Communications in Combinatorics and Optimization
Vol. ??, No. ?? (????), pp. 1-14
DOI:10.22049/cco.2023.28718.1684

CCO
Commun. Comb. Optim.

Research Article

Bounds on Sombor index and inverse sum indeg (ISI) index of graph operations

Fareeha Jamal†**, Muhammed Imran***

Department of Mathematical Sciences, College of Science, United Arab Emirate University
Al Ain 15551, Abu Dhabi, UAE
†202090514@uaeu.ac.ae
*m.imran658@uaeu.ac.ae

Received: 4 June 2023; Accepted: 4 August 2023
Published Online: 8 August 2023

Abstract: Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Denote by $d_G(u)$ the degree of a vertex $u \in V(G)$. The Sombor index of G is defined as $SO(G) = \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2}$, whereas, the inverse sum indeg (ISI) index is defined as $ISI(G) = \sum_{uv \in E(G)} \frac{d_u d_v}{d_u + d_v}$. In this paper, we compute the bounds in terms of maximum degree, minimum degree, order and size of the original graphs G and H for Sombor and ISI indices of several graph operations like corona product, cartesian product, strong product, composition and join of graphs.

Keywords: Sombor index; inverse sum indeg index; graph operations; corona product; cartesian product

AMS Subject classification: 05C90, 05C92

1. Introduction

We consider only simple, finite, undirected and connected graphs. Let G be such a graph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ and edge set $E(G)$. The cardinality $|V(G)| = n_G$ is called order of G and the cardinality $|E(G)| = m_G$ is the size of G. The degree $d_G(u)$ of a vertex u in G is the number of edges incident with u. The minimum degree (resp. maximum degree) of vertex in G is denoted by δ_G (resp. Δ_G). If the vertex u_i is adjacent to vertex u_j, then the edge connecting them is denoted by $u_i u_j$. A regular graph is a graph in which degree of each vertex is same. A complete graph of order n is denoted by K_n.

* *Corresponding Author*

© 2023 Azarbaijan Shahid Madani University

In the chemical and mathematical literature, a topological index is a numerical quantity that is derived from a (chemical) graph such that it remains the same under graph isomorphism. They have several applications in chemistry, pharmacology, materials science, among other, [7, 21]. In 2010, Vukičević and Gašperov [22] introduced the inverse sum indeg index (shortly ISI index) as

$$ISI(G) = \sum_{uv \in E(G)} \frac{d_u d_v}{d_u + d_v}. \tag{1}$$

The ISI index is a well-studied topological index and has several applications in quantitative structure-activity or structure-property relationships (QSAR/QSPR) [10, 11, 15]. Zangi et. al. [24] found basic properties of the ISI matrix. They also gave some bounds for the ISI energy of graphs. The ISI index and ISI energy of the molecular graphs of hyaluronic acid-paclitaxel conjugates was obtained by Havare [10]. For more results on ISI index and ISI energy, see [3, 12].
In [5] Gutman defined new vertex-degree-based topological index called Sombor index, denoted by $SO(G)$ and defined as:

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2}. \tag{2}$$

The study of the Sombor index of graphs has attracted a significant amount of attention within a very short span of time. Redžepović [18] found that the Sombor index has good predictive potential for statistical modeling of enthalpy of vaporization and entropy for alkanes. Cruz et al. [4] investigated the Sombor index of chemical graphs and characterized the extremal graphs with respect to the Sombor index over the sets of (connected) chemical graphs, chemical trees, and hexagonal systems. For more related results, one may refer to [6, 8, 16, 19, 23] and the references therein.
In chemical graph theory, some chemical graphs obtained by the use of different graph operations (graph products) are very interesting to investigate. In 2015, Shetty et al. [20] gave formulae for harmonic index of some graph operations. The exact expressions for first and second Zagreb indices and hyper Wiener index was found by Khalifeh et al. [13, 14]. Akhter et al. [1, 2] computed the exact formulae and the bounds for general sum-connectivity index of some graph operations.
In this paper, we compute some bounds for the Sombor and inverse sum indeg (ISI) index of several graph operations. These graph operations include corona product, cartesian product, strong product, join and composition of graphs.

2. Bounds for the Sombor and inverse sum ideg (ISI) indices

In this section, we derive some bounds for Sombor and inverse sum indeg (ISI) indices of several graph operations. Let G and H be two simple connected graphs whose vertex sets are disjoint. For each $u \in V(G)$ and $v \in V(H)$, we have

$$\Delta_G \geq d_G(u), \quad \delta_G \leq d_G(u), \tag{3}$$
$$\Delta_H \geq d_H(v), \quad \delta_H \leq d_H(v). \tag{4}$$

The equality holds if and only if G and H are regular graphs.

2.1. The corona product

The corona product of G and H, denoted by $G \odot H$, is a graph obtained by taking one copy of G and n_G copies of H and joining the vertex u that is on i-th position in G to every vertex in i-th copy of H. The order and size of $G \odot H$ are $n_G(1+n_H)$ and $m_G + n_G m_H + n_G n_H$, respectively. The degree of a vertex $u \in V(G \odot H)$ is given by

$$d_{G\odot H}(u) = \begin{cases} d_G(u) + n_H & \text{if } u \in V(G), \\ d_H(u) + 1 & \text{if } u \in V(H). \end{cases} \tag{5}$$

In the following theorem, the bounds on the Sombor index of corona product of two graphs are computed.

Theorem 1. *Let G and H be graphs. Then $\alpha_1 \leq SO(G \odot H) \leq \alpha_2$, where*

$$\begin{aligned} \alpha_1 &= \sqrt{2}m_G(\delta_G + n_H) + \sqrt{2}n_G m_H(\delta_H + 1) + n_G n_H\sqrt{(\delta_G + n_H)^2 + (\delta_H + 1)^2}, \\ \alpha_2 &= \sqrt{2}m_G(\Delta_G + n_H) + \sqrt{2}n_G m_H(\Delta_H + 1) + n_G n_H\sqrt{(\Delta_G + n_H)^2 + (\Delta_H + 1)^2}. \end{aligned}$$

The equality holds if and only if G and H are regular graphs.

Proof. Using (3), (4) and (5) in equation (2), we obtain

$$\begin{aligned} SO(G \odot H) = &\sum_{uv\in E(G)} \sqrt{(d_G(u) + n_H)^2 + (d_G(v) + n_H)^2} \\ &+ n_G \sum_{uv\in E(H)} \sqrt{(d_H(u) + 1)^2 + (d_H(v) + 1)^2} \\ &+ \sum_{u\in V(G)} \sum_{v\in V(H)} \sqrt{(d_G(u) + n_H)^2 + (d_H(v) + 1)^2} \\ &\leq \sum_{uv\in E(G)} \sqrt{2(\Delta_G + n_H)^2} + n_G \sum_{uv\in E(H)} \sqrt{2(\Delta_H + 1)^2} \\ &+ \sum_{u\in V(G)} \sum_{v\in V(H)} \sqrt{(\Delta_G + n_H)^2 + (\Delta_H + 1)^2} \\ &= \sqrt{2}m_G(\Delta_G + n_H) + \sqrt{2}n_G m_H(\Delta_H + 1) \\ &+ n_G n_H\sqrt{(\Delta_G + n_H)^2 + (\Delta_H + 1)^2}. \end{aligned} \tag{6}$$

Similarly, we can compute

$$SO(G \odot H) \geq \sqrt{2}m_G(\delta_G + n_H) + \sqrt{2}n_G m_H(\delta_H + 1) + n_G n_H\sqrt{(\delta_G + n_H)^2 + (\delta_H + 1)^2}. \tag{7}$$

The equality in (6) and (7) holds if and only if G and H are regular graphs. □

Let $t \geq 1$ and $\overline{K_t}$ be the complement of K_t. Then t-thorny graph of G is the corona product of G and $\overline{K_t}$. The following corollary is an easy consequence of Theorem 1.

Corollary 1. *For a graph G, the following holds:*

$$\begin{aligned}\sqrt{2}m_G(\delta_G+t)+n_Gt\sqrt{(\delta_G+t)^2+1} &\leq SO(G\odot\overline{K_t})\\ &\leq \sqrt{2}m_G(\Delta_G+t)+n_Gt\sqrt{(\Delta_G+t)^2+1}.\end{aligned}$$

The next theorem gives the bounds on the ISI index of corona product of two graphs.

Theorem 2. *Let G and H be graphs. Then $\alpha_1 \leq ISI(G\odot H) \leq \alpha_2$, where*

$$\begin{aligned}\alpha_1 &= \frac{m_G(\delta_G+n_H)^2}{2(\Delta_G+n_H)}+\frac{n_Gm_H(\delta_H+1)^2}{2(\Delta_H+1)}+\frac{n_Gn_H(\delta_G+n_H)(\delta_H+1)}{\Delta_G+\Delta_H+n_H+1},\\ \alpha_2 &= \frac{m_G(\Delta_G+n_H)^2}{2(\delta_G+n_H)}+\frac{n_Gm_H(\Delta_H+1)^2}{2(\delta_H+1)}+\frac{n_Gn_H(\Delta_G+n_H)(\Delta_H+1)}{\delta_G+\delta_H+n_H+1}.\end{aligned}$$

The equality holds if and only if G and H are regular graphs.

Proof. Using (3), (4) and (5) in equation (1), we obtain

$$\begin{aligned}ISI(G\odot H) &= \sum_{uv\in E(G)}\frac{(d_G(u)+n_H)(d_G(v)+n_H)}{d_G(u)+d_G(v)+2n_H}\\ &+n_G\sum_{uv\in E(H)}\frac{(d_H(u)+1)(d_H(v)+1)}{d_H(u)+d_H(v)+2}\\ &+\sum_{u\in V(G)}\sum_{v\in V(H)}\frac{(d_G(u)+n_H)(d_H(v)+1)}{d_G(u)+d_H(v)+n_H+1}\\ &\leq \sum_{uv\in E(G)}\frac{(\Delta_G+n_H)^2}{2(\delta_G+n_H)}+n_G\sum_{uv\in E(H)}\frac{(\Delta_H+1)^2}{2(\delta_H+1)}\\ &+\sum_{u\in V(G)}\sum_{v\in V(H)}\frac{(\Delta_G+n_H)(\Delta_H+1)}{\delta_G+\delta_H+n_H+1}\\ &= \frac{m_G(\Delta_G+n_H)^2}{2(\delta_G+n_H)}+\frac{n_Gm_H(\Delta_H+1)^2}{2(\delta_H+1)}+\frac{n_Gn_H(\Delta_G+n_H)(\Delta_H+1)}{\delta_G+\delta_H+n_H+1}.\end{aligned} \tag{8}$$

Similarly, we can compute

$$ISI(G\odot H) \geq \frac{m_G(\delta_G+n_H)^2}{2(\Delta_G+n_H)}+\frac{n_Gm_H(\delta_H+1)^2}{2(\Delta_H+1)}+\frac{n_Gn_H(\delta_G+n_H)(\delta_H+1)}{\Delta_G+\Delta_H+n_H+1}. \tag{9}$$

The equality in (8) and (9) holds if and only if G and H are regular graphs. □

The following corollary is an easy consequence of Theorem 2.

Corollary 2. *For a graph G, the following holds:*

$$\frac{m_G(\delta_G+t)^2}{2(\Delta_G+t)}+\frac{n_G t(\delta_G+t)(\delta_H+1)}{\Delta_G+t+1}\leq ISI(G\odot\overline{K_t})$$
$$\leq\frac{m_G(\Delta_G+t)^2}{2(\delta_G+t)}+\frac{n_G t(\Delta_G+t)}{\delta_G+t+1}.$$

2.2. The cartesian product

The cartesian product of G and H, denoted by $G\times H$, is a graph whose vertex set is $V(G\times H)=V(G)\times V(H)$ and two vertices (u_1,v_1) and (u_2,v_2) are adjacent in $G\times H$ whenever [v_1 and v_2 are adjacent in H and $u_1=u_2$] or [u_1 and u_2 are adjacent in G and $v_1=v_2$]. The order of the cartesian product of two graphs is the product of number of vertices of G and H, and the size is $m_Gn_H+m_Hn_G$. If G and H are regular graphs then $G\times H$ is also regular graph. The degree of a vertex $(u,v)\in V(G\times H)$ is

$$d_{G\times H}((u,v))=d_G(u)+d_H(v). \tag{10}$$

In the following theorem, we compute bounds on the Sombor index of $G\times H$.

Theorem 3. *Let G and H be graphs. Then*

$$\sqrt{2}m_{G\times H}(\delta_G+\delta_H)\leq SO(G\times H)\leq\sqrt{2}m_{G\times H}(\Delta_G+\Delta_H).$$

The equality holds if and only if G and H are regular graphs.

Proof. Using (3), (4) and (10) in equation (2), we obtain

$$\begin{aligned}SO(G\times H)&=\sum_{u_1\in V(G)}\sum_{v_1v_2\in E(H)}\sqrt{(d_G(u_1)+d_H(v_1))^2+(d_G(u_1)+d_H(v_2))^2}\\&+\sum_{v_1\in V(H)}\sum_{u_1u_2\in E(G)}\sqrt{(d_G(u_1)+d_H(v_1))^2+(d_G(u_2)+d_H(v_1))^2}\\&\leq\sum_{u_1\in V(G)}\sum_{v_1v_2\in E(H)}\sqrt{2(\Delta_G+\Delta_H)^2}\\&+\sum_{v_1\in V(H)}\sum_{u_1u_2\in E(G)}\sqrt{2(\Delta_G+\Delta_H)^2}\\&=\sqrt{2}(n_Gm_H+n_Hm_G)(\Delta_G+\Delta_H)\\&=\sqrt{2}m_{G\times H}(\Delta_G+\Delta_H).\end{aligned}\tag{11}$$

One can analogously compute the following:

$$SO(G\times H)\geq\sqrt{2}m_{G\times H}(\delta_G+\delta_H). \tag{12}$$

The equality in (11) and (12) obviously holds if and only if G and H are regular graphs. □

Now, we compute the bounds on the ISI index of $G \times H$.

Theorem 4. *Let G and H be graphs. Then*

$$\frac{1}{2}m_{G\times H}\frac{(\delta_G+\delta_H)^2}{\Delta_G+\Delta_H} \le ISI(G\times H) \le \frac{1}{2}m_{G\times H}\frac{(\Delta_G+\Delta_H)^2}{\delta_G+\delta_H}.$$

The equality holds if and only if G and H are regular graphs.

Proof. Using (3), (4) and (10) in equation (1), we obtain

$$\begin{aligned} ISI(G\times H) &= \sum_{u_1\in V(G)}\sum_{v_1v_2\in E(H)} \frac{(d_G(u_1)+d_H(v_1))(d_G(u_1)+d_H(v_2))}{2d_G(u_1)+d_H(v_1)+d_H(v_2)} \\ &+ \sum_{v_1\in V(H)}\sum_{u_1u_2\in E(G)} \frac{(d_G(u_1)+d_H(v_1))(d_G(u_2)+d_H(v_1))}{d_G(u_1)+d_G(u_2)+2d_H(v_1)} \\ &\le \sum_{u_1\in V(G)}\sum_{v_1v_2\in E(H)} \frac{(\Delta_G+\Delta_H)^2}{2(\delta_G+\delta_H)} + \sum_{v_1\in V(H)}\sum_{u_1u_2\in E(G)} \frac{(\Delta_G+\Delta_H)^2}{2(\delta_G+\delta_H)} \\ &= \frac{1}{2}(n_Gm_H+n_Hm_G)\frac{(\Delta_G+\Delta_H)^2}{\delta_G+\delta_H} \\ &= \frac{1}{2}m_{G\times H}\frac{(\Delta_G+\Delta_H)^2}{\delta_G+\delta_H}. \end{aligned} \tag{13}$$

One can analogously compute the following:

$$ISI(G\times H) \ge \frac{1}{2}m_{G\times H}\frac{(\delta_G+\delta_H)^2}{\Delta_G+\Delta_H}. \tag{14}$$

The equality in (13) and (14) obviously holds if and only if G and H are regular graphs. □

2.3. The strong product

The strong product of G and H, denoted by $G \boxtimes H$, is a graph whose vertex set is $V(G\boxtimes H) = V(G)\times V(H)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G\boxtimes H$ whenever [v_1 and v_2 are adjacent in H and $u_1 = u_2$] or [u_1 and u_2 are adjacent in G and $v_1 = v_2$] or [$u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$]. The order of $G \boxtimes H$ is the product of number of vertices of G and H, and the size is $n_Gm_H + n_Hm_G + 2m_Gm_H$. The degree of a vertex $(u, v) \in V(G \boxtimes H)$ is

$$d_{G\boxtimes H}((u,v)) = d_G(u) + d_H(v) + d_G(u)d_H(v). \tag{15}$$

We compute bounds on the Sombor index of $G \boxtimes H$ in the following theorem.

Theorem 5. *Let G and H be graphs. Then*

$$\sqrt{2}m_{G\boxtimes H}(\delta_G+\delta_H+\delta_G\delta_H)\leq SO(G\boxtimes H)\leq\sqrt{2}m_{G\boxtimes H}(\Delta_G+\Delta_H+\Delta_G\Delta_H).$$

The equality holds if and only if G and H are regular graphs.

Proof. Using (3), (4) and (15) in equation (2), we obtain

$$\begin{aligned}
&SO(G\boxtimes H)\\
&=\sum_{u_1\in V(G)}\sum_{v_1v_2\in E(H)}\sqrt{(d_G(u_1)+d_H(v_1)+d_G(u_1)d_H(v_1))^2+(d_G(u_1)+d_H(v_2)+d_G(u_1)d_H(v_2))^2}\\
&+\sum_{v_1\in V(H)}\sum_{u_1u_2\in E(G)}\sqrt{(d_G(u_1)+d_H(v_1)+d_G(u_1)d_H(v_1))^2+(d_G(u_2)+d_H(v_1)+d_G(u_2)d_H(v_1))^2}\\
&+2\sum_{u_1u_2\in E(G)}\sum_{v_1v_2\in E(H)}\sqrt{(d_G(u_1)+d_H(v_1)+d_G(u_1)d_H(v_1))^2+(d_G(u_2)+d_H(v_2)+d_G(u_2)d_H(v_2))^2}\\
&\leq\sum_{u_1\in V(G)}\sum_{v_1v_2\in E(H)}\sqrt{2(\Delta_G+\Delta_H+\Delta_G\Delta_H)^2}\\
&+\sum_{v_1\in V(H)}\sum_{u_1u_2\in E(G)}\sqrt{2(\Delta_G+\Delta_H+\Delta_G\Delta_H)^2}\\
&+2\sum_{u_1u_2\in E(G)}\sum_{v_1v_2\in E(H)}\sqrt{2(\Delta_G+\Delta_H+\Delta_G\Delta_H)^2}\\
&=\sqrt{2}(n_Gm_H+n_Hm_G+2m_Gm_H)(\Delta_G+\Delta_H+\Delta_G\Delta_H)\\
&=\sqrt{2}m_{G\boxtimes H}(\Delta_G+\Delta_H+\Delta_G\Delta_H).
\end{aligned}\tag{16}$$

Analogously, one can compute the following:

$$SO(G\boxtimes H)\geq\sqrt{2}m_{G\boxtimes H}(\delta_G+\delta_H+\delta_G\delta_H).\tag{17}$$

If G and H are regular graphs then the equality in (16) and (17) holds. □

Next, we compute bounds on the ISI index of $G\boxtimes H$.

Theorem 6. *Let G and H be graphs. Then*

$$\frac{1}{2}m_{G\boxtimes H}\frac{(\delta_G+\delta_H+\delta_G\delta_H)^2}{\Delta_G+\Delta_H+\Delta_G\Delta_H}\leq ISI(G\boxtimes H)\leq\frac{1}{2}m_{G\boxtimes H}\frac{(\Delta_G+\Delta_H+\Delta_G\Delta_H)^2}{\delta_G+\delta_H+\delta_G\delta_H}.$$

The equality holds if and only if G and H are regular graphs.

Proof. Using (3), (4) and (15) in equation (1), we obtain

$$\begin{aligned}
ISI(G\boxtimes H)=&\sum_{u_1\in V(G)}\sum_{v_1v_2\in E(H)}\frac{(d_G(u_1)+d_H(v_1)+d_G(u_1)d_H(v_1))(d_G(u_1)+d_H(v_2)+d_G(u_1)d_H(v_2))}{2d_G(u_1)+d_H(v_1)+d_H(v_2)+d_G(u_1)(d_H(v_1)+d_H(v_2))}\\
&+\sum_{v_1\in V(H)}\sum_{u_1u_2\in E(G)}\frac{(d_G(u_1)+d_H(v_1)+d_G(u_1)d_H(v_1))(d_G(u_2)+d_H(v_1)+d_G(u_2)d_H(v_1))}{d_G(u_1)+d_G(u_2)+2d_H(v_1)+d_H(v_1)(d_G(u_1)+d_G(u_2))}\\
&+2\sum_{u_1u_2\in E(G)}\sum_{v_1v_2\in E(H)}\frac{(d_G(u_1)+d_H(v_1)+d_G(u_1)d_H(v_1))(d_G(u_2)+d_H(v_2)+d_G(u_2)d_H(v_2))}{d_G(u_1)+d_G(u_2)+d_H(v_1)+d_H(v_2)+d_G(u_1)d_H(v_1)+d_G(u_2)d_H(v_2)}
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{u_1\in V(G)}\sum_{v_1v_2\in E(H)}\frac{(\Delta_G+\Delta_H+\Delta_G\Delta_H)^2}{2(\delta_G+\delta_H+\delta_G\delta_H)}+\sum_{v_1\in V(H)}\sum_{u_1u_2\in E(G)}\frac{(\Delta_G+\Delta_H+\Delta_G\Delta_H)^2}{2(\delta_G+\delta_H+\delta_G\delta_H)}\\
&+2\sum_{u_1u_2\in E(G)}\sum_{v_1v_2\in E(H)}\frac{(\Delta_G+\Delta_H+\Delta_G\Delta_H)^2}{2(\delta_G+\delta_H+\delta_G\delta_H)}\\
&=\frac{1}{2}(n_Gm_H+n_Hm_G+2m_Gm_H)\frac{(\Delta_G+\Delta_H+\Delta_G\Delta_H)^2}{\delta_G+\delta_H+\delta_G\delta_H}\\
&=\frac{1}{2}m_{G\boxtimes H}\frac{(\Delta_G+\Delta_H+\Delta_G\Delta_H)^2}{\delta_G+\delta_H+\delta_G\delta_H}.
\end{aligned} \tag{18}$$

Analogously, one can compute the following:

$$ISI(G\boxtimes H)\geq\frac{1}{2}m_{G\boxtimes H}\frac{(\delta_G+\delta_H+\delta_G\delta_H)^2}{\Delta_G+\Delta_H+\Delta_G\Delta_H}. \tag{19}$$

If G and H are regular graphs then the equality in (18) and (19) holds. □

2.4. The join of graphs

The join of G and H, denoted by $G+H$, is a union of graphs G and H together with all the edges joining the sets of vertices of G and H. The order and size of $G+H$ are n_Gn_H and $m_G+m_H+n_Gn_H$, respectively. The degree of a vertex u in $G+H$ is given by

$$d_{G+H}(u)=\begin{cases}d_G(u)+n_H & \text{if } u\in V(G),\\ d_H(u)+n_G & \text{if } u\in V(H).\end{cases} \tag{20}$$

We compute bounds on the Sombor index for join of two graphs in the following theorem.

Theorem 7. *Let G and H be graphs. Then $\alpha_1\leq SO(G+H)\leq\alpha_2$, where*

$$\begin{aligned}
\alpha_1 &= \sqrt{2}m_G(\delta_G+n_H)+\sqrt{2}m_H(\delta_H+n_G)+n_Gn_H\sqrt{(\delta_G+n_H)^2+(\delta_H+n_G)^2},\\
\alpha_2 &= \sqrt{2}m_G(\Delta_G+n_H)+\sqrt{2}m_H(\Delta_H+n_G)+n_Gn_H\sqrt{(\Delta_G+n_H)^2+(\Delta_H+n_G)^2}.
\end{aligned}$$

The equality holds if and only if G and H are regular graphs.

Proof. Using (3), (4) and (20) in equation (2), we obtain

$$\begin{aligned}
SO(G+H)=&\sum_{uv\in E(G)}\sqrt{(d_G(u)+n_H)^2+(d_G(v)+n_H)^2}+\\
&\sum_{uv\in E(H)}\sqrt{(d_H(u)+n_G)^2+(d_H(v)+n_G)^2}\\
&+\sum_{u\in V(G)}\sum_{v\in V(H)}\sqrt{(d_G(u)+n_H)^2+(d_H(v)+n_G)^2}\\
&\leq\sum_{uv\in E(G)}\sqrt{2(\Delta_G+n_H)^2}+\sum_{uv\in E(H)}\sqrt{2(\Delta_H+n_G)^2}\\
&+\sum_{u\in V(G)}\sum_{v\in V(H)}\sqrt{(\Delta_G+n_H)^2+(\Delta_H+n_G)^2}\\
&=\sqrt{2}m_G(\Delta_G+n_H)+\sqrt{2}m_H(\Delta_H+n_G)\\
&+n_Gn_H\sqrt{(\Delta_G+n_H)^2+(\Delta_H+n_G)^2}.
\end{aligned} \tag{21}$$

Similarly, we can show that

$$SO(G+H) \geq \sqrt{2}m_G(\delta_G+n_H)+\sqrt{2}m_H(\delta_H+n_G)+n_Gn_H\sqrt{(\delta_G+n_H)^2+(\delta_H+n_G)^2}. \quad (22)$$

If G and H are regular graphs then we obtain the equality in (21) and (22). □

Now, the bounds on the ISI index for join of two graphs is given in the following theorem.

Theorem 8. *Let G and H be graphs. Then $\alpha_1 \leq ISI(G+H) \leq \alpha_2$, where*

$$\begin{aligned}
\alpha_1 &= \frac{1}{2}m_G\frac{(\delta_G+n_H)^2}{\Delta_G+n_H}+\frac{1}{2}m_H\frac{(\delta_H+n_G)^2}{\Delta_H+n_G}+n_Gn_H\frac{(\delta_G+n_H)(\delta_H+n_G)}{\Delta_G+\Delta_H+n_H+n_G},\\
\alpha_2 &= \frac{1}{2}m_G\frac{(\Delta_G+n_H)^2}{\delta_G+n_H}+\frac{1}{2}m_H\frac{(\Delta_H+n_G)^2}{\delta_H+n_G}+n_Gn_H\frac{(\Delta_G+n_H)(\Delta_H+n_G)}{\delta_G+\delta_H+n_H+n_G}.
\end{aligned}$$

The equality holds if and only if G and H are regular graphs.

Proof. Using (3), (4) and (20) in equation (1), we obtain

$$\begin{aligned}
ISI(G+H) &= \sum_{uv\in E(G)}\frac{(d_G(u)+n_H)(d_G(v)+n_H)}{d_G(u)+d_G(v)+2n_H}\\
&+\sum_{uv\in E(H)}\frac{(d_H(u)+n_G)(d_H(v)+n_G)}{d_H(u)+d_H(v)+2n_G}\\
&+\sum_{u\in V(G)}\sum_{v\in V(H)}\frac{(d_G(u)+n_H)(d_H(v)+n_G)}{d_G(u)+d_H(v)+n_H+n_G}\\
&\leq \sum_{uv\in E(G)}\frac{(\Delta_G+n_H)^2}{2(\delta_G+n_H)}+\sum_{uv\in E(H)}\frac{(\Delta_H+n_G)^2}{2(\delta_H+n_G)}\\
&+\sum_{u\in V(G)}\sum_{v\in V(H)}\frac{(\Delta_G+n_H)(\Delta_H+n_G)}{\delta_G+\delta_H+n_H+n_G}\\
&= \frac{1}{2}m_G\frac{(\Delta_G+n_H)^2}{\delta_G+n_H}+\frac{1}{2}m_H\frac{(\Delta_H+n_G)^2}{\delta_H+n_G}+n_Gn_H\frac{(\Delta_G+n_H)(\Delta_H+n_G)}{\delta_G+\delta_H+n_H+n_G}.
\end{aligned} \quad (23)$$

Similarly, we can show that

$$ISI(G+H) \geq \frac{1}{2}m_G\frac{(\delta_G+n_H)^2}{\Delta_G+n_H}+\frac{1}{2}m_H\frac{(\delta_H+n_G)^2}{\Delta_H+n_G}+n_Gn_H\frac{(\delta_G+n_H)(\delta_H+n_G)}{\Delta_G+\Delta_H+n_H+n_G}. \quad (24)$$

If G and H are regular graphs then we obtain the equality in (23) and (24). □

2.5. The composition

The composition or lexicographic product of G and H, denoted by $G[H]$, is the graph whose vertex set is $V(G[H]) = V(G) \times V(H)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G[H]$ whenever $[u_1u_2 \in E(G)]$ or $[v_1$ and v_2 are adjacent in H and $u_1 = u_2]$. The order of $G[H]$ is the product of number of vertices of G and H, and size is $m_G n_H^2 + n_G m_H$. The degree of a vertex $(u, v) \in V(G[H])$ is

$$d_{G[H]}((u, v)) = n_H d_G(u) + d_H(v). \tag{25}$$

In the following theorem, we calculate bounds on the Sombor index for composition of two graphs.

Theorem 9. *Let G and H be graphs. Then*

$$\sqrt{2} m_{G[H]}(n_H \delta_G + \delta_H) \leq SO(G[H]) \leq \sqrt{2} m_{G[H]}(n_H \Delta_G + \Delta_H).$$

The equality holds if and only if G and H are regular graphs.

Proof. Using (3), (4) and (25) in equation (2), we obtain

$$\begin{aligned} SO(G[H]) &= \sum_{u_1 \in V(G)} \sum_{v_1v_2 \in E(H)} \sqrt{(n_H d_G(u_1) + d_H(v_1))^2 + (n_H d_G(u_1) + d_H(v_2))^2} \\ &+ \sum_{v_1 \in V(H)} \sum_{v_2 \in V(H)} \sum_{u_1u_2 \in E(G)} \sqrt{(n_H d_G(u_1) + d_H(v_1))^2 + (n_H d_G(u_2) + d_H(v_2))^2} \\ &\leq \sum_{u_1 \in V(G)} \sum_{v_1v_2 \in E(H)} \sqrt{2(n_H \Delta_G + \Delta_H)^2} \\ &+ \sum_{v_1 \in V(H)} \sum_{v_2 \in V(H)} \sum_{u_1u_2 \in E(G)} \sqrt{2(n_H \Delta_G + \Delta_H)^2} \\ &= \sqrt{2}(n_G m_H + n_H^2 m_G)(n_H \Delta_G + \Delta_H) \\ &= \sqrt{2} m_{G[H]}(n_H \Delta_G + \Delta_H). \end{aligned} \tag{26}$$

Analogously, one can compute the upper bound

$$SO(G[H]) \geq \sqrt{2} m_{G[H]}(n_H \delta_G + \delta_H). \tag{27}$$

The equality in (26) and (27) obviously holds if and only if G and H are regular graphs. □

In the next theorem, we compute bounds on the ISI index for composition of two graphs.

Theorem 10. *Let G and H be graphs. Then*

$$\frac{1}{2}m_{G[H]}\frac{(n_H\delta_G+\delta_H)^2}{n_H\Delta_G+\Delta_H} \leq ISI(G[H]) \leq \frac{1}{2}m_{G[H]}\frac{(n_H\Delta_G+\Delta_H)^2}{n_H\delta_G+\delta_H}.$$

The equality holds if and only if G and H are regular graphs.

Proof. Using (3), (4) and (25) in equation (1), we obtain

$$\begin{aligned}
ISI(G[H]) &= \sum_{u_1\in V(G)}\sum_{v_1v_2\in E(H)}\frac{(n_Hd_G(u_1)+d_H(v_1))(n_Hd_G(u_1)+d_H(v_2))}{2n_Hd_G(u_1)+d_H(v_1)+d_H(v_2)} \\
&+ \sum_{v_1\in V(H)}\sum_{v_2\in V(H)}\sum_{u_1u_2\in E(G)}\frac{(n_Hd_G(u_1)+d_H(v_1))(n_Hd_G(u_2)+d_H(v_2))}{n_H(d_G(u_1)+d_G(u_2))+d_H(v_1)+d_H(v_2)} \\
&\leq \sum_{u_1\in V(G)}\sum_{v_1v_2\in E(H)}\frac{(n_H\Delta_G+\Delta_H)^2}{2(n_H\delta_G+\delta_H)} \\
&+ \sum_{v_1\in V(H)}\sum_{v_2\in V(H)}\sum_{u_1u_2\in E(G)}\frac{(n_H\Delta_G+\Delta_H)^2}{2(n_H\delta_G+\delta_H)} \\
&= \frac{1}{2}(n_Gm_H+n_H^2m_G)\frac{(n_H\Delta_G+\Delta_H)^2}{n_H\delta_G+\delta_H} \\
&= \frac{1}{2}m_{G[H]}\frac{(n_H\Delta_G+\Delta_H)^2}{n_H\delta_G+\delta_H}.
\end{aligned} \tag{28}$$

Analogously, one can compute the upper bound

$$ISI(G[H]) \geq \frac{1}{2}m_{G[H]}\frac{(n_H\delta_G+\delta_H)^2}{n_H\Delta_G+\Delta_H}. \tag{29}$$

The equality in (28) and (29) obviously holds if and only if G and H are regular graphs. □

Remark 1. Recently, Milovanović et al. [17] proved that for a connected graph G

$$SO(G) \geq \frac{\sqrt{2}}{2}M_1(G).$$

The equality holds if and only if G is regular. Here, $M_1(G) = \sum_{uv\in E(G)}(d_u+d_v) = \sum_{u\in V(G)} d_u^2$ is the first Zagreb index [9]. As a consequence of this result, upper bounds on the first Zagreb index follow from the upper bounds on the Sombor index reported in this paper.

3. Conclusion

In this paper, the bounds for Sombor index of several graph operations in terms of maximum degree, minimum degree, order and size of the original graphs G and H are computed. These graph operations include corona product, cartesian product, strong product, composition and join of graphs. Analogously, we calculated the bounds for inverse sum indeg (ISI) index.
It still remains an open problem to compute the bounds for Sombor and ISI indices of graph operations like disjunction, symmetric index of graphs and several others.

Acknowledgment. This research is supported by UPAR grant of United Arab Emirates University (UAEU) via Grant No. G00003739.

Conflict of interest. The authors declare that they have no conflict of interest.

Data Availability. There is no data associated with this article.

References

[1] S. Akhter and R. Farooq, *Computing bounds for the general sum-connectivity index of some graph operations*, Algebra Discrete Math. **29** (2020), no. 2, 147–160
http://dx.doi.org/10.12958/adm281.

[2] S. Akhter, R. Farooq, and S. Pirzada, *Exact formulae of general sum-connectivity index for some graph operations*, Mat. Vesnik **70** (2018), no. 3, 267–282.

[3] A. Altassan, B.A. Rather, and M. Imran, *Inverse sum indeg index (energy) with applications to anticancer drugs*, Mathematics **10** (2022), no. 24, Article ID: 4749
https://doi.org/10.3390/math10244749.

[4] R. Cruz, I. Gutman, and J. Rada, *Sombor index of chemical graphs*, Appl. Math. Comput. **399** (2021), Article ID: 126018
https://doi.org/10.1016/j.amc.2021.126018.

[5] I. Gutman, *Geometric approach to degree-based topological indices: Sombor indices*, MATCH Commun. Math. Comput. Chem. **86** (2021), no. 1, 11–16.

[6] ———, *Some basic properties of Sombor indices*, Open J. Discrete Appl. Math. **4** (2021), no. 1, 1–3
https://doi.org/10.30538/psrp–odam2021.0047.

[7] I. Gutman and B. Furtula (Eds.), *Novel Molecular Structure Descriptors–Theory and Applications I*, Univ. Kragujevac, Kragujevac, 2010.

[8] I. Gutman, N.K. Gürsoy, A. Gürsoy, and A. Ülker, *New bounds on Sombor index*, Commun. Comb. Optim. **8** (2023), no. 2, 305–311
https://doi.org/10.22049/CCO.2022.27600.1296.

[9] I. Gutman and N. Trinajstić, *Graph theory and molecular orbitals. total φ-electron energy of alternant hydrocarbons*, Chem. Phys. Lett. **17** (1972), no. 4, 535–538 https://doi.org/10.1016/0009–2614(72)85099–1.

[10] Ö. Havare, *The inverse sum indeg index (ISI) and ISI energy of Hyaluronic Acid-Paclitaxel molecules used in anticancer drugs*, Open J. Discrete Appl. Math. **4** (2021), no. 3, 72–81
https://doi.org/10.30538/psrp–odam2021.0065.

[11] S.M. Hosamani, B.B. Kulkarni, R.G. Boli, and V.M. Gadag, *QSPR analysis of certain graph theocratical matrices and their corresponding energy*, Appl. Math. Nonlinear Sci. **2** (2017), no. 1, 131–150
https://doi.org/10.21042/AMNS.2017.1.00011.

[12] F. Jamal, M. Imran, and B.A. Rather, *On inverse sum indeg energy of graphs*, Special Matrices **11** (2023), no. 1, Article ID: 20220175
https://doi.org/10.1515/spma–2022–0175.

[13] M.H. Khalifeh, H. Yousefi-Azari, and A.R. Ashrafi, *The hyper-Wiener index of graph operations*, Comput. Math. Appl. **56** (2008), no. 5, 1402–1407
https://doi.org/10.1016/j.camwa.2008.03.003.

[14] ———, *The first and second Zagreb indices of some graph operations*, Discrete Appl. Math. **157** (2009), no. 4, 804–811
https://doi.org/10.1016/j.dam.2008.06.015.

[15] S.A.K. Kirmani, P. Ali, F. Azam, and P.A. Alvi, *On ve-degree and ev-degree topological properties of hyaluronic acid-anticancer drug conjugates with QSPR*, J. Chem. **2021** (2021), Artice ID: 3860856
https://doi.org/10.1155/2021/3860856.

[16] V.R. Kulli, *Sombor indices of certain graph operators*, Int. J. Eng. Sci. Research Tec. **10** (2021), no. 1, 127–134
https://doi.org/10.29121/ijesrt.v10.i1.2021.12.

[17] I. Milovanović, E. Milovanović, A. Ali, and M. Matejić, *Some results on the Sombor indices of graphs*, Contrib. Math. **3** (2021), 59–67
https://doi.org/10.47443/cm.2021.0024.

[18] I. Redžepović, *Chemical applicability of Sombor indices*, J. Serb. Chem. Soc. **86** (2021), no. 5, 445–457
https://orcid.org/0000–0003–4956–0407.

[19] T.A. Selenge and B. Horoldagva, *Extremal Kragujevac trees with respect to Sombor indices*, Commun. Comb. Optim., In press
https://doi.org/10.22049/CCO.2023.28058.1430.

[20] B. Shwetha Shetty, V. Lokesha, and P.S. Ranjini, *On the harmonic index of graph operations*, Trans. Comb. **4** (2015), no. 4, 5–14
https://orcid.org/10.22108/TOC.2015.7389.

[21] R. Todeschini and V. Consonni, *Handbook of Molecular Descriptors*, John Wiley & Sons, 2008.

[22] D. Vukicevic and M. Gasperov, *Bond additive modeling 1. Adriatic indices*, Croatica Chemica Acta **83** (2010), no. 3, 243–260.

[23] Z. Wang, Y. Mao, Y. Li, and B. Furtula, *On relations between Sombor and other*

degree-based indices, J. Appl. Math. Comput. **68** (2022), 1–17
https://doi.org/10.1007/s12190–021–01516–x.

[24] S. Zangi, M. Ghorbani, and M. Eslampour, *On the eigenvalues of some matrices based on vertex degree*, Iranian J. Math. Chem. **9** (2018), no. 2, 149–156
https://doi.org/10.22052/IJMC.2017.93637.1303.