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Zero forcing number for Cartesian product of some graphs

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Abstract: The zero forcing number of a graph G, denoted $Z(G)$, is a graph parameter which is based on a color change rule that describes how to color the vertices. Zero forcing is useful in several branches of science such as electrical engineering, computational complexity and quantum control. In this paper, we investigate the zero forcing number for Cartesian products of some graphs. The main contribution of this paper is to introduce a new presentation of the Cartesian product of two complete bipartite graphs and to obtain the zero forcing number of these graphs. We also introduce a purely graph theoretical method to prove $Z(K_n \Box K_m) = mn - m - n + 2$.

Keywords: Zero forcing number; Rook's graph; Generalized Rook's graph

AMS Subject classification: 05C62; 05C75; 05D99

1. Introduction

Let $G = (V_G, E_G)$ be a graph with vertex set V_G and edge set E_G. The order of G, denoted $|G|$, is the number of vertices of G. Throughout this paper, all graphs are simple (no loops, no multiple edges), undirected, and have finite nonempty vertex sets. The Cartesian product of two graphs G and H, denoted $G \Box H$, is the graph with the vertex set $V_G \times V_H$ and two vertices (g_1, h_1) and (g_2, h_2) are adjacent if and only if either $g_1 = g_2$ and $h_1h_2 \in E_H$, or $h_1 = h_2$ and $g_1g_2 \in E_G$. For other graph theory terminology, we refer to [10].
The zero forcing number of a graph G, denoted $Z(G)$, was introduced in [5] to bound the minimum rank of graphs. Although it is defined as a useful tool to compute the minimum rank of graphs, it becomes an interesting graph parameter that is studied for its own sake (see for example [6, 7, 9]). Zero forcing is also known as graph infection or graph propagation. Actually, it has been described differently and used in many branches of science and mathematics (see for example [1–3, 11]).

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Definition 1. [5]

- Color-change rule:
  Let G be a graph with each vertex colored either white or black. If u is a black vertex of G, and v is the only white neighbor of u, then change the color of v to black.
- A zero forcing set for a graph G is a subset of vertices Z such that if initially the vertices in Z are colored black and the remaining vertices are colored white, all the vertices of G will be turned black after finitely many applications of the color-change rule. The zero forcing number $Z(G)$ is the minimum of $|Z|$ over all zero forcing sets $Z \subseteq V(G)$.

We will call the discrete dynamical process of applying the color-change rule to Z and G, the zero forcing process. For any zero forcing set Z of G, Chilakamarri et al. in [4] introduced the iteration index $I_Z(G)$ of G to be the number of (global) applications of the color-change rule required to turn all vertices of G black. Hogben et al. in [6], studied the iteration index, which they call the propagation time, and characterized graphs that have extreme propagation times. Propagation time was also described implicitly in [3] and explicitly in [8].

Definition 2. [6] Let $G = (V, E)$ be a graph and B a zero forcing set of G. Define $B^{(0)} = B$, and for $t \geq 0$, $B^{(t+1)}$ is the set of vertices w for which there exists a vertex $b \in \bigcup_{s=0}^{t} B^{(s)}$ such that w is the only neighbor of b not in $\bigcup_{s=0}^{t} B^{(s)}$. The propagation time of B in G, denoted $pt(G, B)$, is the smallest integer t_0 such that $V = \bigcup_{t=0}^{t_0} B^{(t)}$.

Definition 3. [6] The minimum propagation time of G is

$$pt(G) = \min\{pt(G, B) \mid B \text{ is a minimum zero forcing set of } G\}.$$

In this paper, we investigate the zero forcing number of some Cartesian products of graphs and also we try to determine some graph parameters associated with the zero forcing number. In Section 2, we obtain the zero forcing number of the Cartesian product of two complete graphs which is called Rook's graph. In Section 3, we introduce a generalization of Rook's graph, and obtain its zero forcing number. Finally, we pose a conjecture about the value of the propagation time of the Cartesian product of two complete bipartite graphs.

2. Zero forcing number of Rook's graph

A graph can be formed from an $m \times n$ chessboard if taking the squares as the vertices and two vertices (exp. v_i and v_j) are adjacent if a chess piece situated on one square (v_i) can be transferred to the other square (v_j) using the chess rules. Rook's graph is an example of this kind of graph. The Rook's graph R_{mn} has mn squares as vertices and two vertices are adjacent if they are on the same row or column. In other words, this graph describes all possible movements of a rook on an $m \times n$ chessboard. We

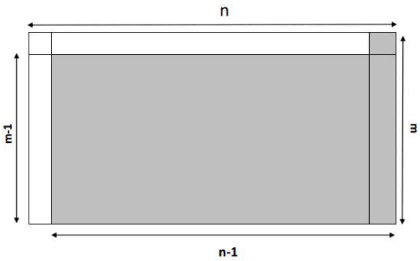

Figure 1. The corresponding $m \times n$ chessboard of graph $K_n \square K_m$ with the black squares represent the zero forcing set B.

can see that the Rook's graph is a $K_n \square K_m$ graph which is the Cartesian product of two complete graphs. The square (i, j) indicates the vertex located on the ith copy of K_n and the jth copy of K_m.
Throughout this section, we use this representation of $K_n \square K_m$. On an $m \times n$ chessboard, as a Rook's graph, where some squares are colored black and the others are colored white, the color change rule of the zero forcing process is: if a square $v = (i, j)$ is black and $w = (i', j)$ or $w = (i, j')$ is the only white square in the row i and the column j, then the color of w has to be changed to black and we say v forces w and write $v \longrightarrow w$.
Now, we are interested in computing the zero forcing number of $K_n \square K_m$ by this representation. Although it is computed in [5] by an algebraic method, here we present a purely graph theoretical method to determine it.

Lemma 1. *For every two integers $m, n \geq 2$,*

$$Z(K_n \square K_m) \leq (m-1)(n-1) + 1.$$

Proof. We assume that the $m \times n$ chessboard represents the graph $K_n \square K_m$. Let $B \subseteq V(K_n \square K_m)$ be the zero forcing set of $K_n \square K_m$ containing all the squares except the squares of the first column and the squares $(1, 2), (1, 3), \ldots, (1, n-1)$, as it is shown in Figure 1. If we apply the color change rule with B, the zero forcing process will finish in two steps. At the first step the vertices $(2, n), (3, n), \ldots, (n, n)$ force the vertices $(2, 1), (3, 1), \ldots, (n, 1)$, respectively, and at the second step the vertices $(n, 1), (n, 2), \ldots, (n, n-1)$ can force the vertices $(1, 1), (1, 2), \ldots, (1, n-1)$, respectively. Therefore $Z(K_n \square K_m) \leq |B| = (m-1)(n-1) + 1$.

□

For a graph G, define $mz(G) = |G| - Z(G)$ [5]. In fact, $mz(G)$ is the number of white vertices in a black-white coloring of G where the black vertices form a minimum zero forcing set. In the following lemma, we give an upper bound for $mz(K_n \square K_m)$.

Lemma 2. *For every two integers $m, n \geq 2$,*

$$mz(K_n \Box K_m) \leq m + n - 2.$$

Proof. We use contradiction and assume that $mz(K_n \Box K_m) = m + n - 1$. Consider a black-white coloring of the $m \times n$ chessboard corresponding to the graph $K_n \Box K_m$ with the black squares representing a minimum zero forcing set of $K_n \Box K_m$. Since the black squares form a zero forcing set, there exists a completely black line (a row or a column with no white square), otherwise each black square has at least two white neighbors which is a contradiction. So the $m + n - 1$ white squares are in a smaller chessboard which the sum of its dimensions is $m + n - 1$ (an $m \times (n - 1)$ or an $(m - 1) \times n$ chessboard). To start the zero forcing process, there must be a line with only one white square. Thus the other $m + n - 2$ white squares are in a smaller chessboard which the sum of its dimensions is $m+n-2$. To continue the zero forcing process, there must be a line with only one white square in this chessboard. So the other $m+n-3$ white squares are in a smaller chessboard which the sum of its dimensions is $m + n - 3$. By continuing this procedure, we reach a chessboard whose sum of its dimensions is 4 and has four white squares. A 3×1 or 1×3 chessboard with four squares is impossible so we end up with a 2×2 chessboard with four white squares. It means the initial chessboard has two rows i, i' and two columns j, j' that have white squares in their crossing. Now, the color change rule cannot change the color of these four squares, which is a contradiction. So $mz(K_n \Box K_m) < m + n - 1$, therefore $mz(K_n \Box K_m) \leq m + n - 2$. □

Lemma 2 gives us a lower bound for $Z(K_n \Box K_m)$ and we can prove that the equality holds.

Theorem 1. *For every two integers $m, n \geq 2$,*

$$Z(K_n \Box K_m) = (m - 1)(n - 1) + 1$$

and also $pt(K_n \Box K_m) = 2$.

Proof. By Lemma 2, $mn - Z(K_n \Box K_m) \leq m + n - 2$ and thus the lower bound follows. The upper bound comes from Lemma 1. So

$$Z(K_n \Box K_m) = (m - 1)(n - 1) + 1.$$

To prove the second part, without loss of generality we assume that $m \leq n$. As it is shown in Lemma 1, there is a zero forcing set B of minimum size for which $pt(K_n \Box K_m, B) = 2$. So $pt(K_n \Box K_m) \leq pt(K_n \Box K_m, B) = 2$ and the upper bound follows. We now prove that $pt(K_n \Box K_m) > 1$. We use contradiction and assume that $pt(K_n \Box K_m) = 1$. So there exists a minimum zero forcing set B with $pt(K_n \Box K_m, B) =$

1. In other words, the vertices of B will force all white vertices in one step. Hence, no vertex that must perform a force, has more than one white neighbor. Consider the $m \times n$ chessboard of graph $K_n \Box K_m$ when the squares corresponding to the vertices of B are black. In this case, the most white squares can only exist in one line. So there exist at most n white squares and we have $|B| \geq mn - n$, which contradicts our assumption that we took B as a minimum zero forcing set. Hence, for every minimum zero forcing set B of $K_n \Box K_m$, $pt(K_n \Box K_m, B) > 1$ and we have $pt(K_n \Box K_m) = 2$. $\Box$

3. Zero forcing number of Generalized Rook's graph

As already mentioned, the Cartesian product of two complete graphs ($K_n \Box K_m$) is a Rook's graph corresponding to an $m \times n$ chessboard. In this section we introduce a similar representation for the Cartesian product of two complete bipartite graphs, saying $K_{m,n} \Box K_{m',n'}$. Here we introduce Generalized Rook's graph as a generalization of Rook's graph, corresponding to this graph and finally obtain the zero forcing number of it during this section. First, we introduce a different checker pattern for this graph. Consider the graph $K_{m,n} \Box K_{m',n'}$. Throughout this section, without loss of generality we assume that $m \geq n$, $m' \geq n'$, and $m \geq m'$. Form an $(m'+n') \times (m+n)$ generalized chessboard from four smaller chessboards denoted C_1 to C_4. We denote the $m' \times m$ chessboard in the upper-left corner by C_1, the $m' \times n$ chessboard in the upper-right corner by C_2, the $n' \times n$ chessboard in the lower-right corner by C_3 and the $n' \times m$ chessboard in the lower-left corner by C_4. So, C_2 and C_4 indicate even chessboards and C_1 and C_3 indicate odd chessboards. The square (i, j) indicates the vertex that is in the ith copy of $K_{m,n}$ and in the jth copy of $K_{m',n'}$ at the same time. For simplicity, we index the rows and columns of the chessboards as shown in Figure 2. So the vertex set of $G = K_{m,n} \Box K_{m',n'}$ can be represented as $V = V_1 \cup V_2 \cup V_3 \cup V_4$, where

$$V_1 = \{(1, i, j) : 1 \leq i \leq m', 1 \leq j \leq m\},$$
$$V_2 = \{(2, i, j) : 1 \leq i \leq m', 1 \leq j \leq n\},$$
$$V_3 = \{(3, i, j) : 1 \leq i \leq n', 1 \leq j \leq n\},$$
$$V_4 = \{(4, i, j) : 1 \leq i \leq n', 1 \leq j \leq m\}.$$

The neighbors of the vertex $(1, i, j)$ when $1 \leq i \leq m'$ and $1 \leq j \leq m$ are the vertices $(2, i, j')$ with $1 \leq j' \leq n$ and the vertices $(4, i', j)$ with $1 \leq i' \leq n'$ and when $1 \leq i \leq n'$ and $1 \leq j \leq n$, the vertices $(4, i, j')$ with $1 \leq j' \leq m$ and the vertices $(2, i', j)$ with $1 \leq i' \leq m'$ are the neighbors of the vertex $(3, i, j)$. Thus the neighbors of each vertex in an odd (even) chessboard are in the even (odd) chessboards. So it is obvious that each chessboard mentioned above, corresponds to an independent set in the graph $K_{m,n} \Box K_{m',n'}$.

In this section, we use this representation of $K_{m,n} \Box K_{m',n'}$, and determine the zero forcing number of this graph. Consider the corresponding $(m'+n') \times (m+n)$ generalized chessboard and every C_i for $i \in \{1, 2, 3, 4\}$. Let all squares be initially colored

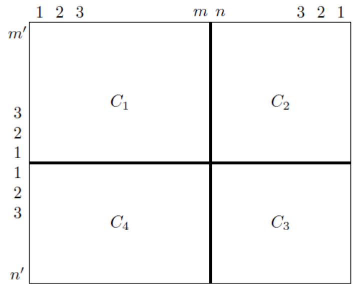

Figure 2. The location of chessboards in the generalized chessboard corresponding to the Cartesian product of two complete bipartite graphs.

black or white. So a black square (k, i, j) of C_k (when k is odd) can force a white square if it is the only white square in the ith row of $C_{(k+1)}$ and jth column of $C_{(k+3)}($ mod 4) (consider $\mathbb{Z}_4 = \{1, 2, 3, 4\}$) and a black square (k, i, j) of C_k (when k is even) can force a white square if it is the only white square in the jth column of $C_{(k+1)}$ (mod 4) and ith row of $C_{(k+3)}($ mod 4). Now, we are ready to give an upper bound for the zero forcing number of the Generalized Rook's graph.

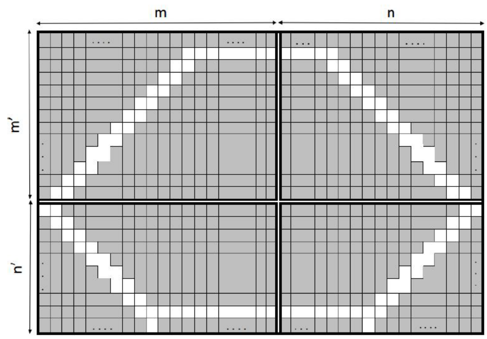

Figure 3. The corresponding generalized chessboard of graph $K_{m,n} \square K_{m',n'}$ (where $m \geq n \geq m' \geq n'$) in which the black squares represent the zero forcing set B.

Lemma 3. *Let m, n, m', and n' be integers greater than 2. Then*

$$Z(K_{m,n} \square K_{m',n'}) \leq (m+n)(m'+n') - 2(m+n+m'+n') + 8.$$

Proof. Consider the generalized chessboard of the graph $K_{m,n} \square K_{m',n'}$. We use a special stairway pattern to color this generalized chessboard. We only show this coloring for the case $m \geq n \geq m' \geq n'$ in Figure 3, as it can be done in a similar fashion for the other cases. In fact, the zero forcing set shown in Figure 3 is $B =$

$\{V_1 \setminus X_1\} \cup \{V_2 \setminus X_2\} \cup \{V_3 \setminus X_3\} \cup \{V_4 \setminus X_4\}$, where

$$X_1 = \{(1,i,i+1),(1,i,i+2) : 1 \le i \le m'-2\} \cup \{(1,m'-1,j) : m' \le j \le m\},$$
$$X_2 = \{(2,i,i+1),(2,i,i+2) : 1 \le i \le m'-2\} \cup \{(2,m'-1,j) : m' \le j \le n\},$$
$$X_3 = \{(3,i,i),(3,i,i+1) : 1 \le i \le n'-2\} \cup \{(3,n'-1,j) : n'-1 \le j \le n\} \cup \{(3,n',n')\},$$
$$X_4 = \{(4,i,i),(4,i,i+1) : 1 \le i \le n'-2\} \cup \{(4,n'-1,j) : n'-1 \le j \le m\} \cup \{(4,n',n')\}.$$

If we apply the color change rule with the set B, the entire graph G will be black after m' steps. The cardinality of B is $(m+n)(m'+n') - 2(m+n+m'+n') + 8$. So the bound follows. □

Now, we present some lemmas in order to prove the equality of Lemma 3.

Lemma 4. *Consider the generalized chessboard of the Cartesian product of two complete bipartite graphs. To start the zero forcing process in every chessboard C_i ($i \in \{1,2,3,4\}$), we need a line (row or column) with only one white square in that chessboard.*

Proof. For each i, all the squares in a line (row or column) of C_i are common neighbors of some squares not in C_i. Assume that for $k \in \{1,2,3,4\}$, every line in C_k having some white squares, has at least two white squares. So even if all the squares in other chessboards are black, no more color change can be done in C_k. So there must be a line with exactly one white square to start the zero forcing process. □

Lemma 5. *Suppose that G is the Cartesian product of two complete bipartite graphs and Z is a zero forcing set of G. Consider the checker pattern of G and color the squares corresponding to the vertices in Z black. Then there exists at least one completely black line in one of the even chessboards and at least one completely black line in one of the odd chessboards.*

Proof. We prove the lemma for the even chessboards. By a similar fashion, this proof is valid for the odd chessboards as well.
Suppose that there dose not exist any completely black line in even chessboards. In this case, each black square in any of the odd chessboards has at least two white neighbors; at least one in each C_2 and C_4. Hence, the zero forcing process cannot start in even chessboards and this contradicts the hypothesis. □

Lemma 6. *Consider the generalized chessboard which represents the graph $G = K_{m,n} \square K_{m',n'}$. In every minimum zero forcing set of G, there exist $t+2$ lines in odd chessboards (also in even chessboards) having exactly t white squares together, for some integers $t \ge 1$.*

Proof. By considering the odd chessboards, according to the previous lemma, there exists at least one completely black line in these chessboards. If there exists another such black line, then these two black lines and one of the lines mentioned in Lemma

4 are the desired lines. So, suppose that there exist no other black lines. without loss of generality, assume that the only black line is a column of C_1. In this case, only the squares of C_4 can change the color of the white squares in odd chessboards. Since the black squares form a minimum zero forcing set of G, we also need color change in odd chessboards by the squares of C_2. On the other hand, none of the squares of C_2 can force any square until there exists a black column in C_3 or a black row in C_1. Without loss of generality, suppose first when the rows of C_3 become black by some squares of C_4, a column in C_3 (say the ith column) also becomes black. This means, if the ith column of C_3 have t white squares, the corresponding rows of these white squares in C_3 have no other white squares. So these t rows of C_3, the ith column of C_3 and the black column of C_1 are the desired lines. By a similar argument for C_2 and C_4, the theorem is also true for even chessboards. □

We are now in a position to obtain an upper bound for $mz(K_{m,n}\Box K_{m',n'})$ and conclude the equality in Lemma 3.

Theorem 2. *Let m, n, m', and n' be integers greater than 2. Then*

$$Z(K_{m,n}\Box K_{m',n'}) = (m+n)(m'+n') - 2(m+n+m'+n') + 8.$$

Proof. We proved the upper bound in Lemma 3. Now, we only need to prove the lower bound. First, we give an upper bound for $mz(G)$ where $G = K_{m,n}\Box K_{m',n'}$. The checker pattern of the graph $K_{m,n}\Box K_{m',n'}$ has four distinct smaller chessboards C_i $(i = 1, 2, 3, 4)$ and a zero forcing process can continue in each C_i at the same time. The sum of dimensions of the even (odd) chessboards is $m+n+m'+n'$. Let $B \subseteq V_G$ be a minimum zero forcing set of G. Consider a black-white coloring of the generalized chessboard of G so that the squares corresponding to the vertices of B are black and the other squares are white. Hence, there exist $Z(G)$ black squares and $mz(G)$ white squares in this coloring. We claim that $mz(G) \leq 2(m+n+m'+n'-4)$. To prove it, we use contradiction and assume that $mz(G) = 2(m+n+m'+n'-4)+1$. So either the odd chessboards or the even chessboards have at least $m+n+m'+n'-3$ white squares. Without loss of generality, assume that the even chessboards have $m+n+m'+n'-3$ white squares. By Lemma 6, there exist $t+2$ lines with exactly t white squares in these chessboards. So there exist $m+n+m'+n'-t-3$ white squares in two chessboards that the sum of their dimensions is $m+n+m'+n'-t-2$. Now, using an argument similar to the proof of Lemma 2, we have two cases:
1. There exist at least $k+1$ white squares in two chessboards which the sum of their dimensions is $k+2$ and one of them is a 1×1 chessboard. Thus we have at least k white squares in a chessboard that the sum of its dimensions is k. We may continue this argument for this chessboard and reach the contradiction as shown in Lemma 2.
2. There exist at least five white squares in two chessboards that the sum of their dimensions is 6 and none of them is a 1×1 chessboard. So each of them is 2×1 or 1×2. Thus these chessboards have totally four squares and cannot have five white squares.

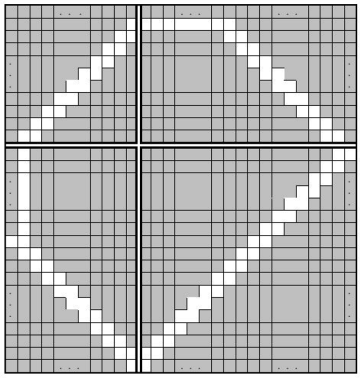

Figure 4. The coloring pattern for $K_{m,n} \square K_{m,n}$ $(m \leq n)$ in which the black squares represent a zero forcing set B for the graph.

The above argument leads to

$$mz(K_{m,n} \square K_{m',n'}) \leq 2(m + n + m' + n' - 4).$$

Therefore

$$\begin{aligned} Z(K_{m,n} \square K_{m',n'}) &\geq (m + n)(m' + n') - 2(m + n + m' + n' - 4) \\ &= (m + n)(m' + n') - 2(m + n + m' + n') + 8. \end{aligned}$$

Thus the lower bound also follows and equality holds. □

Corollary 1. *For every two integers $m, n \geq 3$,*

$$Z(K_{m,n} \square K_{m,n}) = (m + n)^2 - 4(m + n) + 8.$$

We illustrate the statement of Corollary 1 for the case $m \leq n$ in Figure 4.

Corollary 2. *For every integer $n \geq 3$,*

$$Z(K_{n,n} \square K_{n,n}) = 4n^2 - 8n + 8.$$

Example 1. Figure 5 shows the corresponding generalized chessboard of $K_{4,9} \square K_{4,9}$. The black squares demonstrate the vertices of minimum zero forcing set B, which is obtained by Corollary 1, and the numbers in the white squares indicate the steps that they will become black. Easily can be seen, all squares will be black after 9 steps. In other words, $pt(K_{4,9} \square K_{4,9}, B) = 9$.

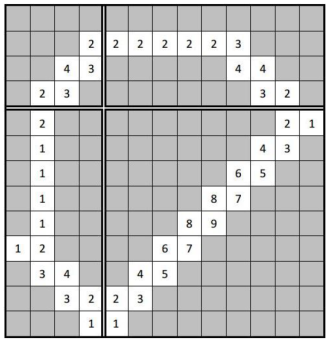

Figure 5. **Corresponding chessboard of** $K_{4,9}\square K_{4,9}$.

Corollary 3. *For every integers* $m, n, m', n' \geq 3$,

$$pt(K_{m,n}\square K_{m',n'}) \leq \min\{\max\{m,n\}, \max\{m',n'\}\}.$$

Proof. As mentioned in the proof of Theorem 2, the propagation time of zero forcing set B, which is introduced in Figure 3, is $m'(= \min\{\max\{m,n\}, \max\{m',n'\}\})$. So

$$pt(K_{m,n}\square K_{m',n'}) \leq pt(K_{m,n}\square K_{m',n'}, B) = \min\{\max\{m,n\}, \max\{m',n'\}\}.$$

□

Example 2. Figure 6 shows a zero forcing set B of size 84 for $K_{4,6}\square K_{5,7}$, which is obtained by Theorem 2. So it can be easily seen that $pt(K_{4,6}\square K_{5,7}, B) = \min\big\{\max\{m,n\}, \max\{m',n'\}\big\} = 6$.

Corollary 4. *The zero forcing number of the Cartesian product of two complete bipartite graphs depends only on the dimensions of its generalized chessboards. In other words, for integers* $m, n, m', n', r, s, r', s' \geq 3$ *which* $m + n = r + s$ *and* $m' + n' = r' + s'$, *we have*

$$Z(K_{m,n}\square K_{m',n'}) = Z(K_{r,s}\square K_{r',s'}).$$

4. Further work

In Section 3, we determined the value of parameter $Z(K_{m,n}\square K_{m',n'})$. For its related parameter, propagation time, we pose the following conjecture.
Conjecture. Let m, n, m', and n' be integers greater than 2. We guess that

$$pt(K_{m,n}\square K_{m',n'}) = \min\big\{\max\{m,n\}, \max\{m',n'\}\big\}.$$

Conflict of Interest

The authors declare no conflict of interest in this paper.

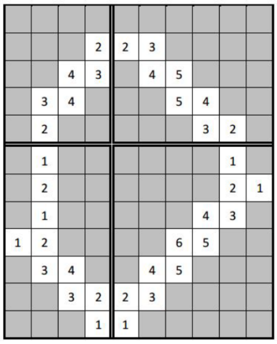

Figure 6. **Corresponding generalized chessboard of $K_{4,6}\square K_{5,7}$.**

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