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Research Article

Game chromatic number of honeycomb related networks

Muhammad Imran[1,*], Syed Ahtsham Ul Haq Bokhary[2,†], Muhammad Shahzad Akhtar[2,‡], Naoki Matsumoto[3]

[1]Department of Mathematical Sciences, United Arab Emirates University, P. O. Box 15551, Al Ain, United Arab Emirates
*imrandhab@gmail.com

[2]Centre for Advanced Studies in Pure and Applied Mathematics, Bahauddin Zakariya University, Multan, Pakistan
†sihtsham@gmail.com
‡shahzadmeo9@gmail.com

[3]Faculty of Education, University of the Ryukyus, Okinawa, Japan
naoki_m@edu.u-ryukyu.ac.jp

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Abstract: Let G be a simple connected graph having finite number of vertices (nodes). Let a coloring game is played on the nodes of G by two players, Alice and Bob alternately assign colors to the nodes such that the adjacent nodes receive different colors with Alice taking first turn. Bob wins the game if he is succeeded to assign k distinct colors in the neighborhood of some vertex, where k is the available number of colors, otherwise Alice wins. The game chromatic number of G is the minimum number of colors that are needed for Alice to win this coloring game and is denoted by $\chi_g(G)$. In this paper, the game chromatic number $\chi_g(G)$ for some interconnecting networks such as infinite honeycomb network, elementary wall of infinite height and infinite octagonal network is determined. Also, the bounds for the game chromatic number $\chi_g(G)$ of infinite oxide network are explored.

Keywords: Coloring; game chromatic number (GCN); infinite honeycomb network; infinite oxide network; elementary wall of infinite height; infinite octagonal network

AMS Subject classification: 05C15, 05C30

1. Introduction

Let G be a graph. The minimum number of colors that are needed to color the vertices (or nodes) of a graph G such that the adjacent nodes receive different colors,

* *Corresponding Author*

is known as the chromatic number of G and is denoted by $\chi(G)$ [10]. The game chromatic number (GCN) is defined as follows: For a simple finite connected graph G, a coloring game is played on the nodes of G by two players Alice and Bob with Alice taking first turn. They alternately allocate the colors from a set of k colors to the nodes of G in such a way that the adjacent nodes receive different colors. Bob wins the game for k colors provided he succeeds to allocate k different colors in the adjacency of an uncolored node of G before the complete coloring of G, otherwise, Alice wins for k colors after the complete coloring of G. The game chromatic number of G is the minimum number of colors that are needed for Alice to win this coloring game and is denoted by $\chi_g(G)$ [10].

The *game chromatic number (GCN)* is defined as follows: Let G be a simple connected graph having finite number of nodes. A coloring game is played on the nodes of G by two players Alice and Bob with Alice taking first turn. They alternately allocate the colors from a set of k colors to the nodes of the graph G in such a way that the connected nodes receive distinct colors. Bob wins the coloring game for k colors if Bob is succeeded to allocate k different colors in the adjacency of an uncolored node of G before the complete coloring of G, otherwise, Alice wins for k colors after the complete coloring of G.

The *game chromatic number* of G is the minimum number of colors that are needed for Alice to win this coloring game and is denoted by $\chi_g(G)$ [18]. The well-known result for the GCN of any graph G is:

$$\chi(G) \leq \chi_g(G) \leq \Delta(G) + 1 \tag{1}$$

where $\chi(G)$ denotes the chromatic number and $\Delta(G)$ represents the maximum degree of G.

This game coloring problem is an interesting topic, which has attracted the researchers in network science and game theory and a lot of work in this interesting field has done in the recent years. Game theory plays an important role in the World of Business and trading and makes strong impact on the economy of the country. Coloring of World Map is one of the important application of graph coloring. This research on game chromatic number work was started by Faigle et al. [12], who computed GCN for different families of graphs. Later, Kierstead and Trotter [16] (also, see [20]) showed that the upper bound for the GCN of a forest is 4 and the maximum GCN of planar graphs is 33. In [5], Bodlaender computed that the GCN of the cartesian product graph is less than or equal to some number for the class of planar graphs. Later, authors in [3], computed the exact values of $\chi_g(G\Box H)$, when G and H are in some special classes of graphs and proved that in general $\chi_g(G\Box H)$ is not bounded above by some function of game chromatic numbers of G and H, where $G\Box H$ represent the cartesian product of G and H. Sia and Gallian [18], computed the exact values of GCN for cartesian product of graphs like $S_m\Box P_n, S_m\Box C_n$, and $P_2\Box W_n$. Enomoto, Fujisawa and Matsumoto [11] studied the game chromatic number for strong product of graphs. Furtado et al. [13] identified caterpillars with game chromatic number 4, Akthar et al. [2] established results on game chromatic number for splitting graphs of

path and cycle. Chakraborti et al. [9] generalized concept of game chromatic number to hypergraphs. In [6], Bokhary et al. found the GCN of some convex polytope graphs of degree 4, see also [7]. Guan and Zhu [14], gave the results related to game chromatic number of outerplanar graphs. In this work, we have extended this study to some interconnecting networks and computed the exact value of the GCN of infinite honeycomb network, elementary wall of infinite height and infinite octagonal network. Also, the bounds for the GCN of infinite oxide network are determined.
We give definitions/notations, which are used in the results later on. Assume that the coloring game is played by Alice and Bob with k colors. We say that an uncolored node v is under a threat if there are k-1 colors in the neighbor of v, and it may be possible to color a node that is adjacent to v with the remaining color, so that all the k colors would occur in the neighborhood of v. The threat to v is said to be blocked or dealt provided v is allocated a color, or it is imppossible for v to get all k colors in its neighborhood. Also, we note that the color numbers are only used to differentiate different colors. Like, if colors 1 and 2 are already used, so far a new color, 3 (say), can be assigned to any color that is different from 1 or 2.

2. Preliminaries

A simple finite graph G consists of a set $V(G)$ of elements called vertices (or nodes), and a finite set $E(G)$ called edges. A graph G is said to be connected, if we can find a path between any two vertices of G. The degree of a node $v \in V(G)$ is the number of edges incident with v in G. The maximum and the minimum degree v in G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. The distance between two different vertices u and v in G is the length of a shortest path between them. The diameter of G, denoted by D, is defined as the maximum pairwise distance between the vertices of G. A simple connected graph G is said to be k-colorable, if we can assign one of k colors to each vertex of G so that adjacent vertices have different colors. A k-colorable graph is said to be k-chromatic if it is not $(k-1)$ colorable [8].

3. Methodology

In this paper, the game chromatic number denoted by $\chi_g(G)$ for some interconnecting networks such as infinite honeycomb network, elementary wall of infinite height and infinite octagonal network is explored. Since these networks are 3-regular, so by using Equation (1) and Bobs winning strategy for 3 colours, we find the exact value of the game chromatic number of these networks. Also, the bounds for the game chromatic number $\chi_g(G)$ of infinite oxide network are determined by using the Bobs winning strategy for 3 colours.

4. Results and Discussion

Game chromatic number of an infinite Honeycomb network

Honeycomb network is an interesting network which we see in daily life on the trees in which honeybees make honey and it has a lot of practical applications. In Holub, et al. [15] found the extremal graphs in the honeycomb network with restrictions on its valency and diameter. In this section, we have explored another graph property GCN of an infinite Honeycomb network.

Definition 1. A honeycomb network can be constructed from a hexagon. The honeycomb network $HC(1)$ is a single hexagon. The honeycomb network $HC(2)$ is designed from $HC(1)$ when six hexagons are connected to the boundary edges of $HC(1)$. By induction, honeycomb network $HC(n)$ is constructed from $HC(n-1)$ when a layer of hexagons is attached to the boundary of $HC(n-1)$.

The following is a key lemma for our results. Prepare an even cycle $v_1 v_2 \ldots v_k$ $(k \geq 4)$ and add a pendant vertex u_i to v_i for every even number i, where all vertices are distinct. Let B_k denote the resulting graph; see Figure 1. As shown in the following lemma, B_k is good for Bob.

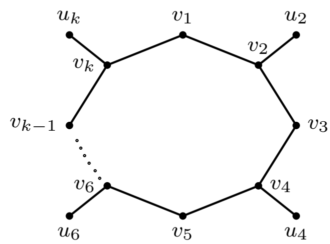

Figure 1. **The graph B_k**

Lemma 1. *For an integer $k \geq 4$, let $G = B_k$. Suppose that Alice first allocates some color to v_1. Then Bob can force Alice to use a fourth color.*

Proof. It suffices to show that Bob can win the game using three colors. Without loss of generality, Alice allocates color 1 to v_1. After that, Bob alternately allocates colors 2 and 3 to $v_3, v_5, \ldots, v_{k-1}$ in this order. By Bob's coloring, $v_2, v_4, \ldots, v_{k-2}, v_k$requires colors 3 and 2 alternately. For each even $i = 2, 4, \ldots, k-2$, if Alice allocates any color to neither u_i nor v_i just after Bob colors v_{i+1}, Bob can force Alice to use a fourth color by allocating a third color (2 or 3) to u_i. However, after Bob allocates a color to $v_k - 1$, Alice needs to allocate a third color to both one of u_{k-2} and v_{k-2} and that of u_k and v_k, which is impossible. $\square$

By the above lemma, we have the following important corollary.

Corollary 1. *Let G be a graph. In some Bob's turn, if G contains an induced subgraph S isomorphic to B_k for some even $k \geq 4$ where only one vertex v in S is coloured and v is of degree 2, then Bob can force Alice to use a fourth color.*

Theorem 1. *Let H be the infinite Honeycomb network then*
$\chi_g(H) = 4$

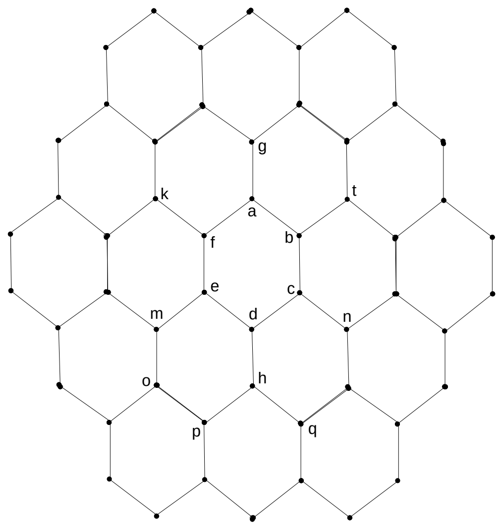

Figure 2. **A small part R of infinite Honeycomb network H**

Proof. The infinite honeycomb network H consists of infinite number of hexagons. We select a small part R of dimension 3 of infinite honeycomb network H as shown in the Figure 2 and find the GCN of H. Since each hexagon in H is surrounded by infinite number of layers of hexagons and symmetry can be seen in the network therefore Bob has the same winning strategy for each hexagon that is selected by Alice to start the game. Some nodes of the small part R of H are labelled for our convenience as shown in the Figure 2. Since the maximum degree of H is 3, Equation (1) implies that $\chi_g(H) \leq 4$. Hence it suffices to show that $\chi_g(H) > 3$. Without loss of generality, Alice first allocates color 1 to the vertex a. In this case, the nine vertices $\{a, b, c, d, e, f, t, h, k\}$ induce B_6 in which only one vertex of degree 2 in the B_6, a, is coloured. Thus, we have $\chi_g(H) > 3$ by Corollary 1. □

Game chromatic number of an infinite oxide network

Chemical graph theory is an important branch of Mathematics which deals with chemical structures. Silicate and oxide networks are very interesting structures of chemistry which we study in the chemical graph theory. A lot of work has done on these structures and properties of these networks are computed by the Mathematicians. In [17, 19], Manuel et al. and Simonraj studied silicate and oxide networks in different aspects. In this section, we explored the GCN of an infinite oxide network.

Definition 2. Silicate network can be designed in many ways. Here, we have designed the graph of silicate network from a honeycomb network. Take a honeycomb network $HC(n)$ which has dimension n. All the nodes of $HC(n)$ are replaced by the silicon atoms. Every edge of $HC(n)$ is subdivided exactly once. Oxygen atoms are placed on new nodes. 6n new pendant edges at each of 2-degree silicon atoms of $HC(n)$ are introduced and oxygen atoms are placed at the pendant nodes (see the Figure 3). With each silicon atom, relate the three connected oxygen atoms and make a tetrahedron unit. The resulting network obtained is then the silicate network of dimension n, which is denoted by $SL(n)$ having $15n2 + 3n$ nodes and $36n^2$ edges. When all the silicon vertices are removed from a silicate network, new network is formed which is called the oxide network. oxide network of dimension n is designated by $OX(n)$ with $9n^2 + 3n$ nodes and $18n^2$ edges [17]. Oxide network of dimension two is shown in the Figure 3 (b).

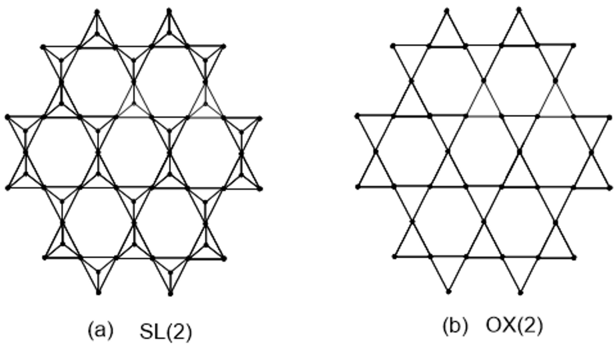

Figure 3. **Silicate network** $SL(2)$ **and Oxide network** $OX(2)$

Theorem 2. *Let X be the infinite Oxide network. Then*

$$4 \leq \chi_g(X) \leq 5.$$

Proof. The infinite oxide network X contains infinite number of triangles. We select a small part T of dimension 2 (See Figure 4) of infinite oxide network X to play the

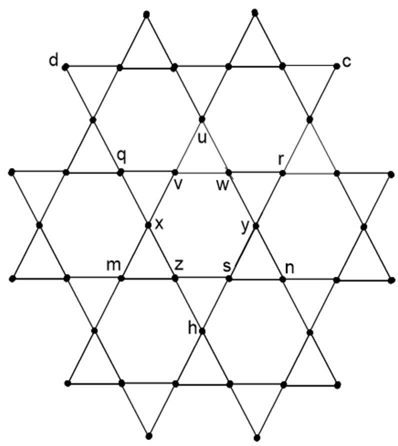

Figure 4. **A small part T of infinite Oxide network X**

game and find the bounds for the GCN of X. Since each triangle in X is attached with three triangles and there is symmetry in the network therefore Bob has the same winning strategy for every triangle, which, is selected by Alice to start the game. Some nodes of the small part T are labelled to play the game as shown in the Figure 4. Since maximum degree of X is 4, from Equation 1, we have $\chi_g(X) \leq 5$. Hence it suffices to show that $\chi_g(X) > 3$. Without loss of generality, Alice first allocates color 1 to the vertex w. In this case, we consider the subgraph S induced by the nine vertices $\{w, y, s, z, x, v, n, m, u\}$. Observe that S is obtained from B_6 by adding three additional edges sn, xm and wu. As in the proof of Lemma 1, Bob allocates colors 2 and 3 to s and x in this order, which requires the third colors 3 and 2 for y and $\{z, v\}$. It is not prevented by the existence of the additional three edges that Bob allocates colors 3 or 2 to vertices n or m,u (after Alice's move). Therefore, similarly to the proof of Theorem 1, we have $\chi_g(X) > 3$. □

Game chromatic number of elementary walls of infinite height

The grid graph is an important network which is designed from the cartesian product of paths P_m and P_n. The grid graph is studied in different aspects in graph theory and many other networks are designed from the grid graph such as Enhanced mesh network and enhanced grid network. Elementary wall is also designed from the grid graph which is defined below. The structure of W-4-Immersion-Free Graphs such as elementary walls of different heights is studied by Belmonte, Giannopoulou, Lokshtanov and Thilikos in [4]. In this section, we find the GCN of the elementary wall of infinite height.

Definition 3. Let r be a natural number. The (s,s)-grid is the graph which is obtained from the cartesian product of path P_s with P_s. The (elementary) wall of height s is designed from the (s,s)-grid and is denoted by W_s whose node set is $V(W_s) = \{(x,y)|x \in [s+1], y \in [2s+2]\}$ in which we make two nodes (x,y) and $(x^{'}, y^{'})$ are adjacent if and only if either $x = x^{'}$ and $y^{'} \in \{y-1, y+1\}$ or $y^{'} = y$ and $x^{'} = x + (-1)^{x+y}$, and then return all nodes of valency 1; see the Figure 5 for elementary wall of height 4. The nodes of this node set are called original nodes of the wall. From the construction, it is observed that the graph W_s has number of nodes $2s^2+4s$ and number of edges $3s^2+4s-1$ [4].

Theorem 3. *Let W be the elementary wall of infinite height. Then*

$$\chi_g(W) = 4.$$

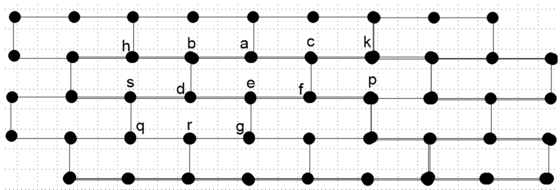

Figure 5. **A small part P of elementary wall of infinite height W**

Proof. The elementary wall of infinite height W in the Euclidean plane has maximum degree 3. We choose a small part P (See Figure 5) of height 4 of W to play the game and find the exact values for the GCN of W. Some nodes of the small part P are labeled for our convenience as shown in the Figure 5. Since each node of W has degree 3 and there is a symmetry in the network, therefore, Bob has the same strategy for every node of W , which, is selected by Alice to start the game. Since maximum degree of W is 3, from Equation (1) we have $\chi_g(W) \leq 4$. Hence it suffices to show that $\chi_g(W) > 3$. Without loss of generality, Alice first allocates color 1 to the vertex a. In this case, the nine vertices $\{a, c, f, e, d, b, k, g, h\}$ induce B_6 in which only one vertex of degree 2 in the B_6, a, is coloured. Thus, we have $\chi_g(H) > 3$ by Corollary 1. □

5. Game chromatic number of an infinite octagonal network

The Octagonal network is also an important network which is obtained by arranging octagons row wise and column wise as defined below. In [1], Ahmad, Afzal, Nazeer and Kang found the topological indices of octagonal network. In this section we find the GCN of an infinite Octagonal network.

Definition 4. Octagonal network by $O_{n,m}$ for $n, m \geq 2$. Its planer representation of $O_{n,m}$ is shown in [1] with m rows and n columns of octagonal. The vertex set V and edge E of $O_{n,m}$ are $V = \{x_i^j, 1 \leq i \leq 2n-1,\ i \text{ is odd and } 1 \leq j \leq 3m+1\} \bigcup \{x_i^{3j-2}; 1 \leq i \leq 2n,\ i \text{ is even and } 1 \leq j \leq m+1\} \bigcup \{x_{2n}^{3j-1} x_{2n}^{3j}, 1 \leq j \leq m\}$ and $E = \{x_i^j x_i^{j+1}, 1 \leq i \leq 2n-1,\ i \text{ is odd and } 1 \leq j \leq 3m\} \bigcup \{x_i^{3j-2} x_{i+1}^{3j-2}, 1 \leq i \leq 2n-1,$ $i \text{ is odd and } 1 \leq j \leq m+1\} \bigcup \{x_i^{3j-2} x_{i+1}^{3j-1}, 1 \leq i \leq 2n-2,\ i \text{ is even and } 1 \leq j \leq m\} \bigcup \{x_i^{3j} x_{i-1}^{3j+1}, 3 \leq i \leq 2n-1,\ i \text{ is odd and } 1 \leq j \leq m\} \bigcup \{x_{2n}^j x_{2n}^{j+1}, 1 \leq j \leq 3m\}$.

The number of nodes in octagonal network is $(4m+2)n+2m$ and number of edges in an octagonal network is $(6m+1)n+m$ [1].

Theorem 4. *Let O be the infinite Octagonal network. Then*

$$\chi_g(O) = 4.$$

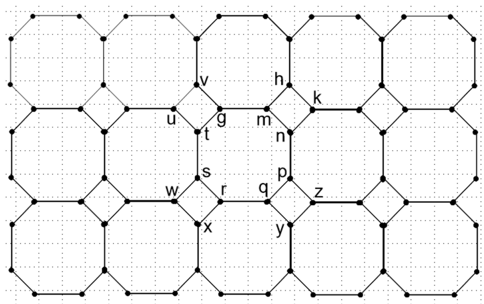

Figure 6. **A small part $O_{5,3}$ of infinite Octagonal network O**

Proof. The infinite octagonal network O contains an infinite number of octagons. We select a small part $O_{5,3}$ (See Figure 6) of infinite Octagonal network O as shown in the Figure 6 and find the GCN of Octagonal network O. Since each octagon in O is surrounded by infinite number of layers of Octagons and symmetry can be seen in the network therefore Bob has the same winning strategy for each Octagon that is selected by Alice to start the game. Some nodes of the small part $O_{5,3}$ of O are labelled for our convenience as shown in the Figure 6. Since maximum degree of O is 3, from equation 1 we have $\chi_g(O) \leq 4$. Hence it suffices to show that $\chi_g(O) > 3$. Without loss of generality, Alice first allocates color 1 to the vertex m. In this case, the twelve vertices $\{m, n, p, q, r, s, t, g, k, y, w, v\}$ induce B_8 in which only one vertex of degree 2 in the B_8, m, is coloured. Thus, we have $\chi_g(H) > 3$ by Corollary 1. $\square$

6. Conclusion

Game theory plays an important role in the world of business and makes strong and significant impact on the economy of the country. In this paper, we have found the game chromatic number of three important classes of networks, that is, infinite honeycomb network, elementary wall of infinite height and infinite octagonal network. We have also explored the bounds for the game chromatic number of infinite oxide network. This study can be further extended for the silicate network and other chemically important networks.

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Conflicts of Interest
The authors declare that they have no conflicts of interest.

Data Availability
No data were associated with this article.

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