Communications in Combinatorics and Optimization
Vol. 9, No. 1 (2024), pp. 119-130
https://doi.org/10.22049/cco.2023.28642.1647

CCO
Commun. Comb. Optim.

Research Article

The energy and edge energy of some Cayley graphs on the abelian group $\mathbb{Z}_n^4$

Fateme Movahedi

Department of Mathematics, Faculty of Sciences, Golestan University, Gorgan, Iran.
f.movahedi@gu.ac.ir

Received: 8 April 2023; Accepted: 2 June 2023
Published Online: 5 June 2023

Abstract: Let $G = (V, E)$ be a simple graph such that $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of G. The energy of graph G is denoted by $E(G)$ and is defined as $E(G) = \sum_{i=1}^{n} |\lambda_i|$. The edge energy of G is the energy of line graph G. In this paper, we investigate the energy and edge energy for two Cayley graphs on the abelian group $\mathbb{Z}_n^4$, namely, the Sudoku graph and the positional Sudoku graph. Also, we obtain graph energy and edge energy of the complement of these two graphs.

Keywords: Graph energy, abelian group, spectrum, complement, line graph

AMS Subject classification: 05C50, 05C25

1. Introduction

Throughout this paper, we consider $G = (V, E)$ a simple and finite graph with the vertex set $V = \{v_1, \ldots, v_n\}$ and the edge set $E = \{e_1, \ldots, e_m\}$. For $v_i \in V$, the degree of v_i is the number of edges connected to v_i and is denoted by d_i. The complement of a graph G, denoted by $\overline{G}$, is the graph with all the vertices of graph G, and two vertices in $\overline{G}$ are adjacent if they are not adjacent in G.

The adjacency matrix of G denoted by $A(G) = (a_{ij})$ is the matrix of the order n where $a_{ij} = 1$ if the vertices v_i and v_j are adjacent and $a_{ij} = 0$ otherwise. The eigenvalues of $A(G)$ are the eigenvalues of graph G. Let $\lambda_1 > \lambda_2 > \cdots > \lambda_t$ be a non-increasing sequence of eigenvalues of G with multiplicities $m_1, m_2, \ldots, m_t$, respectively. The spectrum of G denoted by, $Spec(G)$ is written by

$$Spec(G) = \begin{pmatrix} \lambda_1 & \ldots & \lambda_t \\ m_1 & \ldots & m_t \end{pmatrix}.$$

Assume that $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of graph G. The energy of graph G is defined as $E(G) = \sum_{i=1}^{n} |\lambda_i|$ [10]. The energy of a graph is a mathematical concept that has many applications in theoretical chemistry. Accordingly, much research on graph energy and its numerous variants have been done. The line graph $L(G)$ of the graph G is the graph whose vertex set of $L(G)$ is the set of G and two vertices of $L(G)$ are adjacent if and only if the corresponding edges in G have a common vertex [12]. The edge energy of a graph G, denoted by $EE(G)$ is the energy of the line graph of G [2]. Some results are obtained on the edge energy of graphs that can be found in [8, 11, 16, 17].

The connection between algebraic structures and graph theory is one of the important branches of mathematics. In 1878, first was introduced the Cayley graphs on finite groups [3]. Let Γ be a finite group and $S \subseteq \Gamma \setminus \{1\}$ with condition $S = S^{-1}$. The $G = Cay(\Gamma, S)$ is an undirected and simple graph that contains the vertex set $V(G) = \Gamma$ and the edge set $E(G) = \{(x, y) \,|\, xy^{-1} \in S\}$. A Cayley graph $Cay(\Gamma, S)$ is connected if and only if $\Gamma =< S >$ [1].

There are extensive studies devoted to computing graph energies of Cayley graphs. For more literature, on Cayley graphs and their energies, readers might refer to [4, 5, 9, 13, 15].

In this paper, we consider two Cayley graphs on group $\mathbb{Z}_n^4$, namely, the Sudoku graphs and the positional Sudoku graphs. The energy of these graphs is obtained. Also, by computing the eigenvalues of line graphs of the Sudoku graph and the positional Sudoku graph, their edge energy is obtained. Finally, we obtain the edge energy and graph energy of the complement of the Sudoku graph and the positional Sudoku graph.

2. Energies of the Sudoku graph $\mathbf{Sud(n)}$

In this section, we obtain the graph energy and edge energy of the Sudoku graph that is the Cayley graph on group $\mathbb{Z}_n^4$.

An n-Sudoku is an arrangement of $n \times n$ square blocks each consisting of $n \times n$ cells and each cell contains a number from $\{1, \ldots, n^2\}$ such that every block, row or column contain all numbers $1, 2, \ldots, n^2$. The Sudoku graph $Sud(n)$ is a simple graph that has the vertices the n^4 cells of an n-Sudoku and two distinct vertices are adjacent if and only if they are in the same block, row or column [14].

The $Sud(n)$ is the $(3n^2 - 2n - 1)$-regular graph with $\frac{n^2(3n^2-2n-1)}{2}$ edges. In [14], is proved that graph $Sud(n)$ is a Cayley graph for $\mathbb{Z}_n^4$. Therefore, $Sud(n) = Cay(\mathbb{Z}_n^4, S)$ in which $S = S_1 \cup S_2 \cup S_3$ and S_i's are defined as follows

$$\begin{aligned}
S_1 &= \{(0, 0, x_3, x_4) \,:\, x_3, x_4 \in \mathbb{Z}_n \,,\, (x_3, x_4) \neq (0, 0)\}, \\
S_2 &= \{(0, x_2, 0, x_4) \,:\, x_2, x_4 \in \mathbb{Z}_n \,,\, x_2 \neq 0\}, \\
S_3 &= \{(x_1, 0, x_3, 0) \,:\, x_1, x_3 \in \mathbb{Z}_n \,,\, x_1 \neq 0\}.
\end{aligned} \tag{1}$$

The eigenvalues of $Sud(n) = Cay(\mathbb{Z}_n^4, S)$ are computed in [14]. We state the spectrum of graph $Sud(n)$ in the following lemma.

Lemma 1. [14] *For $n \geq 2$, the spectrum of the Sudoku graph $Sud(n)$ is as follows*

$$\begin{pmatrix} 3n^2-2n-1 & 2n^2-2n-1 & n^2-n-1 & n^2-2n-1 & -1 & -1-n \\ 1 & 2(n-1) & 2n(n-1) & (n-1)^2 & n^2(n-1)^2 & 2n(n-1)^2 \end{pmatrix}.$$

In the following theorem, we obtain the graph energy of the Cayley graph $Sud(n)$ on group $\mathbb{Z}_n^4$ for $n \geq 2$.

Theorem 1. *Let $Sud(n) = Cay(\mathbb{Z}_n^4, S)$ be the Cayley graph on the abelian group $\mathbb{Z}_n^4$ and $S = \bigcup_{i=1}^3 S_i$ that S_i's are defined in (1).*

i) If $n = 2$, then $E(Sud(n)) = 34$.

ii) If $n \geq 3$, then $E(Sud(n)) = 2n(n-1)^2(3n+2)$.

Proof. According to Lemma 1 and the definition of the graph energy, we get

$$\begin{aligned} E(Sud(n)) &= \sum_{i=1}^{n^4} |\lambda_i| \\ &= |3n^2-2n-1| + 2(n-1)|2n^2-2n-1| + 2n(n-1)|n^2-n-1| \\ &+ (n-1)^2|n^2-2n-1| + n^2(n-1)^2|-1| + 2n(n-1)^2|-1-n| \\ &= (-1+n)^2(5n^2+6n+1) + (n-1)^2|n^2-2n-1| \end{aligned}$$

For $n = 2$, we obtain $E(Sud(n)) = (n-1)^2(5n^2+6n+2) = 34$. Also if $n \geq 3$, we have $E(Sud(n)) = (-1+n)^2(5n^2+6n+1) + (n-1)^2(n^2-2n-1) = 2n(n-1)^2(3n+2)$. □

Lemma 2. [6] *Let G be a regular graph of degree $r \geq 2$ with n vertices and $m = \frac{nr}{2}$ edges. Then the following relations hold.*

(i) For $1 \leq i \leq n$, $\lambda_i(L(G)) = \lambda_i(G) + r - 2$,

(ii) for $n+1 \leq i \leq m$, $\lambda_i(L(G)) = -2$.

Now, we obtain the eigenvalues of the line graph of Sudoku graph $Sud(n)$.

Theorem 2. *The spectrum of the line graph of the Sudoku graph $Sud(n)$ for $n \geq 2$ is as follows*

$$\begin{pmatrix} 6n^2-4n-4 & 5n^2-4n-4 & 4n^2-3n-4 & 4n^2-4n-4 & 3n^2-2n-4 & 3n^2-3n-4 & -2 \\ 1 & 2(n-1) & 2n(n-1) & (n-1)^2 & n^2(n-1)^2 & 2n(n-1)^2 & \frac{1}{2}n^4(3n^2-2n-3) \end{pmatrix}.$$

Proof. Let $G = Sud(n)$ be the Sudoku graph. Since graph $Sud(n)$ for $n \geq 2$ is the $(3n^2-2n-1)$-regular graph, thus using Lemmas 1 and 2, we obtain the eigenvalues of the line graph of $Sud(n)$ as follows
i) $\lambda_1\big(L(Sud(n))\big) = (3n^2-2n-1) + (3n^2-2n-1) - 2 = 6n^2-4n-4$.
ii) For $2 \leq i \leq 2n-1$, then

$$\lambda_i\big(L(Sud(n))\big) = (2n^2-2n-1) + (3n^2-2n-1) - 2 = 5n^2-4n-4.$$

iii) For $2n \leq i \leq 2n^2 - 1$, then

$$\lambda_i\big(L(Sud(n))\big) = (n^2 - n - 1) + (3n^2 - 2n - 1) - 2 = 4n^2 - 3n - 4.$$

iv) For $2n^2 \leq i \leq 3n^2 - 2n$, then

$$\lambda_i\big(L(Sud(n))\big) = (n^2 - 2n - 1) + (3n^2 - 2n - 1) - 2 = 4n^2 - 4n - 4.$$

v) For $3n^2 - 2n + 1 \leq i \leq n^4 - 2n^3 + 4n^2 - 2n$, then

$$\lambda_i\big(L(Sud(n))\big) = -1 + (3n^2 - 2n - 1) - 2 = 3n^2 - 2n - 4.$$

vi) For $n^4 - 2n^3 + 4n^2 - 2n + 1 \leq i \leq n^4$, then

$$\lambda_i\big(L(Sud(n))\big) = (-1 - n) + (3n^2 - 2n - 1) - 2 = 3n^2 - 3n - 4.$$

vii) For $n^4 + 1 \leq i \leq m = \frac{n^4(3n^2-2n-1)}{2}$, then $\lambda_i\big(L(Sud(n))\big) = -2$.

Therefore, the spectrum of graph $L(Sud(n))$ for $n \geq 2$ is obtained from the eigenvalues of $\lambda_i\big(L(Sud(n))\big)$ for $1 \leq i \leq m$. □

Theorem 3. *The edge energy of the Sudoku graph $Sud(n)$ for $n \geq 2$ is as follows*

$$EE(Sud(n)) = 2n^4(3n^2 - 2n - 3).$$

Proof. Let $G = Sud(n)$ be the Sudoku graph for $n \geq 2$ of the order n^4 and size $m = \frac{n^4(3n^2-2n-1)}{2}$. Since the edge energy of graph $Sud(n)$ is the energy of the line graph of $Sud(n)$, therefore using Theorem 2, we get

$$\begin{aligned}
EE(G) = E(L(G)) &= \sum_{i=1}^{m} |\lambda_i(L(G))| \\
&= |6n^2 - 4n - 4| + 2(n-1)|5n^2 - 4n - 4| + 2n(n-1)^2|4n^2 - 3n - 4| \\
&+ (n-1)^2|4n^2 - 4n - 4| + n^2(n-1)^2|3n^2 - 2n - 4| \\
&+ 2n(n-1)^2|3n^2 - 3n - 4| + \frac{n^4(3n^2 - 2n - 3)}{2}| - 2|
\end{aligned}$$

By simplifying the above relation and since $n \geq 2$, the result is completed. □

Now, we investigate to obtain the graph energy and the edge energy of the complement of the Cayley graph $Sud(n)$. To do this, we first recall the following known result about the characteristic polynomial of the complement of a graph.

Lemma 3. [7] *If G is a r-regular with n vertices then*

$$P_{\overline{G}}(x) = (-1)^n \frac{x - n + r + 1}{x + r + 1} P_G(-x - 1),$$

where $P_{\overline{G}}$ is the characteristic polynomial of the complement of the graph G.

Theorem 4. *Let $Sud(n)$ be the Sudoku graph of the order n^4 where $n \geq 2$. Then, the spectrum of the complement of graph $Sud(n)$ is as follows*

$$\begin{pmatrix} n^4-3n^2+2n & -2n^2+2n & -n^2+n & -n^2+2n & 0 & n \\ 1 & 2(n-1) & 2n(n-1) & (n-1)^2 & n^2(n-1)^2 & 2n(n-1)^2 \end{pmatrix}.$$

Proof. Let G be the Sudoku graph of the order n^4 with the degree of all vertices $3n^2-2n-1$. Thus, using Lemmas 1 and 3, we get the eigenvalues of $\overline{G}$ as follows

$$\begin{aligned}
P_{\overline{G}}(\lambda) &= (-1)^{n^4}\Big(\frac{\lambda-n^4+3n^2-2n-1+1}{\lambda+3n^2-2n-1+1}\Big)P_G(-\lambda-1)\\
&= (-1)^{n^4}\Big(\frac{\lambda-n^4+3n^2-2n}{\lambda+3n^2-2n}\Big)\big(-\lambda-3n^2+2n+1-1\big)\\
&\quad \big(-\lambda-2n^2+2n+1-1\big)^{2(n-1)}\big(-\lambda-n^2+n+1-1\big)^{2n(n-1)}\\
&\quad \big(-\lambda-n^2+2n+1-1\big)^{(n-1)^2}\big(-\lambda+1-1\big)^{n^2(n-1)^2}\big(-\lambda+1+n-1\big)^{2n(n-1)^2}\\
&= (-1)^{2(n-1)}\big(\lambda+2n^2-2n\big)^{2(n-1)}(-1)^{2n(n-1)}\big(\lambda+n^2-n\big)^{2n(n-1)}(-1)^{(n-1)^2}\\
&\quad \big(\lambda+n^2-2n\big)^{(n-1)^2}(-1)^{n^2(n-1)^2}\big(\lambda\big)^{n^2(n-1)^2}(-1)^{2n(n-1)^2}\big(\lambda-n\big)^{2n(n-1)^2}\\
&= (-1)\big(\lambda+2n^2-2n\big)^{2(n-1)}\big(\lambda+n^2n\big)^{2n(n-1)}\big(\lambda+n^2-2n\big)^{(n-1)^2}\\
&\quad \big(\lambda\big)^{n^2(n-1)^2}\big(\lambda-n\big)^{2n(n-1)^2}.
\end{aligned}$$

Since the roots of the characteristic polynomial of a graph G are the eigenvalues of G, therefore, the eigenvalues of the complement of the Sudoku graph $Sud(n)$ for $n \geq 2$ are easily obtained from the above. □

Theorem 5. *The energy of the complement of the Sudoku graph $Sud(n)$ for $n \geq 2$, is equal to $2n(n-1)^2(3n+2)$.*

Proof. Let G be the Sudoku graph of the order n^4. According to the definition of graph energy and using Theorem 4, for $n \geq 2$ we get

$$\begin{aligned}
E(\overline{G}) &= \sum_{i=1}^{n^4}\big|\lambda_i(\overline{G})\big|\\
&= (n^4-3n^2+2n)+2(n-1)(2n^2-2n)+2n(n-1)(n^2-n)\\
&+(n-1)^2(n^2-2n)+n^2(n-1)^2(n-1)^2(0)+2n(n-1)^2(n)\\
&= 6n^4-8n^3-2n^2+4n.
\end{aligned}$$

By simplifying the above relation, the result holds. □

Finally, we obtain the edge energy of the complement of graph $Sud(n)$, for $n \geq 2$.

Theorem 6. *The edge energy of the complement of the Sudoku graph $Sud(n)$ of the order n^4, where $n \geq 2$ is equal to $2n^4(n^4-3n^2+2n-2)$.*

Proof. Let G be the Sudoku graph $Sud(n)$ of the order n^4 for $n \geq 2$. In Theorem 4, the eigenvalues of the complement graph G are computed. Since G is the $(3n^2-2n-1)$-regular graph, thus the degree of all vertices of $\overline{G}$ is equal to n^4-3n^2+2n. So, $\overline{G}$ is the (n^4-3n^2+2n)-regular graph.
Therefore, using Lemma 2 and Theorem 4 we get the eigenvalues of $\overline{G}$ as follows.
i) $\lambda_1\big(L(\overline{G})\big) = (n^4-3n^2+2n)+(n^4-3n^2+2n)-2 = 2n^4-6n^2+4n-2$.
ii) For $2 \leq i \leq 2n-1$,

$$\lambda_i\big(L(\overline{G})\big) = (-2n^2+2n)+(n^4-3n^2+2n)-2 = n^4-5n^2+4n-2.$$

iii) For $2n \leq i \leq 2n^2-1$,

$$\lambda_i\big(L(\overline{G})\big) = (-n^2+n)+(n^4-3n^2+2n)-2 = n^4-4n^2+3n-2.$$

iv) For $2n^2 \leq i \leq 3n^2-2n$,

$$\lambda_i\big(L(\overline{G})\big) = (-n^2+2n)+(n^4-3n^2+2n)-2 = n^4-4n^2+4n-2.$$

v) For $3n^2-2n+1 \leq i \leq n^4-2n^3+4n^2-2n$,

$$\lambda_i\big(L(\overline{G})\big) = 0+(n^4-3n^2+2n)-2 = n^4-3n^2+2n-2.$$

vi) For $n^4-2n^3+4n^2-2n+1 \leq i \leq n^4$,

$$\lambda_i\big(L(\overline{G})\big) = n+(n^4-3n^2+2n)-2 = n^4-3n^2+3n-2.$$

vii) For $n^4+1 \leq i \leq m = \frac{n^4(3n^2-2n-1)}{2}$, $\lambda_i\big(L(\overline{G})\big) = -2$.

According to obtained eigenvalues above, the edge energy of the complement of $Sud(n)$ for $n \geq 2$ is as follows

$$\begin{aligned}
EE(\overline{G}) = E(L(\overline{G})) &= \sum_{i=1}^{m} |\lambda_i(L(\overline{G}))| \\
&= (2n^4-6n^2+4n-2)+2(n-1)(n^4-5n^2+4n-2) \\
&+ 2n(n-1)(n^4-4n^2+3n-2)+(n-1)^2(n^4-4n^2+4n-2) \\
&+ n^2(n-1)^2(n^4-3n^2+2n-2)+2n(n-1)^2(n^4-3n^2+3n-2) \\
&+ \frac{n^4(3n^2-2n-3)}{2}|-2|.
\end{aligned}$$

By simplifying the above relation, the result holds. $\square$

3. Energies of the positional Sudoku graph SudP(n)

In this section, we focus on investigating the graph energy and edge energy of the positional Sudoku graph that is the Cayley graph on group $\mathbb{Z}_n^4$.

The positional Sudoku graph $SudP(n)$ is a Cayley graph $Cay(\mathbb{Z}_n^4, \tilde{S})$ that has more edges than $Sud(n)$ and $\tilde{S} = S_1 \cup S_2 \cup S_3 \cup S_4$ where S_i's for $i = 1, 2, 3$ are defined in (1) and

$$S_4 = \{(x_1, x_2, 0, 0) : x_1, x_2 \in \mathbb{Z}_n,\ x_1, x_2 \neq 0\}. \tag{2}$$

The eigenvalues of $Cay(\mathbb{Z}_n^4, \tilde{S}) = SudP(n)$ are obtained in [14]. In the following Lemma, the spectrum of $SudP(n)$ is stated.

Lemma 4. [14] *For $n \geq 2$, the spectrum of the positional Sudoku graph $SudP(n)$ is as follows*

$$\begin{pmatrix} 4n(n-1) & 2n^2-3n & n(n-2) & 0 & -n & -2n \\ 1 & 4(n-1) & 4(n-1)^2 & (n-1)^4 & 4(n-1)^3 & 2(n-1)^2 \end{pmatrix}.$$

First, we obtain the graph energy of the Cayley graph $SudP(n)$ on group $\mathbb{Z}_n^4$ for $n \geq 2$.

Theorem 7. *Let $SudP(n) = Cay(\mathbb{Z}_n^4, \tilde{S})$ be the Cayley graph on the abelian group $\mathbb{Z}_n^4$ and $\tilde{S} = \bigcup_{i=1}^4 S_i$ that S_i's are defined in (1) and (2). Then for $n \geq 2$,*

$$E(SudP(n)) = 8n^2(n-1)^2.$$

Proof. Similar to Theorem 1 and using Lemma 4, we get

$$\begin{aligned} E(SudP(n)) &= \sum_{i=1}^{n^4} |\lambda_i| \\ &= |4n(n-1)| + 4(n-1)|2n^3 - 3n| + 4n(n-1)^2|n(n-2)| \\ &+ (n-1)^4|0| + 4(n-1)^3|-n| + 2(n-1)^2|-2n| \\ &= 4n(n-1) + 4n(n-1)^2 + 4n(n-2)(n-1)^2 \\ &+ 4n(n-1)^3 + (4n-4)(2n^2-3n). \end{aligned}$$

By simplification the above relation, the result is completed. □

For computing the edge energy of graph $SudP(n)$, we need to obtain the eigenvalues of the line graph of the positional Sudoku graph $SudP(n)$. To do this, we first show that $SudP(n)$ is an $(4n^2 - 4n)$-regular graph.

Theorem 8. *The positional Sudoku graph $SudP(n)$ for $n \geq 2$ is an $(4n^2 - 4n)$-regular graph.*

Proof. According to the structure of the positional Sudoku graph $SudP(n)$, in addition to the vertices that are adjacent in the Sudoku graph $Sud(n)$, the vertices that have the same numbers are also adjacent. Therefore, for the vertex $v \in V(SudP(n))$, the number of the same digits is equal to $n^2 - (2n-1)$. Thus, the degree of all $v \in V(SudP(n))$, we get $\deg(v) = (3n^2 - 2n - 1) + (n^2 - 2n) + 1 = 4n^2 - 4n$. Consequently, graph $SudP(n)$ is the regular graph of degree $(4n^2 - 4n)$. □

Theorem 9. *The spectrum of the line graph of the positional Sudoku graph $SudP(n)$ for $n \geq 2$ is as follows*

$$\begin{pmatrix} 8n^2-8n-2 & 6n^2-7n-2 & 5n^2-6n-2 & 4n^2-4n-2 & 4n^2-5n-2 & 4n^2-6n-2 & -2 \\ 1 & 4(n-1) & 4(n-1)^2 & (n-1)^4 & 4(n-1)^3 & 2(n-1)^2 & \frac{1}{2}n^4(4n^2-4n-2) \end{pmatrix}.$$

Proof. Let $G = SudP(n)$ be the positional Sudoku graph. By applying Theorem 8, G is the $(4n^2-4n)$-regular graph. Thus using Lemmas 4 and 2, we compute the eigenvalues of the line graph of $SudP(n)$ for $n \geq 2$ as follows

i) $\lambda_1(L(G)) = (4n^2 - 4n) + (4n^2 - 4n) - 2 = 8n^2 - 4n - 2$.

ii) For $2 \leq i \leq 4n-3$, then $\lambda_i(L(G)) = (2n^2 - 3) + (4n^2 - 4n) - 2 = 6n^2 - 7n - 2$.

iii) For $4n - 2 \leq i \leq 2n^2 - 1$, then

$$\lambda_i(L(G)) = (n^2 - n - 1) + (3n^2 - 2n - 1) - 2 = 4n^2 - 3n - 4.$$

iv) For $2n^2 \leq i \leq 4n^2 - 4n + 1$, then

$$\lambda_i(L(G)) = (n^2 - 2n) + (4n^2 - 4n) - 2 = 5n^2 - 6n - 2.$$

v) For $4n^2 - 4n + 2 \leq i \leq n^4 - 4n^3 + 10n^2 - 8n + 2$, then

$$\lambda_i(L(G)) = 0 + (4n^2 - 4n) - 2 = 4n^2 - 4n - 2.$$

vi) For $n^4 - 4n^3 + 10n^2 - 8n + 3 \leq i \leq n^4 - 2n^2 + 4n - 2$, then

$$\lambda_i(L(G)) = -n + (4n^2 - 4n) - 2 = 4n^2 - 5n - 2.$$

vii) For $n^4 - 2n^2 + 4n - 1 \leq i \leq n^4$, then

$$\lambda_i(L(G)) = -2n + (4n^2 - 4n) - 2 = 4n^2 - 6n - 2.$$

viii) For $n^4 + 1 \leq i \leq (n^4(4n^2 - 4n - 2))/2$, then $\lambda_i(L(G)) = -2$.

Therefore, according to the above eigenvalues and their multiplicities, the spectrum of graph $L(SudP(n))$ is obtained. □

Theorem 10. *The edge energy of the positional Sudoku graph $SudP(n)$, for $n \geq 2$ is as follows*

$$EE(SudP(n)) = 4n^2(2n^6 - 4n^5 + 4n^4 - 2n^3 - 3n^2 + 4n - 2).$$

Proof. Let $G = SudP(n)$ be the positional Sudoku graph for $n \geq 2$ of the order n^4 and size $m = (n^4(4n^2 - 3n - 2))/2$. Similar to the proof Theorem 3, we have

$$\begin{aligned}
EE(G) = E(L(G)) &= \sum_{i=1}^{m} |\lambda_i(L(G))| \\
&= |8n^2 - 8n - 2| + 4(n-1)|6n^2 - 7n - 2| + 4(n-1)^2|5n^2 - 6n - 2| \\
&+ (n-1)^4|4n^2 - 4n - 2| + 4(n-1)^3|4n^2 - 5n - 2| \\
&+ 2(n-1)^2|4n^2 - 6n - 2| + \frac{n^4(4n^2 - 4n - 2)}{2}| - 2|
\end{aligned}$$

By simplifying the above relation and since $n \geq 2$, the result is completed. $\square$

Similar to the results for the complement of the Sudoku graph in Section 2, we are interested in determining the graph energy and the edge energy of the complement of the Cayley graph $SudP(n)$.

Theorem 11. *Let $SudP(n)$ be the positional Sudoku graph of the order n^4 where $n \geq 2$. Then, the spectrum of the complement of graph $SudP(n)$ is as follows*

$$\begin{pmatrix} n^4 - 4n^2 + 4n - 1 & -2n^2 + 3n - 1 & -n^2 + 2n - 1 & -1 & n - 1 & 2n - 1 \\ 1 & 4(n-1) & 4(n-1)^2 & (n-1)^4 & 4(n-1)^3 & 2(n-1)^2 \end{pmatrix}.$$

Proof. Let G be the positional Sudoku graph of the order n^4 for $n \geq 2$. Since G is $(4n^2 - 4n)$-regular graph, using Lemmas 3 and 4 we get

$$\begin{aligned}
P_{\overline{G}}(\lambda) &= (-1)^{n^4}\Big(\frac{\lambda - n^4 + 4n^2 - 4n + 1}{\lambda + 4n^2 - 4n + 1}\Big)P_G(-\lambda - 1) \\
&= (-1)^{n^4}\Big(\frac{\lambda - n^4 + 4n^2 - 4n + 1}{\lambda + 4n^2 - 4n + 1}\Big)\big(-\lambda - 4n^2 + 4n - 1\big) \\
&\quad \big(-\lambda - 2n^2 + 3n - 1\big)^{4(n-1)}\big(-\lambda - n^2 + 2n - 1\big)^{4(n-1)^2} \\
&\quad \big(-\lambda - 1\big)^{(n-1)^4}\big(-\lambda + n - 1\big)^{4(n-1)^3}\big(-\lambda + 2n - 1\big)^{2(n-1)^2} \\
&= (-1)\big(\lambda - n^4 + 4n^2 - 4n + 1\big)\big(\lambda + 2n^2 - 3n + 1\big)^{4(n-1)}\big(\lambda + n^2 - 2n + 1\big)^{4(n-1)^2} \\
&\quad \big(\lambda + 1\big)^{(n-1)^4}\big(\lambda - n + 1\big)^{4(n-1)^3}\big(\lambda - 2n + 1\big)^{2(n-1)^2}.
\end{aligned}$$

By computing the roots of the characteristic polynomial $P_{\overline{G}}(\lambda)$ where G is the positional Sudoku graph, the eigenvalues of the complement of the Sudoku graph $Sud(n)$ for $n \geq 2$ are obtained. $\square$

Theorem 12. *The energy of the complement of the positional Sudoku graph $SudP(n)$ for $n \geq 2$, is equal to $2(n-1)^2(5n^2 - 2n + 1)$.*

Proof. Assume that $G = SudP(n)$ is the positional Sudoku graph of the order n^4. Similar to the proof of Theorem 5 and by applying Theorem 11, we have

$$\begin{aligned}
E(\overline{G}) &= \sum_{i=1}^{n^4} |\lambda_i(\overline{G})| \\
&= (n^4 - 4n^2 + 4n - 1) + 4(n-1)(2n^2 - 3n + 1) + 4(n-1)^2(n^2 - 2n + 1) \\
&+ (n-1)^4(-1) + 4(n-1)^3(n-1) + 2(n-1)^2(2n-1) \\
&= 10n^4 - 24n^3 + 20n^2 - 8n + 2.
\end{aligned}$$

By simplifying the above relation, the result holds. □

The edge energy of the complement of graph $SudP(n)$, where $n \geq 2$, is computed in the following theorem.

Theorem 13. *The edge energy of the complement of the positional Sudoku graph $SudP(n)$ of the order n^4, where $n \geq 2$ is equal to $2n^4(n^4 - 4n^2 + 4n - 3)$.*

Proof. Suppose that G is the positional Sudoku graph $SudP(n)$ of the order n^4 for $n \geq 2$. Similar to the proof of Theorem 6, it is sufficient to obtain the eigenvalues of the line graph of $\overline{G}$. Since the complement of the positional Sudoku graph is $(n^4 - 4n^2 + 4n - 1)$-regular graph, thus using Lemma 2 and Theorem 9, we get the eigenvalues of the graph $\overline{G}$.

i) $\lambda_1\big(L(\overline{G})\big) = (n^4 - 4n^2 + 4n - 1) + (n^4 - 4n^2 + 4n - 1) - 2 = 2n^4 - 8n^2 + 8n - 4.$

ii) For $2 \leq i \leq 4n - 3$, then

$$\lambda_i\big(L(\overline{G})\big) = (-n^2 + 2n - 1) + (n^4 - 4n^2 + 4n - 1) - 2 = n^4 - 5n^2 + 6n - 4.$$

iii) For $4n^2 - 4n + 2 \leq i \leq n^24 - 4n^3 + 10n^2 - 8n + 2$, then

$$\lambda_i\big(L(\overline{G})\big) = (-1) + (n^4 - 4n^2 + 4n - 1) - 2 = n^4 - 4n^2 + 4n - 4.$$

iv) For $n^4 - 4n^3 + 10n^2 - 8n + 3 \leq i \leq n^4 - 2n^2 + 4n - 2$, then

$$\lambda_i\big(L(\overline{G})\big) = (n-1) + (n^4 - 4n^2 + 4n - 1) - 2 == n^4 - 4n^2 + 5n - 4.$$

v) For $3n^2 - 2n + 1 \leq i \leq n^4 - 2n^3 + 4n^2 - 2n$, then

$$\lambda_i\big(L(\overline{G})\big) = 0 + (n^4 - 3n^2 + 2n) - 2 = n^4 - 3n^2 + 2n - 2.$$

vi) For $n^4 - 2n^2 + 4n - 1 \leq i \leq n^4$, then

$$\lambda_i\big(L(\overline{G})\big) = (2n-1) + (n^4 - 4n^2 + 4n - 1) - 2 = n^4 - 4n^2 + 6n - 4.$$

vii) For $n^4 + 1 \leq i \leq m = \frac{n^4(n^4 - 4n^2 + 4n - 1)}{2}$, then $\lambda_i\big(L(\overline{G})\big) = -2$.

Therefore, the edge energy of the complement of the positional Sudoku graph $SudP(n)$ for $n \geq 2$ is as follows

$$
\begin{aligned}
EE(\overline{G}) &= \sum_{i=1}^{m} |\lambda_i(L(\overline{G}))| \\
&= (2n^4 - 8n^2 + 8n - 4) + 4(n-1)(n^4 - 6n^2 + 7n - 4) \\
&+ 4(n-1)^2(n^4 - 5n^2 + 6n - 4) + (n-1)^4(n^4 - 4n^2 + 4n - 4) \\
&+ 4(n-1)^3(n^4 - 4n^2 + 5n - 4) + 2(n-1)^2(n^4 - 4n^2 + 6n - 4) \\
&+ \frac{n^4(n^4 - 4n^2 + 4n - 3)}{2}| - 2|
\end{aligned}
$$

By simplifying the above relation, the result is completed. □

Conflict of interest. The authors declare that they have no conflict of interest.

Data Availability. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

References

[1] L.W. Beineke, R.J. Wilson, and P.J. Cameron, *Topics in Algebraic Graph Theory*, Cambridge University Press, 2004.

[2] Ş.B. Bozkurt and D. Bozkurt, *On incidence energy*, MATCH Commun. Math. Comput. Chem. **71** (2014), no. 1, 215–225.

[3] A. Cayley, *Desiderata and suggestions: No. 2. The Theory of groups: graphical representation*, Amer. J. Math. **1** (1878), no. 2, 174–176 https://doi.org/10.2307/2369306.

[4] S. Chokani, F. Movahedi, and S.M. Taheri, *Graph energies of zero-divisor graphs of finite commutative rings*, Int. J. Nonlinear Anal. Appl. **14** (2023), no. 7, 207–216 https://doi.org/10.22075/ijnaa.2022.7136.

[5] ______, *The minimum edge dominating energy of the Cayley graphs on some symmetric groups*, Algebr. Struct. their Appl. **10** (2023), no. 2, 15–30 https://doi.org/10.22034/as.2023.3001.

[6] D. Cvetković, M. Doob, and H. Sachs, *Spectra of Graphs-Theory and Application*, Academic Press, New York, 1980.

[7] D. Cvetković, P. Rowlinson, and S. Simić, *An Introduction to the Theory of Graph Spectra*, Cambridge University Press, New York, 2010.

[8] K.C. Das, S.A. Mojallal, and I. Gutman, *On energy of line graphs*, Linear Algebra Appl. **499** (2016), 79–89 https://doi.org/10.1016/j.laa.2016.03.003.

[9] M. Ghorbani, *On the energy and Estrada index of Cayley graphs*, Discrete Math. Algorithms Appl. **7** (2015), no. 1, Atricle ID: 1550005 https://doi.org/10.1142/S1793830915500056.

[10] I. Gutman, *The energy of a graph*, Ber. Math. Statist. Sekt. Forschungsz. Graz. **103** (1978), 1–22.

[11] I. Gutman, M. Robbiano, E.A. Martins, D.M. Cardoso, L. Medina, and O. Rojo, *Energy of line graphs*, Linear Algebra Appl. **433** (2010), no. 7, 1312–1323 https://doi.org/10.1016/j.laa.2010.05.009.

[12] F. Harary, *Graph Theory*, Addison-Wesley, 1969.

[13] A. Ilić, *The energy of unitary Cayley graphs*, Linear Algebra Appl. **431** (2009), no. 10, 1881–1889 https://doi.org/10.1016/j.laa.2009.06.025.

[14] W. Klotz and T. Sander, *Integral Cayley graphs over abelian groups*, Electron. J. Combin. **17** (2010), no. 1, ID: #R81 https://doi.org/10.37236/353.

[15] N. Palanivel and A.V. Chithra, *Energy and Laplacian energy of unitary addition Cayley graphs*, Filomat **33** (2019), no. 11, 3599–3613.

[16] B.R. Rakshith, *On Zagreb energy and edge-Zagreb energy*, Commun. Comb. Optim. **6** (2021), no. 1, 155–169 https://doi.org/10.22049/cco.2020.26901.1160.

[17] H Shooshtary and J Rodriguez, *New bounds on the energy of a graph*, Commun. Comb. Optim. **7** (2022), no. 1, 81–90 https://doi.org/10.22049/cco.2021.26999.1179.