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Research Article

On the Roman Domination Polynomials

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Abstract: A Roman dominating function (RDF) on a graph G is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the condition that every vertex u with $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of an RDF f is the sum of the weights of the vertices under f. The Roman domination number, $\gamma_R(G)$ of G is the minimum weight of an RDF in G. The Roman domination polynomial of a graph G of order n is the polynomial $RD(G,x) = \sum_{i=\gamma_R(G)}^{2n} d_R(G,i)x^i$, where $d_R(G,i)$ is the number of RDFs of G with weight i. In this paper we prove properties of Roman domination polynomials and determine $RD(G,x)$ in several classes of graphs G by new approaches. We also present bounds on the number of all Roman domination polynomials in a graph.

Keywords: Roman domination polynomial, Roman dominating function, Roman domination number

AMS Subject classification: 05C69

1. Introduction

For notations and definitions not given here we refer to [13]. We consider simple and finite graphs $G = (V, E)$, where $V = V(G)$ is the vertex set and $E = E(G)$ is the edge set. The *order* of G, denoted $|V(G)| = n$, is the number of vertices in G and the size of G, denoted $|E(G)| = m$, is the number of edges in G. For any two vertices $x, y \in V(G)$, x and y are *adjacent* if the edge $xy \in E(G)$. The *degree* of a vertex v, denoted by $\deg(v)$ (or $\deg_G(v)$), is the number of vertices adjacent to v. A vertex of degree zero is called an *isolated vertex*. We denote by Δ and δ, respectively, the *maximum degree* and *minimum degree* among the vertices of G. An *induced subgraph*

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of a graph G is a graph formed from a subset D of vertices of G and all of the edges in G connecting pairs of vertices in that subset, denoted by $\langle D\rangle$. An *independent set* is a set of vertices any two of which are not adjacent. A graph G is *bipartite* if $V(G)$ can be partitioned into two independent sets called *partite sets*. The *join* of two graphs G_1 and G_2, denoted by $G_1 \vee G_2$ is a graph with vertex set $V(G_1)\cup V(G_2)$ and edge set $E(G_1)\cup E(G_2)\cup\{uv|u\in V(G_1)\ and\ v\in V(G_2)\}$.

A *dominating set* of a graph G is a subset D of vertices such that every vertex outside D has a neighbor in D. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality amongst all dominating sets of G. Cockayne et al. [9] introduced the mathematical definition of Roman domination. This concept was subsequently developed very vastly, and to see the latest progress until 2020 we refer to [6–8]. A function $f : V \longrightarrow \{0,1,2\}$ is called a *Roman dominating function* or just an RDF for G if for every vertex $v\in V$ with $f(v)=0$ there exists a vertex $u\in N(v)$ such that $f(u)=2$. The *weight* of an RDF f is the sum $f(V)=\sum_{v\in V} f(v)$. The minimum weight of an RDF on G is called the *Roman domination number* of G and is denoted by $\gamma_R(G)$.

Graph polynomials play an important role in studying the structure of a graph, and there are some polynomials associated to graphs such as Chromatic polynomial, clique polynomial, characteristic polynomial and Tutte polynomial. Alikhani and Peng [4] introduced the concept of domination polynomials in graphs. This concept was further studied in [1, 3] and has been considered for some other types of dominating sets, for example, for total dominating sets ([2]), connected dominating sets ([14]) and hope dominating sets ([15]).

Gangabylaiah et al. [12] introduced the concept of Roman domination polynomial of a graph. For a graph G of order n with Roman domination number $\gamma_R(G)$, the Roman domination polynomial of a graph G, denoted $RD(G,x)$, is defined as follows

$$RD(G,x)=\sum_{i=\gamma_R(G)}^{2n} d_R(G,i)x^i,$$

where, $d_R(G,i)$ is the number of all Roman dominating functions on the graph G with weight i. They presented several basic properties and exact values of the Roman domination polynomial of a graph. This concept was further studied by Deepak et al. [10, 11].

In this paper we prove some further properties of Roman domination polynomial in graphs. We prove some previous results given in [11, 12] by new and easier approach. We also present bounds for the number of all RDFs of graph G.

We recall that the number of solutions of the equation $x_1+x_2+\cdots+x_n=r$, $x_i\in\mathbb{Z}^+$, is

$$\binom{r+n-1}{r}=\binom{r+n-1}{n-1}$$

(see e.g. [5]), and thus we have the following proposition:

Proposition 1. *The number of integral solutions of $x_1+x_2+...+x_n=r$, $a\le x_i\le b$, is*

$$\binom{r-na+n-1}{n-1}+\sum_{k=1}^{n}(-1)^k\binom{n}{k}\binom{r-na-k(b-a+1)+n-1}{n-1}.$$

We also make use of the following.

Proposition 2 ([9]). *For a path P_n, $\gamma_R(P_n)=\lceil\frac{2n}{3}\rceil$.*

2. Roman domination polynomial in join of graphs

Roman domination polynomial in join of two graphs was studied in [12]. In this section, we determine the Roman domination polynomial in join of two graphs by a new approach and then using it we determine the Roman domination polynomial in the complete and complete bipartite graphs. For this purpose, we first introduce some notations. For a graph G of order n, let:

- $D_R(G,k)$ stands for the set of all RDFs on the graph G with weight k, and let $d_R(G,k)=|D_R(G,k)|$.
- $D_{nR}(G,k)$ stands for the set of all functions $f:V(G)\to\{0,1,2\}$ on the graph G with weight k such that f is not an RDF, and let $d_{nR}(G,k)=|D_{nR}(G,k)|$.
- $D(G,k)$ stands for the set of all functions $f:V(G)\to\{0,1,2\}$ on the graph G with weight k, and let $d(G,k)=|D(G,k)|$.
- $P(G,x)=\sum_{i=0}^{2|V(G)|}d(G,i)x^i$.

Clearly, $d(G,k)=d_R(G,k)+d_{nR}(G,k)$. Furthermore, the following is easily verified.

Observation 1. If G_1 and G_2 are two graphs of order n_1 and n_2, respectively, then

$$P(G_1\vee G_2,x)=P(G_1,x)P(G_2,x).$$

We now determine the Roman domination polynomial in join of two graphs.

Theorem 2. *If G_1 and G_2 are two connected graphs of order n_1 and n_2, respectively, then*

$$\begin{aligned}RD(G_1\vee G_2,x) &= \sum_{p=1}^{n_1}\sum_{r=0}^{n_1-p}\sum_{q=1}^{n_2}\sum_{s=0}^{n_2-q}\binom{n_1}{p}\binom{n_1-p}{r}\binom{n_2}{q}\binom{n_2-q}{s}x^{2p+r+2q+s}\\ &+ RD(G_1,x)\sum_{i=0}^{n_2}\binom{n_2}{i}x^i-x^{n_1}\sum_{i=0}^{n_2-1}\binom{n_2}{i}x^i\\ &+ RD(G_2,x)\sum_{i=0}^{n_1}\binom{n_1}{i}x^i-x^{n_2}\sum_{i=0}^{n_1-1}\binom{n_1}{i}x^i-x^{n_1+n_2}.\end{aligned}$$

Proof. For an RDF f in a graph G, we denote by V_i the set of all vertices of G with label i under f. Thus an RDF f can be represented by a triplet $(V_0,\ V_1,\ V_2)$, and we use the notation $f = (V_0,\ V_1,\ V_2)$. In order to enumerate the RDFs of the graph $G_1 \vee G_2$, for any RDF $f : V(G_1 \vee G_2) \to \{0, 1, 2\}$ put $p = |\{v : v \in V(G_1),\ f(v) = 2\}|$ and $q = |\{v : v \in V(G_2),\ f(v) = 2\}|$. Now we enumerate all RDFs on $G_1 \vee G_2$ by dividing them into the following types:

Type-1: RDFs $f = (V_0, V_1, V_2)$, where $V_2 = \emptyset$.

Note that there is only one Type-1 RDF assigning 1 to every vertex of $G_1 \vee G_2$. Thus we obtain the term $x^{n_1+n_2}$ of the Roman domination polynomial.

Type-2: RDFs $f = (V_0, V_1, V_2)$, where $V_2 \cap V(G_1) \neq \emptyset$ and $V_2 \cap V(G_2) = \emptyset$.

Observe that f is Type-2 RDF for $G_1 \vee G_2$ if and only if $f|_{V(G_1)}$ is an RDF for G_1. Note that a typical RDF of G_1 is a Type-2 RDF of $G_1 \vee G_2$ with exception that all the vertices of G_1 assigned value 1 and there is at least one vertex in G_2 with weight 0. Thus, we obtain the following terms of the Roman domination polynomial.

$$RD(G_1, x) \sum_{i=0}^{n_2} \binom{n_2}{i} x^i - x^{n_1} \sum_{i=0}^{n_2} \binom{n_2}{i} x^i = RD(G_1, x) \sum_{i=0}^{n_2} \binom{n_2}{i} x^i - x^{n_1} \sum_{i=0}^{n_2-1} \binom{n_2}{i} x^i - x^{n_1+n_2},$$

where i is the number of vertices of G_2 with weight one.

Type-3: RDFs $f = (V_0, V_1, V_2)$, where $V_2 \cap V(G_1) = \emptyset$ and $V_2 \cap V(G_2) \neq \emptyset$.

Similar to Type-2 RDFs, we find the following terms of the Roman domination polynomial.

$$RD(G_2, x) \sum_{i=0}^{n_1} \binom{n_1}{i} x^i - x^{n_2} \sum_{i=0}^{n_1} \binom{n_1}{i} x^i = RD(G_2, x) \sum_{i=0}^{n_1} \binom{n_1}{i} x^i - x^{n_2} \sum_{i=0}^{n_1-1} \binom{n_1}{i} x^i - x^{n_1+n_2},$$

where i is the number of vertices of G_1 with weight one.

Type-4: RDFs $f = (V_0, V_1, V_2)$, where $V_2 \cap V(G_1) \neq \emptyset$ and $V_2 \cap V(G_2) \neq \emptyset$.

We enumerate the number of Type-4 RDFs on $G_1 \vee G_2$ by summing all such RDFs that assign 2 to p vertices of G_1 and q vertices of G_2, where $1 \leq p \leq n_1$ and $1 \leq q \leq n_2$. For a fixed $p \in \{1, \ldots, n_1\}$ and fixed $q \in \{1, \ldots, n_2\}$, and a fixed Type-4 RDF f on $G_1 \vee G_2$, it may be possible that f assign 1 to some vertices of G_1 or G_2. We enumerate Type-4 RDFs on $G_1 \vee G_2$ assigning 2 to p vertices of G_1 and q vertices of G_2, by summing all such RDFs assigning 1 to r vertices of G_1 and s vertices of G_2, where $0 \leq r \leq n_1 - p$ and $0 \leq s \leq n_2 - q$.

There are $\binom{n_1}{p}\binom{n_1-p}{r}$ functions on G_1 such that p vertices are assigned 2 and r vertices are assigned 1. For each such choice, there are $\binom{n_2}{q}\binom{n_2-q}{s}$ functions on the graph G_2, such that q vertices are assigned 2 and s vertices are assigned 1. Thus we obtain the term

$$\sum_{p=1}^{n_1} \sum_{r=0}^{n_1-p} \sum_{q=1}^{n_2} \sum_{s=0}^{n_2-q} \binom{n_1}{p} \binom{n_1 - p}{r} \binom{n_2}{q} \binom{n_2 - q}{s} x^{2p+r+2q+s}.$$

Therefore

$$\begin{aligned}
RD(G_1 \vee G_2, x) &= x^{n_1+n_2} + RD(G_1,x)\sum_{i=0}^{n_2}\binom{n_2}{i}x^i - x^{n_1}\sum_{i=0}^{n_2-1}\binom{n_2}{i}x^i - x^{n_1+n_2}\\
&+ RD(G_2,x)\sum_{i=0}^{n_1}\binom{n_1}{i}x^i - x^{n_2}\sum_{i=0}^{n_1-1}\binom{n_1}{i}x^i - x^{n_1+n_2}\\
&+ \sum_{p=1}^{n_1}\sum_{r=0}^{n_1-p}\sum_{q=1}^{n_2}\sum_{s=0}^{n_2-q}\binom{n_1}{p}\binom{n_1-p}{r}\binom{n_2}{q}\binom{n_2-q}{s}x^{2p+r+2q+s}\\
&= \sum_{p=1}^{n_1}\sum_{r=0}^{n_1-p}\sum_{q=1}^{n_2}\sum_{s=0}^{n_2-q}\binom{n_1}{p}\binom{n_1-p}{r}\binom{n_2}{q}\binom{n_2-q}{s}x^{2p+r+2q+s}\\
&+ RD(G_1,x)\sum_{i=0}^{n_2}\binom{n_2}{i}x^i - x^{n_1}\sum_{i=0}^{n_2-1}\binom{n_2}{i}x^i\\
&+ RD(G_2,x)\sum_{i=0}^{n_1}\binom{n_1}{i}x^i - x^{n_2}\sum_{i=0}^{n_1-1}\binom{n_1}{i}x^i - x^{n_1+n_2}.
\end{aligned}$$

Thus the proof is complete. □

Since $RD(\overline{K_n}, x) = x^n(1+x)^n$ (see [12]), $K_n = \underbrace{K_1 \vee K_1 \vee \cdots \vee K_1}_{n\ times}$, and $K_{m,n} = \overline{K_m} \vee \overline{K_n}$, we obtain the following.

Corollary 1 ([12]).

$$\begin{aligned}
1.\ RD(K_n,x) &= (1+x+x^2)^n - (1+x)^n + x^n.\\
2.\ RD(K_{m,n},x) &= x^{m+n} + (1+x)^{m+n}(x^m+x^n) - x^m(1+x)^n - x^n(1+x)^m\\
&+ \sum_{p=1}^{m}\sum_{r=0}^{m-p}\sum_{q=1}^{n}\sum_{s=0}^{n-q}\binom{m}{p}\binom{m-p}{r}\binom{n}{q}\binom{n-q}{s}x^{2p+r+2q+s}.
\end{aligned}$$

3. Roman domination polynomial in Paths

The Roman domination polynomial for a path P_n is determined in [11] in which the authors employed the results of Alikhani and Peng [3] on the domination polynomials of graphs. In this section we determine the Roman domination polynomial for a path P_n without the need of domination polynomials. For this purpose, we need some notations. For any vertex v of a graph G, let:

- $d_k^0(G,v) = |\{f : V(G) \to \{0,1,2\} \mid f\ is\ an\ RDF\ with\ f(V) = k, f(v) = 0\}|$.
- $d_k^1(G,v) = |\{f : V(G) \to \{0,1,2\} \mid f\ is\ an\ RDF\ with\ f(V) = k, f(v) = 1\}|$.

- $d_k^2(G,v) = |\{f : V(G) \to \{0,1,2\} \mid f \text{ is an RDF with } f(V) = k, f(v) = 2\}|$.

Then $d_k^0(G,v) + d_k^1(G,v) + d_k^2(G,v)$ is equal to the number of all RDFs with weight k. That is,

$$d_R(G,k) = d_k^0(G,v) + d_k^1(G,v) + d_k i^2(G,v).$$

So the coefficients of x^i in the polynomial $\sum\limits_{i=\gamma_R(G)}^{2n} (d_i^0(G,v)+d_i^1(G,v)+d_i^2(G,v))x^i$, are independent of the choice of the vertex v, and so the Roman domination polynomial of graph G can be written as

$$RD(G,x) = \sum_{i=\gamma_R(G)}^{2n} (d_i^0(G,v) + d_i^1(G,v) + d_i^2(G,v))x^i,$$

where v is a vertex in $V(G)$.

Lemma 1. *Let $G = P_n$, $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(P_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\}$. Then*

$$\begin{aligned} d_i^0(P_n,v_1) &= d_i^2(P_{n-1},v_1), \\ d_i^1(P_n,v_1) &= d_{i-1}^0(P_{n-1},v_1) + d_{i-1}^1(P_{n-1},v_1) + d_{i-1}^2(P_{n-1},v_1), \\ d_i^2(P_n,v_1) &= d_{i-2}^0(P_{n-2},v_1) + d_{i-2}^1(P_{n-2},v_1) + d_{i-2}^2(P_{n-2},v_1) + d_{i-2}^1(P_{n-1},v_1) \\ &+ d_{i-2}^2(P_{n-1},v_1), \end{aligned}$$

where $P_{n-1} = G[\{v_1, v_2, \ldots, v_{n-1}\}]$ and $P_{n-2} = G[\{v_1, v_2, \ldots, v_{n-2}\}]$.

Proof. Let $Q_{n-1} = G[\{v_2, v_3, \ldots, v_n\}]$ and $R_{n-2} = G[\{v_3, v_4, \ldots, v_n\}]$. For any RDF f for P_n, if $f(v_1) = 0$ then $f(v_2) = 2$. So the number of RDFs on P_n with $f(v_1) = 0$ is equal to the number of RDFs on Q_{n-1} with $f(v_2) = 2$. Therefore $d_i^0(P_n,v_1) = d_i^2(Q_{n-1},v_2)$.
Assume that f is an RDF for P_n with weight i and $f(v_1) = 1$. Then $g = f|_{\{v_2,v_3,\ldots,v_n\}}$ is an RDF for Q_{n-1}, $g(V) = i-1$ and $g(v_2) \in \{0,1,2\}$. Conversely, suppose that g is an RDF for Q_{n-1} with weight $i-1$. Then the function $f : V(P_n) \to \{0,1,2\}$ defined by $f(v_1) = 1$ and $f(v_j) = g(v_j)$ for $2 \leq j \leq n$, is an RDF for P_n with weight i. Thus in this case, $d_i^1(P_n,v_1) = d_{i-1}^0(Q_{n-1},v_2) + d_{i-1}^1(Q_{n-1},v_2) + d_{i-1}^2(Q_{n-1},v_2)$.
Assume that f is an RDF for P_n with weight i and $f(v_1) = 2$. Assume that $f(v_2) = 0$. With a similar argument to that used for the calculation of $d_i^1(P_n,v_1)$, we obtain that $d_i^2(P_n,v_1) = d_{i-2}^0(R_{n-2},v_3) + d_{i-2}^1(R_{n-2},v_3) + d_{i-2}^2(R_{n-2},v_3)$. Next assume that $f(v_2) = 1$. It is evident that the restriction of g to $V(P_n) - \{v_1\}$ is an RDF for Q_{n-1}, $g(V) = i-2$ and $g(v_2) = 1$. Conversely, if g is an RDF for Q_{n-1} with weight $i-2$ and $g(v_2) = 1$, then $f(v_1) = 2$ and $f(v_j) = g(v_j)$ for $2 \leq j \leq n$, and so f is an RDF for P_n

with weight i. Thus in this case $d_i^2(P_n, v_1) = d_{i-2}^1(Q_{n-1}, v_2)$. Similarly, if $f(v_2) = 2$, then $d_i^2(P_n, v_1) = d_{i-2}^2(Q_{n-1}, v_2)$. We thus have the following:

$$\begin{aligned} d_i^2(P_n, v_1) &= d_{i-2}^0(R_{n-2}, v_3) + d_{i-2}^1(R_{n-2}, v_3) + d_{i-2}^2(R_{n-2}, v_3) \\ &+ d_{i-2}^1(Q_{n-1}, v_2) + d_{i-2}^2(Q_{n-1}, v_2). \end{aligned}$$

Now, let P_{n-1} and P_{n-2} be obtained by relabeling the vertices of Q_{n-1} and R_{n-2} as $V(P_{n-1}) = \{u_1 = v_2, u_2 = v_3, \ldots, u_{n-1} = v_n | v_i \in V(Q_{n-1}), i = 2, 3, \ldots, n\}$, and $V(P_{n-2}) = \{u_1 = v_3, u_2 = v_4, \ldots, u_{n-2} = v_n | v_i \in V(R_{n-2}), i = 3, 4, \ldots, n\}$. Then $d_i^0(P_n, v_1) = d_i^2(P_{n-1}, u_1)$, $d_i^1(P_n, v_1) = d_{i-1}^0(P_{n-1}, u_1) + d_{i-1}^1(P_{n-1}, u_1) + d_{i-1}^2(P_{n-1}, u_1)$ and $d_i^2(P_n, v_1) = d_{i-2}^0(P_{n-2}, u_1) + d_{i-2}^1(P_{n-2}, u_1) + d_{i-2}^2(P_{n-2}, u_1) + d_{i-2}^1(P_{n-1}, u_1) + d_{i-2}^2(P_{n-1}, u_1)$, as desired. □

Note that Lemma 1 expresses a recursive relation to obtain the values $d_i^0(P_n, v_1)$, $d_i^1(P_n, v_1)$ and $d_i^2(P_n, v_1)$. In a recursive relationship, it is essential to know the initial conditions as well as the end condition. By Proposition 2 we obtain the following.

Proposition 3. *If P_n is a path graph with n vertices then*

1) $d_{2n-2}^0(P_n, v_1) = 1$, $d_m^0(P_n, v_1) = 0$ for $m > 2n - 2$ or $m < \lceil \frac{2n}{3} \rceil$.

2) $d_{2n-1}^1(P_n, v_1) = 1$, $d_m^1(P_n, v_1) = 0$ for $m > 2n - 1$ or $m < \lceil \frac{2n}{3} \rceil$.

3) $d_{2n}^2(P_n, v_1) = 1$, $d_m^2(P_n, v_1) = 0$ for $m > 2n$ or $m < \lceil \frac{2n}{3} \rceil$.

Theorem 3. *For $n \geq 4$, let $G = P_n$ with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\}$. Then,*

$$RD(P_n, x) = RD(P_1, x)RD(P_{n-1}, x) + x^2 RD(P_{n-2}, x) + xRD(P_1, x)RD(P_{n-3}, x),$$

with the initial values $RD(P_1, x) = x + x^2$, $RD(P_2, x) = 3x^2 + 2x^3 + x^4$, $RD(P_3, x) = x^2 + 5x^3 + 6x^4 + 3x^5 + x^6$.

Proof. It is evident that

$$\begin{aligned} RD(P_n, x) &= \sum_{i=\gamma_R(P_n)}^{2n} (d_i^0(P_n, v_1) + d_i^1(P_n, v_1) + d_i^2(P_n, v_1))x^i \\ &= \sum_{i=\lceil \frac{2n}{3} \rceil}^{2n} (d_i^0(P_n, v_1) + d_i^1(P_n, v_1) + d_i^2(P_n, v_1))x^i. \end{aligned}$$

Using Lemma 1, it can be written:

$$\begin{aligned}
RD(P_n,x) &= \sum_{i=\lceil (2n)/3\rceil}^{2n} (d_i^2(P_{n-1},v_1)+d_{i-1}^0(P_{n-1},v_1)+d_{i-1}^1(P_{n-1},v_1)+d_{i-1}^2(P_{n-1},v_1)\\
&+ d_{i-2}^0(P_{n-2},v_1)+d_{i-2}^1(P_{n-2},v_1)+d_{i-2}^2(P_{n-2},v_1)+d_{i-2}^1(P_{n-1},v_1)\\
&+ d_{i-2}^2(P_{n-1},v_1))x^i\\
&= \sum_{i=\lceil (2n)/3\rceil}^{2n} (d_{i-1}^0(P_{n-1},v_1)+d_{i-1}^1(P_{n-1},v_1)+d_{i-1}^2(P_{n-1},v_1))x^i\\
&+ \sum_{i=\lceil (2n)/3\rceil}^{2n} (d_{i-2}^0(P_{n-2},v_1)+d_{i-2}^1(P_{n-2},v_1)+d_{i-2}^2(P_{n-2},v_1))x^i\\
&+ \sum_{i=\lceil (2n)/3\rceil}^{2n} (d_i^2(P_{n-1},v_1)+d_{i-2}^1(P_{n-1},v_1)+d_{i-2}^2(P_{n-1},v_1))x^i\\
&= xRD(P_{n-1},x)+x^2RD(P_{n-2},x)+\sum_{i=\lceil (2n)/3\rceil}^{2n} (d_i^2(P_{n-1},v_1)+d_{i-2}^1(P_{n-1},v_1)\\
&+ d_{i-2}^2(P_{n-1},v_1))x^i\\
&= xRD(P_{n-1},x)+x^2RD(P_{n-2},x)+\sum_{i=\lceil (2n)/3\rceil}^{2n} (d_i^2(P_{n-1},v_1)+d_{i-2}^1(P_{n-1},v_1)\\
&+ d_{i-2}^2(P_{n-1},v_1)+d_{i-2}^0(P_{n-1},v_1)-d_{i-2}^0(P_{n-1},v_1))x^i\\
&= xRD(P_{n-1},x)+x^2RD(P_{n-2},x)+x^2RD(P_{n-1},x)+\sum_{i=\lceil (2n)/3\rceil}^{2n} (d_i^2(P_{n-1},v_1)\\
&- d_{i-2}^0(P_{n-1},v_1))x^i\\
&= (x+x^2)RD(P_{n-1},x)+x^2RD(P_{n-2},x)+\sum_{i=\lceil (2n)/3\rceil}^{2n} (d_i^2(P_{n-1},v_1)\\
&- d_{i-2}^0(P_{n-1},v_1))x^i\\
&= RD(P_1,x)RD(P_{n-1},x)+x^2RD(P_{n-2},x)+\sum_{i=\lceil (2n)/3\rceil}^{2n} (d_{i-2}^0(P_{n-3},v_1)\\
&+ d_{i-2}^1(P_{n-3},v_1)+d_{i-2}^2(P_{n-3},v_1)+d_{i-2}^1(P_{n-2},v_1)+d_{i-2}^2(P_{n-2},v_1)\\
&- d_{i-2}^2(P_{n-2},v_1))x^i\\
&= RD(P_1,x)RD(P_{n-1},x)+x^2RD(P_{n-2},x)\\
&+ \sum_{i=\lceil (2n)/3\rceil}^{2n} (d_{i-2}^0(P_{n-3},v_1)+d_{i-2}^1(P_{n-3},v_1)+d_{i-2}^2(P_{n-3},v_1)+d_{i-2}^1(P_{n-2},v_1))x^i\\
&= RD(P_1,x)RD(P_{n-1},x)+x^2RD(P_{n-2},x)+x^2RD(P_{n-3},x)\\
&+ \sum_{i=\lceil \frac{2n}{3}\rceil}^{2n} (d_{i-3}^0(P_{n-3},v_1)+d_{i-3}^1(P_{n-3},v_1)+d_{i-3}^2(P_{n-3},v_1))x^i\\
&= RD(P_1,x)RD(P_{n-1},x)+x^2RD(P_{n-2},x)+x^2RD(P_{n-3},x)+x^3RD(P_{n-3},x)\\
&= RD(P_1,x)RD(P_{n-1},x)+x^2RD(P_{n-2},x)+(x^2+x^3)RD(P_{n-3},x)\\
&= RD(P_1,x)RD(P_{n-1},x)+x^2RD(P_{n-2},x)+xRD(P_1,x)RD(P_{n-3},x),
\end{aligned}$$

as desired. □

4. A bound for the total number of Roman domination polynomial

In this section, we prove an upper bound as well as a lower bound for the number of all RDFs of a graph and characterize graphs achieving equality for the lower bound.

Theorem 4. *Let G be an arbitrary graph with n vertices. Then*

$$2^n \leq d_R(G) \leq 3^n - \sum_{k=0}^{n}(-1)^k \binom{n}{k}\binom{2n-2k-1}{n}.$$

Equality for the lower bound holds if and only if $G = \bar{K_n}$.

Proof. Any function that assigns the values 1 or 2 to the vertices of a graph is an RDF, and so the number of such functions is equal to 2^n. So $2^n \leq d_R(G)$. On the other hand, the total number of functions such as $f : V(G) \to \{0,1,2\}$ is equal to 3^n, as a result $d_R(G) \leq 3^n - |A|$, where $A = \{f : V(G) \to \{0,1,2\} | f \textit{ is not an RDF}\}$. It is enough to calculate the value of $|A|$. For every function $f : V(G) \to \{0,1,2\}$, let $x_1 = f(v_1), x_2 = f(v_2), \ldots, x_n = f(v_n)$. Then, $|A|$ is at least equal to the number of non-negative integer solutions of the inequality

$$x_1 + x_2 + \cdots + x_n \leq n-1,$$

where $0 \leq x_i \leq 1$, $0 \leq i \leq n$. The number of solutions of $x_1 + x_2 + \cdots + x_n \leq n-1$, $0 \leq x_i \leq 1$, $0 \leq i \leq n$ is

$$\binom{2n-1}{n} + \sum_{k=1}^{n}(-1)^k \binom{n}{k}\binom{2n-2k-1}{n} = \sum_{k=0}^{n}(-1)^k \binom{n}{k}\binom{2n-2k-1}{n}.$$

Thus $\sum_{k=0}^{n}(-1)^k \binom{n}{k}\binom{2n-2k-1}{n} \leq |A|$.
We next prove the equality part. Let $G = \bar{K_n}$ and f be an arbitrary RDF on G. Then $f(v) = 1$ or $f(v) = 2$ for each vertex v. The number of these functions is equal to 2^n. Conversely, suppose that G is an arbitrary graph with $d_R(G) = 2^n$. If $G \neq \bar{K_n}$, then G has at least one edge $v_t v_s$. Set $A = \{f : V(G) \to \{0,1,2\} \mid f(v_i) \geq 1,\ 1 \leq i \leq n\} \cup \{g : V(G) \to \{0,1,2\} \mid f(v_t) = 0, f(v_s) = 2, f(v_i) = 1, 1 \leq i \leq n, i \neq t, i \neq s\}$. Clearly, every member of A is an RDF for G and $|A| = 2^n + 1$. This contradicts the assumption $d_R(G) = 2^n$. This completes the proof. □

Note that the upper bound of Theorem 4 is sharp, as can be seen in the graph K_3.

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Conflict of Interest
The authors declare no conflict of interest in this paper.
Data Availability
Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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