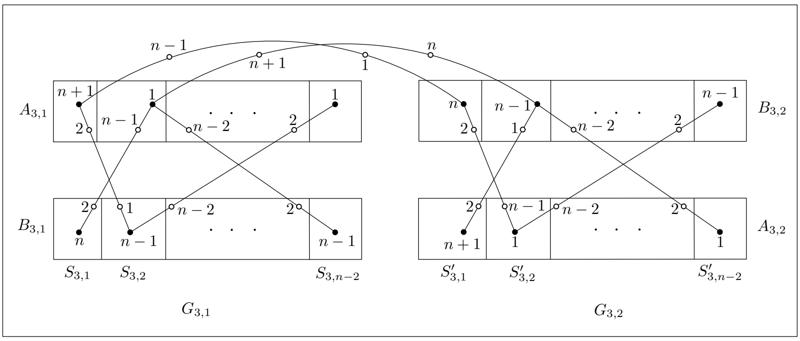
The only uncolored vertices of Q_n are all i-vertices on the edges between 4 copies $G_1, \ldots, G_4$. To color these i-vertices do as follows. Note that, the indices are in module 4.

- For $1 \leq i \leq 4$ if $v \in S_{i,1} \cap A_{i,1}$ and $u \in S_{i+1,1} \cap A_{i+1,1}$, then color two i-vertices (v,u) and (u,v) with colors α and β, where α is the color of all t-vertices in $S_{i+1,1} \cap B_{i+1,1}$ and β is the color of all t-vertices in $S_{i,1} \cap B_{i,1}$.

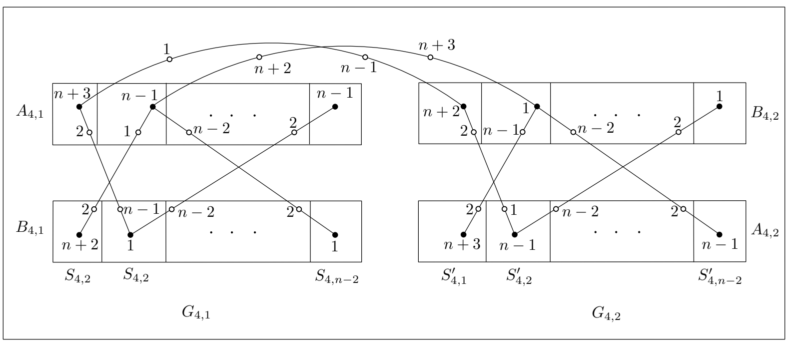
- For $1 \leq i \leq 4$ if $v \in S_{i,1} \cap B_{i,1}$ and $u \in S_{i+1,1} \cap B_{i+1,1}$, then color two i-vertices (v,u) and (u,v) with colors α and β, where α is the color of all t-vertices in $S_{i+1,1} \cap A_{i+1,1}$ and β is the color of all t-vertices in $S_{i,1} \cap A_{i,1}$.



G_2



G_3



G_4

Figure 2. $(n+3)$**-coloring of** $G_2^{\frac{3}{3}}$**,** $G_3^{\frac{3}{3}}$ **and** $G_4^{\frac{3}{3}}$

- For $1 \leq i \leq 4$ if $v \in S'_{i,1} \cap A_{i,2}$ and $u \in S'_{i+1,1} \cap A_{i+1,2}$, then color two i-vertices (v,u) and (u,v) with colors α and β, where α is the color of all t-vertices in $S'_{i+1,1} \cap B_{i+1,2}$ and β is the color of all t-vertices in $S'_{i,1} \cap B_{i,2}$.

- For $1 \leq i \leq 4$ if $v \in S'_{i,1} \cap B_{i,2}$ and $u \in S'_{i+1,1} \cap B_{i+1,2}$, then color two i-vertices

(v,u) and (u,v) with colors α and β, where α is the color of all t-vertices in $S'_{i+1,1} \cap A_{i+1,2}$ and β is the color of all t-vertices in $S'_{i,1} \cap A_{i,2}$.

- For $j > 1$ and $1 \le i \le 4$ if $v \in S_{i,j} \cap A_{i,1}$ and $u \in S_{i+1,j} \cap A_{i+1,1}$, then color two t-vertices (v,u) and (u,v) as same as t-vertices $c((u',u))$ and $c((v',v))$, where $u' \in S'_{i+1,j} \cap B_{i+1,2}$ and $v' \in S'_{i,j} \cap B_{i,2}$.
- For $j > 1$ and $1 \le i \le 4$ if $v \in S_{i,j} \cap B_{i,1}$ and $u \in S_{i+1,j} \cap B_{i+1,1}$, then color two t-vertices (v,u) and (u,v) as same as t-vertices $c((u',u))$ and $c((u,u'))$, where $u' \in S'_{i+1,j} \cap A_{i+1,2}$ and $v' \in S'_{i,j} \cap A_{i,2}$.
- For $j > 1$ and $1 \le i \le 4$ if $v \in S'_{i,j} \cap A_{i,2}$ and $u \in S'_{i+1,j} \cap A_{i+1,2}$, then color two t-vertices (v,u) and (u,v) as same as t-vertices $c((u',u))$ and $c((v',v))$, where $u' \in S_{i+1,j} \cap B_{i+1,1}$ and $v' \in S_{i,j} \cap B_{i,1}$.
- For $j > 1$ and $1 \le i \le 4$ if $v \in S'_{i,j} \cap B_{i,2}$ and $u \in S'_{i+1,j} \cap B_{i+1,2}$, then color two t-vertices (v,u) and (u,v) as same as t-vertices $c((u',u))$ and $c((u,u'))$, where $u' \in S_{i+1,j} \cap A_{i+1,1}$ and $v' \in S_{i,j} \cap A_{i,1}$.

We can check that each color class is an independent set and the given coloring is a proper $(n+3)$-coloring of $Q_n^{\frac{3}{3}}$. ■

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Conflict of interest. The authors declare that they have no conflict of interest.

Data Availability. There is no data associated with this article.

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