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Short Note

Characterization of product cordial dragon graphs

Mukti Acharya† **and Joseph Varghese Kureethara***

Department of Mathematics, Christ University, Bangalore, India
†mukti1948@gmail.com
*frjoseph@christuniversity.in

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Abstract: The vertices of a graph are to be labelled with 0 or 1 such that each edge gets the label as the product of its end vertices. If the number of vertices labelled with 0's and 1's differ by at most one and if the number of edges labelled with 0's and 1's differ by at most by one, then the labelling is called product cordial labelling. Complete characterizations of product cordial dragon graphs is given. We also characterize dragon graphs whose line graphs are product cordial.

Keywords: Cordial graph, Binary labeling, Line graph, Dragon graph

AMS Subject classification: 05C78

1. Introduction

Our world has a vast amount of binaries. Prey-Predator, Male-Female, Rich-Poor, Day-Night, On-Off, Plus-Minus, Like-Dislike, Hire-Fire, Life-Death etc., are some of the common binaries we encounter daily. Graphs represent objects and their relations. Graphs are very good tools for representing binaries. For example, a group of people can be represented by the vertices. Their COVID-19 statuses, such as COVID Negative or COVID Positive, could be represented by one and zero, respectively. The interaction between two COVID Negative people will be safe. However, in all other cases, the interaction is unsafe. Hence, this could be modelled using a graph with edges labelled as the product of the labels of its vertices. This leads to a binary labelling of the edges with zero and one.
A natural question can be about the number of vertices labelled with zeros and ones. If the zeros and ones are almost equally distributed among the vertices, the resultant labelling is known as cordial labelling. In this paper, our search is to find cordial labelling of the vertices and edges of the dragon graphs.

* *Corresponding Author*

2. Cordial Graphs

Cahit introduced cordial labelling of graphs in 1987 [1]. Let each vertex be labelled with a zero or one. The label of each edge is the absolute difference between the labels of its end vertices. Now, if the absolute difference between the number of vertices labelled with one and the number of vertices labelled with zero differs by at most one and if the absolute difference between the number of edges labelled with one and the number of edges labelled with zero also differs by at most one, then the labelling is said to be a cordial labelling. A graph for which we can find cordial labelling is known as a cordial graph. Cahit proved that trees, complete bipartite graphs, pinwheels, ladders etc., are all cordial, as reported in [1].
Cordiality in various types of graphs has been extensively studied since 1987. For the signed graphs, the plus and minus signs are counted for cordiality. Signed graphs are graphs with edges labelled '+' or '-'. In the place of absolute difference, several authors tried other operations. In [2], Sundaram et al. proposed the product cordial labelling of graphs as stated below.
Let $G = (V, E)$ be a graph. Let $f : V(G) \to \{0, 1\}$ and $g : E(G) \to \{0, 1\}$ be such that for every $e = uv \in E, g(uv) = f(u)f(v)$. Let $v_f(x)$ and $e_g(x)$ be the number of vertices and edges of G labelled with x, where x is either 0 or 1, respectively. The graph G is known as product cordial if and only if $|v_f(1) - v_f(0)| \leq 1$ and $|e_g(1) - e_g(0)| \leq 1$.
Sundaram et al. [2] had identified several graphs that are product cordial, including trees and unicyclic graphs of odd order.

3. Cordiality of Dragon Graphs

Dragon graphs, $D(n, k)$, $n \geq 3, k \geq 1$, are unicyclic graphs formed by joining a cycle C_n to a path, P_k by a bridge. The dragon graph, $D(7, 6)$, is shown in Figure 1. It

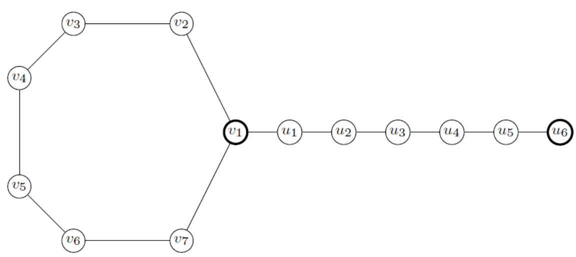

Figure 1. Dragon graph, $D(7, 6)$

has been proved by Sundaram et al.[2] that dragon graphs are product cordial. This result is incorrect as $D(5, 3)$ shown in Figure 2 is not product cordial. Hence, we characterize dragon graphs that are product cordial.

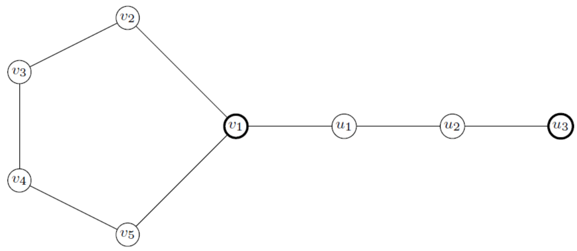

Figure 2. **Dragon graph,** $D(5, 3)$

4. Characterization of Dragon Graph

Theorem 1. *The dragon graph* $D := D(n, k)$ *of order* $n+k$ *where* $n \geq 3, k \geq 1$ *is product cordial if and only if* $n + k$ *is odd or* $k > n - 2$.

Proof. Given a dragon graph $D := D(n, k)$, let the vertices of the cycle be $v_1, v_2, \ldots, v_n$ and the vertices of the path be $u_1, u_2, \ldots, u_k$ where dragon graph is formed by joining the vertices v_1 and u_1 by an edge. All vertices of the dragon graph are of degree 2 except the vertices v_1 and u_k with degrees 3 and 1, respectively.
We now proceed with the sufficiency part of the proof by analysing various cases.

Case 1. $k \geq n - 1$.
Let $k = n - 1$ or $k = n$ or $n + 1$. Obviously, $|n - k| \leq 1$. Assign the label 1 to all the n vertices of the cycle and 0 to all the path vertices. i.e., if $f : \{0, 1\} \to V(G)$, and $g : \{0, 1\} \to E(G)$, we have, $f(v) = 1$ for $v \in V(C_n)$ and $f(v) = 0$ for $v \in V(P_k)$. Thus $v_f(1) = n$, $v_f(0) = k$, $e_g(1) = n$ and $e_g(0) = k$. We get a product cordial graph since $|n - k| \leq 1$. Let $k > n + 1$ and let $k - n = m$. Assign the label 1 to all the n vertices of the cycle and $\lfloor \frac{m}{2} \rfloor$ vertices of the path, viz., $u_1, u_2, \ldots, u_{\lfloor \frac{m}{2} \rfloor}$ and label the remaining vertices with 0. Clearly, $v_f(1) = n+\lfloor \frac{m}{2} \rfloor$, $v_f(0) = k-\lfloor \frac{m}{2} \rfloor$, $e_g(1) = n+\lfloor \frac{m}{2} \rfloor$ and $e_g(0) = k - \lfloor \frac{m}{2} \rfloor$. It follows that the given labelling is a product cordial labelling. Hence, the graph $D := D(n, k)$ is product cordial if $k \geq n - 1$.

Case 2. $k < n - 1$ and $n + k$ is odd.
Let $n - k = m$. Assign the label 0 to all the k vertices of the path and the $\lfloor \frac{m}{2} \rfloor$ vertices of the cycle starting from v_1. That is the vertices of the cycle that are labelled 0 are $v_1, v_2, \ldots v_{\lfloor \frac{m}{2} \rfloor}$. Label all the other vertices of the cycle with 1. Clearly, $v_f(1) = n - \lfloor \frac{m}{2} \rfloor$, $v_f(0) = k + \lfloor \frac{m}{2} \rfloor$, $e_g(1) = n - (\lfloor \frac{m}{2} \rfloor + 1)$ and $e_g(0) = k + \lfloor \frac{m}{2} \rfloor + 1$. It follows that the given labelling is a product cordial labelling. Hence sufficiency follows.
For the necessary part, let us assume the contrapositive. i. e., let $n + k$ is even and $k < n - 1$. Suppose the dragon graph $G = D(n, k)$ is product cordial with vertex labelling f and edge labelling g. Then $v_f(1) = v_f(0) = e_g(1) = e_g(0) = \frac{k+n}{2}$.
Let $S = \{v \in V : f(v) = 1\}$. Since $k < n - 1$, $|S| = \frac{k+n}{2} < n$, S does not contain all the vertices of the cycle C_n. Hence, the induced subgraph $G[S]$ is acyclic and

$e_g(1) = |E(G[S])| < |S|$, which is a contradiction. Let $n - k = m$. Here, m is an even number. Therefore, at least $\frac{m}{2}$ vertices of the cycle should be labelled 0. To minimize the edges with the label 0 in the cycle, we exclude the vertex v_1 and label with 0 consecutive vertices of the cycle starting from v_2. If all the $\frac{n+k}{2}$ consecutive vertices are labelled 0, then there are $\frac{n+k}{2} + 1$ edges that get the label 0. If we go for fewer vertices of the cycle with label 0, in optimal labelling, the number of cycle edges labelled 0 will be one more than that of the vertices. This extra one-edge labelling can be compensated only by labelling the path's edges with 0 one less than that of the number of vertices labelled with 0. This is impossible as any minimal labelling of the path's vertices with 0 gives the same number of edges with label 0. Hence, no product cordial labelling of the dragon graph exists when $n+k$ is even and $k < n-1$. This completes the necessary part. Hence the theorem is proved. □

The dragon graph given in Figure 2 is not product cordial.

5. Product Cordial Line Graph of a Dragon Graph

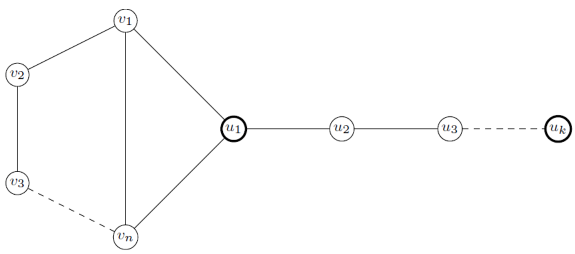

Figure 3. **Line graph of a Dragon graph,** $D(n, k)$

The line graph of a dragon graph has three parts. They are, a C_3, a C_n and a path P_k. The C_3 is formed from the three edges incident to the only vertex of three degree in the dragon graph. We call this C_3 as the permanent C_3. The C_n shares an edge with the C_3 and can follow the same labeling as that of C_n in the dragon graph. P_k is formed from the path of length k in the dragon graph joined by an edge to the cycle C_n in the dragon graph. Hence, the part P_k follows the same labeling as that of the dragon graph.

Hence, let the labels of the vertices of permanent C_3 be v_1, v_n and u_1. Let the labels of the vertices of the C_n be v_1, v_2 $\ldots$, v_n. Let the labels of the vertices of the path be u_1, u_2, $\ldots$, u_k.

The line graph $L(D(5, 3))$ of $D(5, 3)$ is given in Figure 4.

Now, we characterize $D(n, k)$ such that $L(D(n, k))$ is product cordial.

Theorem 2. *The line graph of the dragon graph $D := D(n, k)$ of order $n + k$ where $n \geq 3$, $k \geq 1$ is product cordial if and only if $n > 3$ or $k > 1$.*

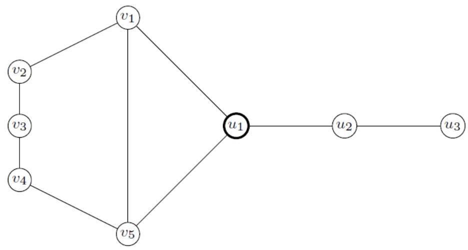

Figure 4. $L(D(5,3))$

Proof. Let the line graph of the dragon graph $D := D(n,k)$ of order $n+k$ where $n \geq 3, k \geq 1$ is product cordial. Then we need to show that $n > 3$ or $k > 1$. This is same as showing that if $n = 3$ and $k = 1$, the dragon graph is not product cordial. When $n = 3$ and $k = 1$, the dragon graph is 3-pan. Its line graph is $K_4 - e$. The graph $K_4 - e$ has four vertices and five edges. Hence, at least two vertices are to be labelled with 0. This forces at least four edges to have label 0. Hence, the line graph of the dragon graph $D(3,1)$ is not product cordial.

Conversely, assume that $n > 3$ or $k > 1$. Now, we have the following three cases.

Case 1. $k > 1$ and $n = 3$.
When $k = 2$ or $k = 3$, label the vertices of the permanent C_3 with 1 and label the other vertices with 0. When $k > 3$, label the $\lceil \frac{(n+k)}{2} \rceil$ vertices of $L(D(3,k))$ with 1 such that the induced subgraph of those vertices form a dragon graph with the permanent C_3 as the cycle. Label the remaining vertices with 0. Then, we can easily verify that $L(D(3,k))$ is product cordial.

Case 2. $k = 1$ and $n > 3$.
When $n = 4$ or $n = 5$, label the vertices of the permanent C_3 with 1 and label the other vertices with 0. When $n > 5$, label the $\lceil \frac{(n+k)}{2} \rceil$ vertices of $L(D(n,1))$ with 1 such that the induced subgraph of those vertices forms a dragon graph with the permanent C_3 as the cycle. Label the remaining vertices with 0. This gives a product cordial labelling for $L(D(n,1))$.

Case 3. $k > 1$ and $n > 3$. We distinguish two situations.
Subcase 3.1. $\lfloor \frac{(n+k)}{2} \rfloor \leq n - 2$.
Label the first $\lfloor \frac{(n+k)}{2} \rfloor$ vertices of $\{v_2, v_3, \ldots v_{n-1}\}$ of $L(D(n,k))$ with 0. Label all the remaining vertices with 1. i.e., $f(v) = 0$ for $v \in \{v_i \mid 2 \leq i \leq \lfloor \frac{(n+k)}{2} \rfloor\}$ and $f(v) = 1$ for the remaining vertices. We have, $v_f(1) = \lfloor \frac{(n+k)}{2} \rfloor - 1$, $v_f(0) = n+k+1-\lfloor \frac{(n+k)}{2} \rfloor$, $e_g(1) = \lfloor \frac{(n+k)}{2} \rfloor - 1$ and $e_g(0) = n+k-\lfloor \frac{(n+k)}{2} \rfloor$. This labeling gives a product cordial labeling.
Subcase 3.2. $\lfloor \frac{(n+k)}{2} \rfloor > n - 2$.
Label the vertices v_2, v_3 $\ldots v_{n-1}$ with 0. Then, label exactly $\lfloor \frac{(n+k)}{2} \rfloor - (n-2)$ consecutive vertices u_k, u_{k-1} $\ldots$etc. with the label 0. Label all the other vertices with 1. i.e., $f(v) = 0$ for $v \in \{v_i \mid 2 \leq i \leq n-1\} \cup \{u_i \mid \lfloor \frac{(n+k)}{2} \rfloor - (n-1) \leq i \leq k\}$

and $f(v) = 1$ for the remaining vertices. This gives a product cordial labeling of $L(D(n,k))$. Hence, the proof is complete. □

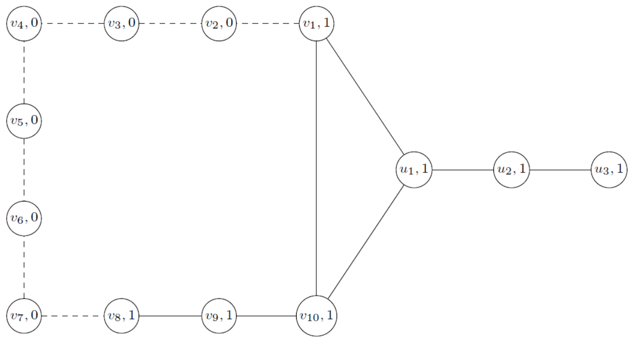

Figure 5. $L(D(10,3))$

An illustration of subcase 3.1 of the above theorem is given in Figure 5. Here thick edges indicate 1-labelling and dashed lines indicate 0-labelling. Subcase 3.2 of the

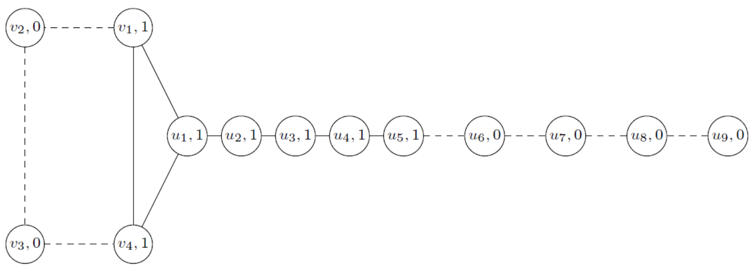

Figure 6. $L(D(4,9))$

above theorem is illustrated in Figure 6.

6. Conclusion

Dragon graph is a unicyclic graph. It is the cycle in the dragon graph that plays a crucial role in it being a product cordial graph or not. Thus it is worth exploring the product cordial nature of graphs by analyzing the cycles in them. Are k-cyclic graphs product cordial? By k-cyclic graph, we mean to identify a graph with k induced cycles.

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Data Availability. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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