

Coalition of cubic graphs of order at most 10

Saeid Alikhani^{1,*}, Hamidreza Golmohammadi^{2,3,†}, Elena V. Konstantinova^{2,3,‡}

¹Department of Mathematical Sciences, Yazd University, 89195-741, Yazd, Iran
alikhani@yazd.ac.ir

²Novosibirsk State University, Pirogova str. 2, Novosibirsk, 630090, Russia

³Sobolev Institute of Mathematics, Ak. Koptuyug av. 4, Novosibirsk, 630090, Russia

[†]h.golmohammadi@ng.nsu.ru

[‡]e_konsta@math.nsc.ru

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Abstract: The coalition in a graph G consists of two disjoint sets of vertices V_1 and V_2 , neither of which is a dominating set but whose union $V_1 \cup V_2$, is a dominating set. A coalition partition in a graph G is a vertex partition $\pi = \{V_1, V_2, \dots, V_k\}$ such that every set $V_i \in \pi$ is not a dominating set but forms a coalition with another set $V_j \in \pi$ which is not a dominating set. The coalition number $C(G)$ equals the maximum k of a coalition partition of G . In this paper, we compute the coalition numbers of all cubic graphs of order at most 10.

Keywords: coalition; cubic graphs; Petersen graph.

AMS Subject classification: 05C69

1. Introduction

Let $G = (V, E)$ be a simple graph of order n . The open neighborhood (closed neighborhood) of a vertex $v \in V$ is the set $N(v) = \{u \mid uv \in E\}$, (the set $N[v] = N(v) \cup \{v\}$). The number of vertices in $N(v)$ is the degree of v , denoted by $deg(v)$. A vertex of degree $n - 1$ is called a full or universal vertex, while a vertex of degree 0 is an isolate one. As usual the minimum degree and the maximum degree of G is denoted by $\delta(G)$ and $\Delta(G)$, respectively. A subset $V_i \subseteq V$ is called a singleton set if $|V_i| = 1$ and a non-singleton set, if $|V_i| \geq 2$. A set $S \subseteq V$ is a dominating set of G , if every vertex

* *Corresponding Author*

in $V \setminus S$ has at least one neighbor in S . A set $S \subseteq V$ is a total dominating set of G with no isolated vertex, if every vertex in V has at least one neighbour in S ; in other words $N(S) = V$. The literature on this subject has been surveyed and detailed in the two excellent so-called domination books by Haynes, Hedetniemi, and Slater [9, 10]. A domatic partition is a partition of the vertex set into dominating sets. Formally, the domatic number $d(G)$ equals the maximum order k of a vertex partition, called a domatic partition, $\pi = \{V_1, V_2, \dots, V_k\}$ such that every set V_i is a dominating set in G . The domatic number of a graph was introduced by Cockayne and Hedetniemi [4]. For more details on the domatic number refer to e.g., [12–14].

Coalitions and coalition partitions were introduced in [5], and are now studied in graph theory (see for example [6–8]). The concepts were defined in terms of general graph properties but focused on the property of being a dominating set. A coalition π in a graph G consists of two disjoint sets of vertices V_1 and V_2 , neither of which is a dominating set of G but whose union $V_1 \cup V_2$ is a dominating set of G . We say that the sets V_1 and V_2 form a coalition and that they are coalition partners in π .

A coalition partition, henceforth called a c -partition, in a graph G is a vertex partition $\pi = \{V_1, V_2, \dots, V_k\}$ such that every set V_i of π is either a singleton dominating set of G , or is not a dominating set of G but forms a coalition with another non-dominating set $V_j \in \pi$. The coalition number $C(G)$ equals the maximum order k of a c -partition of G . Associated with every coalition partition π of a graph G is a graph called the coalition graph of G with respect to π , denoted $CG(G, \pi)$, the vertices of which correspond one-to-one with the sets V_1, V_2, \dots, V_k of π and two vertices are adjacent in $CG(G, \pi)$ if and only if their corresponding sets in π form a coalition ([8]). In [8] the coalition graphs (focusing on the coalition graphs of paths, cycles, and trees) have studied. Recently Bakhshesh, Henning and Pradhan in [3] characterized all graphs G of order n with $\delta(G) \leq 1$ and $C(G) = n$, and also characterized all trees T of order n with $C(T) = n - 1$.

The class of cubic graphs is especially interesting for mathematical applications because of various important open problems in graph theory. Cubic graphs are the smallest or simplest possible potential counterexamples, and so this creates motivation to study coalition parameter for the cubic graphs of order at most 10. Alikhani and Peng have studied the domination polynomials (which is the generating function for the number of dominating sets of a graph) of cubic graphs of order 10 in [2]. As a consequence, they have shown that the Petersen graph is determined uniquely by its domination polynomial.

In the next section, we compute the coalition numbers of cubic graphs of order 6 and 8. Also, we compute the coalition numbers of the cubic graphs of order 10 in Section 3.

2. Coalition numbers of cubic graphs of order 6 and 8

In this section, we study the coalition numbers of the cubic graphs of order 6 and 8. We need the following theorems.

Theorem 1. [5] *If G is a graph of order n , then $1 \leq C(G) \leq n$.*

Theorem 2. [5] *If G is a graph with no full vertex and minimum degree $\delta(G) \geq 1$, then $C(G) \geq \delta(G) + 2$.*

Theorem 3. [6] *For any graph G , $C(G) \leq \frac{(\Delta(G)+3)^2}{4}$.*

Theorem 4. [5] *Let G be a graph with maximum degree $\Delta(G)$, and let π be a $C(G)$ -partition. If $X \in \pi$, then X is in at most $\Delta(G) + 1$ coalitions.*

First we determine the coalition number of the cubic graphs of order 6. There are exactly two cubic graphs of order 6 which are denoted by G_1 and G_2 in Figure 1.

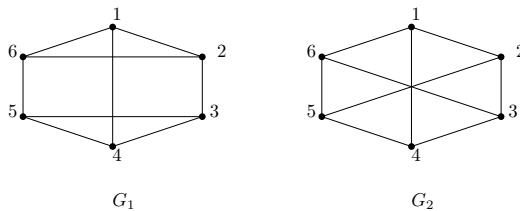


Figure 1. Cubic graphs of order 6.

Theorem 5. *The coalition number of the cubic graphs G_1 and G_2 (Figure 1) of order 6 is 6.*

Proof. Theorems 1 and 2 show that $5 \leq C(G) \leq 6$. Let compute $C(G)$ for the graph G_1 . We create a partition of order 6. Let $\pi = \{V_1 = \{1, 4\}, V_2 = \{2, 5\}, V_3 = \{3, 6\}\}$ be a domatic partition of a graph G_1 , where $d(G_1) = 3$. Since any partition of a non-singleton, minimal dominating set into two nonempty sets creates two non-dominating sets whose union forms a coalition, so we can partition each of the minimal dominating sets $V_1 = \{1, 4\}$, $V_2 = \{2, 5\}$ and $V_3 = \{3, 6\}$ into two sets such as $V_{1,1} = \{1\}$, $V_{1,2} = \{4\}$, $V_{2,1} = \{2\}$, $V_{2,2} = \{5\}$, $V_{3,1} = \{3\}$, and $V_{3,2} = \{6\}$, which each of $V_{1,1}$, $V_{2,1}$ and $V_{3,1}$ form a coalition with each of $V_{1,2}$, $V_{2,2}$ and $V_{3,2}$, respectively. Therefore, we have $C(G_1) \geq 6$. Moreover, as seen early, $C(G_1) \leq 6$, and so we have $C(G_1) = 6$. So, we can create a maximal c -partition of G_1 of order 6 as follows:
 $\pi_1 = \{V_{1,1} = \{1\}, V_{1,2} = \{4\}, V_{2,1} = \{2\}, V_{2,2} = \{5\}, V_{3,1} = \{3\}, V_{3,2} = \{6\}\}$. An identical argument shows that $C(G_2) = 6$. \square

Now, we compute the coalition numbers of cubic graphs of order 8. There are exactly 6 cubic graphs of order 8 which are denoted by G_1, G_2, \dots, G_6 in Figure 2.

The following theorem gives the coalition numbers of cubic graphs of order 8:

Theorem 6. *For the cubic graphs G_1, G_2, \dots, G_6 of order 8 (Figure 2) we have:*

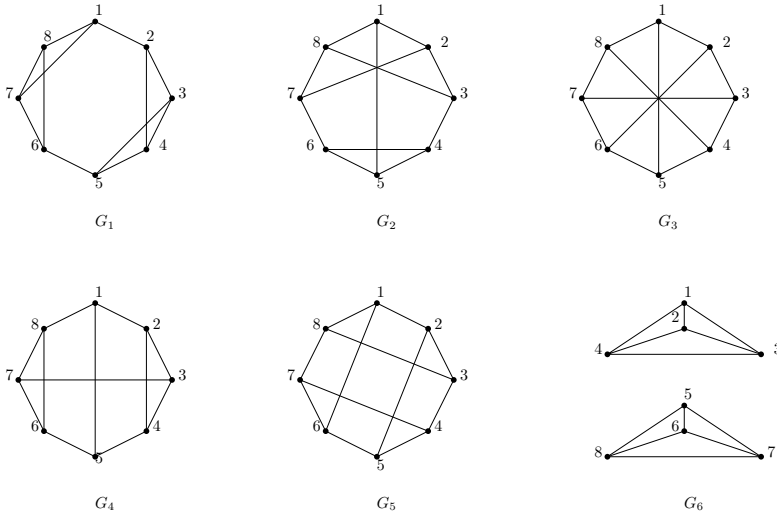


Figure 2. Cubic graphs of order 8.

- (i) $C(G_1) = C(G_5) = C(G_6) = 8$.
- (ii) $C(G_4) = 7$.
- (iii) $C(G_2) = C(G_3) = 6$.

Proof. (i) Using Theorems 1 and 2, we have $5 \leq C(G) \leq 8$. Without loss of generality, we compute $C(G)$ for the graph G_1 (with similar argument we can determine the coalition number for other graphs in this part). We show that there is a partition of order 8 for the graph G_1 . Let $\pi = \{V_1 = \{1, 5\}, V_2 = \{2, 6\}, V_3 = \{3, 8\}, V_4 = \{4, 7\}\}$ be a domatic partition of the graph G_1 , where $d(G_1) = 4$. Since any partition of a non-singleton, minimal dominating set into two nonempty sets creates two non-dominating sets whose union forms a coalition, so we can partition each of the minimal dominating sets in π into two sets such as $V_{1,1} = \{1\}, V_{1,2} = \{5\}, V_{2,1} = \{2\}, V_{2,2} = \{6\}, V_{3,1} = \{3\}, V_{3,2} = \{8\}, V_{4,1} = \{4\}$, and $V_{4,2} = \{7\}$, which each of $V_{1,1}, V_{2,1}, V_{3,1}$ and $V_{4,1}$ form a coalition with each of $V_{1,2}, V_{2,2}, V_{3,2}$, and $V_{4,2}$, respectively. So, we have $C(G_1) \geq 8$. Since, $C(G_1) \leq 8$, so we have $C(G_1) = 8$. Therefore, we can construct a maximal c -partition of G_1 of order 8 as follows:

$$\pi_1 = \{V_{1,1} = \{1\}, V_{1,2} = \{5\}, V_{2,1} = \{2\}, V_{2,2} = \{6\}, V_{3,1} = \{3\}, V_{3,2} = \{8\}, V_{4,1} = \{4\}, V_{4,2} = \{7\}\}.$$

- (ii) We show that there is no partition of order 8 for G_4 . Let $\pi' = \{V'_1, V'_2, \dots, V'_8\}$ be a c -partition of G_4 . Since $|V(G_4)| = 8$, so the only possible partition of eight sets is eight singleton sets. Since the graph G_4 has exactly two dominating sets of size 2, without loss of generality, we may assume that there are only four singleton sets V'_2, V'_4, V'_6 and V'_8 , in which V'_2 forms a coalition with V'_6 , and V'_4 is

in a coalition with V_8' . It follows that none of two singleton sets of the remaining singleton sets V_1' , V_3' , V_5' and V_7' form a coalition of size 2. Therefore, there is no partition of order 8. So, $C(G_4) \leq 7$. Now we show that there is a partition of order 7. Let $\pi'' = \{V_1'' = \{1, 3, 5, 7\}, V_2'' = \{2, 6\}, V_3'' = \{4, 8\}\}$ be a domatic partition of a graph G_4 , where $d(G_4) = 3$. We can partition each of the minimal dominating sets $V_2'' = \{2, 6\}$ and $V_3'' = \{4, 8\}$ into two sets such as $V_{2,1}'' = \{2\}$, $V_{2,2}'' = \{6\}$, $V_{3,1}'' = \{4\}$, and $V_{3,2}'' = \{8\}$, in which $V_{2,1}'' = \{2\}$ form a coalition with $V_{2,2}'' = \{6\}$, and $V_{3,1}'' = \{4\}$ is in a coalition with $V_{3,2}'' = \{8\}$. Now we create a partition π_1'' of sets and put the sets $V_{2,1}'' = \{2\}$, $V_{2,2}'' = \{6\}$, $V_{3,1}'' = \{4\}$, and $V_{3,2}'' = \{8\}$ in this partition. To obtain other sets of partition π_1'' , let $W = \{1, 5, 7\} \subset V_1''$ be a minimal dominating set contained in V_1'' . So, we shall partition it into two non-dominating sets $W_1 = \{1\}$ and $W_2 = \{5, 7\}$, add these two sets to π_1'' . The set $T = \{3\}$ remains which is not a dominating set, else there are at least 4 disjoint dominating sets in G_4 , a contradiction, because $d(G_4) = 3$. The set T forms a coalition with W_2 , so we can add T to π_1'' . Therefore, we have $C(G_4) \geq 7$. We know that $C(G_4) \leq 7$, and so $C(G_4) = 7$. Therefore, we can create a maximal c -partition of G_4 of order 7 as follows:

$$\pi_1'' = \{W_1 = \{1\}, W_2 = \{5, 7\}, T = \{3\}, V_{2,1}'' = \{2\}, V_{2,2}'' = \{6\}, V_{3,1}'' = \{4\}, V_{3,2}'' = \{8\}\}.$$

- (iii) Without loss of generality, we compute $C(G)$ for the graph G_2 . First, we show there is no partition of order 7 for G_2 . We suppose $\psi = \{U_1, U_2, \dots, U_7\}$. Since $|V(G_2)| = 8$, so the only possible partition of $V(G_2)$ is six singleton sets and one doubleton set. Since $\gamma(G_2) = 3$, no two singleton sets form a coalition and by Theorem 4, the doubleton set can be in a coalition with at most four singleton sets. Therefore, $C(G_2) \leq 6$. In order to show that there is a partition of order 6, we make a domatic partition. Assume that $\psi' = \{U_1' = \{1, 2, 4, 7\}, U_2' = \{3, 5, 6, 8\}\}$ be a domatic partition of the graph G_2 , where $d(G_2) = 2$. Since any partition of a non-singleton, minimal dominating set into two nonempty sets creates two non-dominating sets whose union forms a coalition, so we may assume that each of the minimal dominating sets $U_1'' = \{1, 4, 7\} \subset U_1'$ and $U_2'' = \{3, 5, 8\} \subset U_2'$ into two sets such as $U_{1,1}'' = \{1, 4\}$, $U_{1,2}'' = \{7\}$, $U_{2,1}'' = \{3, 8\}$ and $U_{2,2}'' = \{5\}$, in which $U_{1,1}'' = \{1, 4\}$ form a coalition with $U_{1,2}'' = \{7\}$, and $U_{2,1}'' = \{3, 8\}$ is in a coalition with $U_{2,2}'' = \{5\}$. Now we create a partition ψ'' of sets and put the sets $U_{1,1}'' = \{1, 4\}$, $U_{2,1}'' = \{7\}$, $U_{2,1}'' = \{3, 8\}$ and $U_{2,2}'' = \{5\}$ in this partition. The sets $U_1''' = \{2\} \subset U_1'$ and $U_1''' = \{2\} \subset U_2'' = \{6\} \subset U_2'$ remain which are not dominating sets, else there are at least 4 disjoint dominating sets in G_2 , a contradiction. The set $U_1''' = \{2\}$ forms a coalition with $U_{1,1}''$ and $U_2''' = \{6\}$ is in coalition with $U_{2,1}'' = \{3, 8\}$, and so we can add U_1''' and U_2''' to ψ'' . Therefore, we have $C(G_2) \geq 6$. Moreover, we observed that $C(G_2) \leq 6$ and so we have $C(G_2) = 6$. Therefore, we have a maximal c -partition of G_2 of order 6 as follows:

$$\psi'' = \{U_{1,1}'' = \{1, 4\}, U_{1,2}'' = \{7\}, U_{2,1}'' = \{3, 8\}, U_{2,2}'' = \{5\}, U_1''' = \{2\}, U_2''' = \{6\}\}.$$

Therefore, we have the result. □

3. Coalition numbers of cubic graphs of order 10

In this section, we study the coalition numbers of cubic graphs of order 10. In particular, we obtain the coalition number of the Petersen graph.

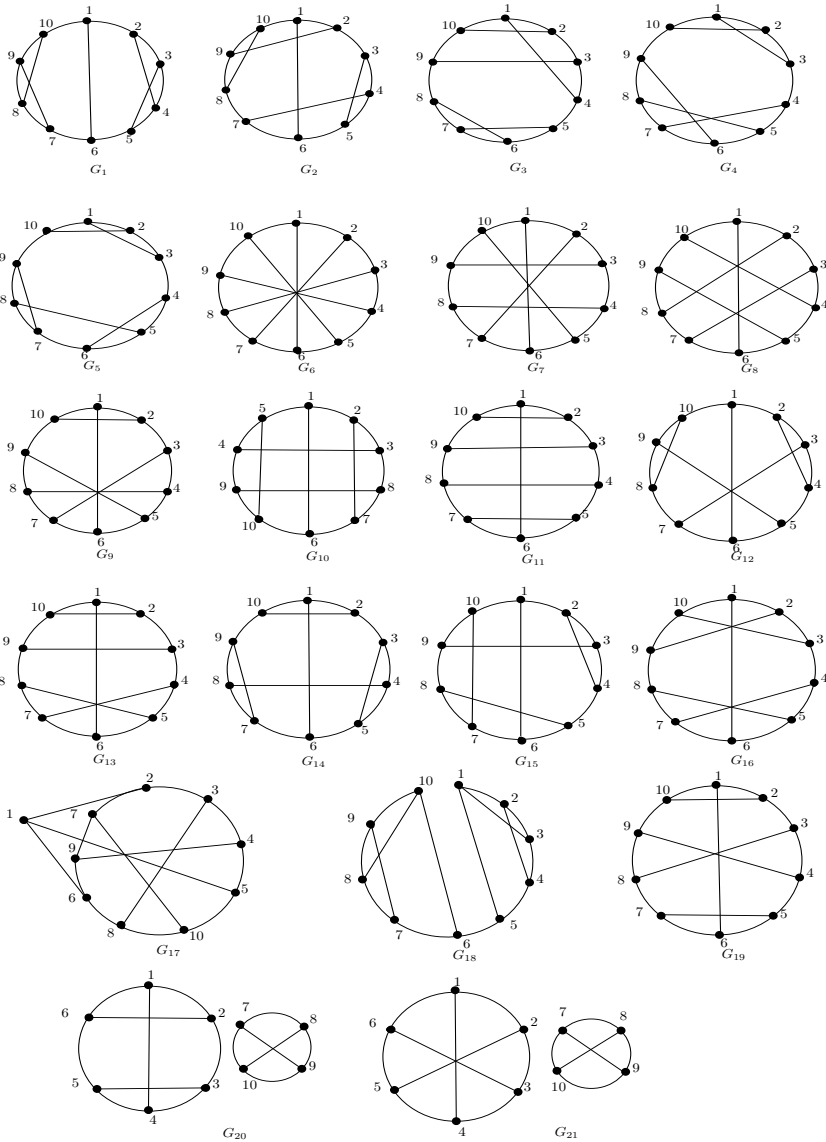


Figure 3. Cubic graphs of order 10.

There are exactly 21 cubic graphs of order 10 denoted by G_1, G_2, \dots, G_{21} in Figure 3 (see [11]). In particular, the graph G_{17} is isomorphic to the Petersen graph P . Now we state and prove the following theorem.

Theorem 7. *For the cubic graphs G_1, G_2, \dots, G_{21} of order 10 (Figure 3) we have:*

(i) $C(G_1) = C(G_{18}) = 8$.

(ii) $C(G_i) = 7$ for $i \in \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 19, 20, 21\}$.

(iii) $C(G_j) = 6$ for $j \in \{16, 17\}$.

Proof. By Theorems 2 and 3, for the cubic graphs of order 10, $5 \leq C(G) \leq 9$.

(i) We show that $C(G_1) \neq 9$. Suppose that there exists a c -partition $\pi = \{V_1, \dots, V_9\}$ for the graph G_1 with nine classes. So, every set V_k of π is either a singleton dominating set of G_1 , or is not a dominating set of G but forms a coalition with another non-dominating set $V_l \in \pi$. Since the graph G_1 has order 10, exactly one class of π must contain two vertices. Without loss of generality, let $|V_1| = 2$ and $|V_j| = 1$ for $2 \leq j \leq 9$. Let S be a dominating set in the graph G_1 . Since $|S| \geq 3$, no two singleton sets form a coalition. Now every set V_j for $2 \leq j \leq 9$ must be in a coalition with V_1 . Moreover, by Theorem 4, V_1 is in a coalition with at most $\Delta(G) + 1 = 4$ sets. That means V_1 is a coalition partner of at most 4 singleton sets. Therefore, remain at least 4 singleton sets, neither of which can form a coalition with V_1 , and it is a contradiction. So, we cannot create a partition of order 9. Therefore, $C(G_1) \leq 8$.

Now we show that there is a partition of order 8 for the graph G_1 . First construct a domatic partition. Let $\pi' = \{V_1 = \{1, 5, 9\}, V_2 = \{3, 6, 10\}, V_3 = \{2, 4, 7, 8\}\}$ be a domatic partition of a graph G_1 , where $d(G_1) = 3$. We can partition each of the minimal dominating sets $V_1 = \{1, 5, 9\}$ and $V_2 = \{3, 6, 10\}$ into two sets such as $V_{1,1} = \{1, 5\}$, $V_{1,2} = \{9\}$, $V_{2,1} = \{3\}$, and $V_{2,2} = \{6, 10\}$, which each of $V_{1,1}$ and $V_{2,1}$ form a coalition with each of $V_{1,2}$ and $V_{2,2}$, respectively. Now we create a partition π'' of sets and put the sets $V_{1,1} = \{1, 5\}$, $V_{1,2} = \{9\}$, $V_{2,1} = \{3\}$, and $V_{2,2} = \{6, 10\}$ in this partition. The set $V_3 = \{2, 4, 7, 8\}$ remains which is a dominating set of order 4, however does not contain a minimal dominating set. We partition the set $V_3 = \{2, 4, 7, 8\}$ into four singleton sets such as $V_{3,1} = \{2\}$, $V_{3,2} = \{4\}$, $V_{3,3} = \{7\}$, and $V_{3,4} = \{8\}$, which each of $V_{3,1} = \{2\}$ and $V_{3,2} = \{4\}$ is in coalition with $V_{2,2} = \{6, 10\}$ while $V_{1,1} = \{1, 5\}$ is in coalition with each of $V_{3,3} = \{7\}$ and $V_{3,4} = \{8\}$. So, we have $C(G_1) \geq 8$. Furthermore, as seen early, $C(G_1) \leq 8$ and so, $C(G_1) = 8$. Therefore, we can establish a maximal c -partition of G_1 of order 8 as follows: $\pi'' = \{V_{1,1} = \{1, 5\}, V_{1,2} = \{9\}, V_{2,1} = \{3\}, V_{2,2} = \{6, 10\}, V_{3,1} = \{2\}, V_{3,2} = \{4\}, V_{3,3} = \{7\}, V_{3,4} = \{8\}\}$.

(ii) Without loss of generality, we compute $C(G)$ for the graph G_2 . Similar to Part (i) there is no c -partition of order 9 for G_2 . Now we show that there is no

partition of order 8 for G_2 . Let $\pi_1 = \{V_1, V_2, \dots, V_8\}$ be a c -partition of G_2 . We consider two cases.

Case 1. There are exactly two sets of π_1 which consist of two vertices. Without loss of generality, let $|V_1| = |V_2| = 2$ and $|V_j| = 1$ for $3 \leq j \leq 8$. Let S be a dominating set in the graph G_2 . Since $|S| \geq 3$, no two singleton sets form a coalition. Now every set V_j for $3 \leq j \leq 8$ must be in a coalition with at least one of V_1 or V_2 . By Theorem 4, each of V_1 and V_2 is in at most $\Delta(G) + 1 = 4$ coalitions. Let V_1 and V_2 form a coalition, then each of V_1 and V_2 can be in at most three additional coalitions. Without loss of generality, suppose there are two collections such as $\pi_{1,1} = \{V_1, V_3, V_4, V_5\} \subset \pi_1$, which V_1 is in a coalition with each of singleton sets V_3, V_4 and V_5 and $\pi_{1,2} = \{V_2, V_6, V_7, V_8\} \subset \pi_1$, which V_2 is in a coalition with each of singleton sets V_6, V_7 and V_8 such that $\pi_{1,1} \cup \pi_{1,2} = \pi_1$ and $\pi_{1,1} \cap \pi_{1,2} = \emptyset$. Note that V_1 and V_2 form dominating sets of cardinality 3 with singleton sets. All dominating sets of cardinality 3 of the graph G_2 are: $\{1, 3, 8\}$, $\{1, 4, 8\}$, $\{1, 4, 9\}$, $\{1, 4, 10\}$, $\{1, 5, 8\}$, $\{2, 5, 8\}$, $\{3, 6, 8\}$, $\{3, 6, 9\}$, $\{3, 6, 10\}$, $\{3, 7, 10\}$, $\{4, 6, 9\}$, $\{5, 6, 9\}$.

Now we show there are no two collections such as $\pi_{1,1}$ and $\pi_{1,2}$ such that $\pi_{1,1} \cup \pi_{1,2} = \pi_1$ and $\pi_{1,1} \cap \pi_{1,2} = \emptyset$. Since V_1 and V_2 are in a coalition, without loss of generality, we can suppose $V_1 = \{1, 4\}$ and $V_2 = \{6, 9\}$, and so V_1 is a coalition partner with each of $V_3 = \{8\}$, $V_4 = \{9\}$ and $V_5 = \{10\}$, and also V_2 is in a coalition with each of $V_6 = \{8\}$, $V_7 = \{9\}$ and $V_8 = \{10\}$. Therefore, $\pi_{1,1} \cup \pi_{1,2} \neq \pi_1$ and $\pi_{1,1} \cap \pi_{1,2} \neq \emptyset$, and it is a contradiction. So, there are no two collections such as $\pi_{1,1}$ and $\pi_{1,2}$ when V_1 and V_2 are coalition partners. Implying that, there is no a partition of order 8. Hence, $C(G_2) \leq 7$. Similarly, an identical argument shows the same result holds when V_1 and V_2 form a coalition, and V_1 form a coalition with any three singleton sets of π_1 and V_2 form a coalition with each of the other three singleton sets of π_1 . If V_1 and V_2 do not form a coalition, by Theorem 4, each of V_1 and V_2 can be in at most four coalitions. Without loss of generality, let two collections such as $\pi_{1,1} = \{V_1, V_3, V_4\} \subset \pi_1$, which V_1 is in a coalition with each of singleton sets V_3, V_4 and $\pi_{1,2} = \{V_2, V_5, V_6, V_7, V_8\} \subset \pi_1$, which V_2 is in a coalition with each of singleton sets V_5, V_6, V_7 and V_8 such that $\pi_{1,1} \cup \pi_{1,2} = \pi_1$ and $\pi_{1,1} \cap \pi_{1,2} = \emptyset$. Without loss of generality, assume that $V_1 = \{5, 8\}$ is a coalition partner with each of $V_3 = \{1\}$ and $V_4 = \{2\}$. It follows that the remaining vertices such as 3, 4, 6, 7, 9, 10 must create the collection $\pi_{1,2} = \{V_2, V_5, V_6, V_7, V_8\}$, which V_2 is in a coalition with each of singleton sets V_5, V_6, V_7 and V_8 . Without loss of generality, let $V_2 = \{3, 9\}$, $V_5 = \{6\}$, $V_6 = \{4\}$, $V_7 = \{7\}$ and $V_8 = \{10\}$. It can be seen, V_2 forms a coalition with V_5 and does not form a coalition with each of V_6, V_7 and V_8 . Then, there is no the collection $\pi_{1,2}$. Implying that, $\pi_{1,1} \cup \pi_{1,2} \neq \pi_1$ and $\pi_{1,1} \cap \pi_{1,2} \neq \emptyset$, and it is a contradiction. Therefore, there is no a partition of order 8 and so $C(G_2) \leq 7$. Similarly, an identical argument shows for any two sets of π_1 such as V_1 and V_2 , which V_1 and V_2 do not form a coalition, and V_1 form a coalition with any two singleton sets of π_1 and V_2 form

a coalition with each of the other four singleton sets of π_1 the same result holds. To complete the proof in the Case 1, when V_1 and V_2 are not coalition partners, we also may assume that $\pi_{1,1} = \{V_1, V_3, V_4, V_5\} \subset \pi_1$, which V_1 is in a coalition with each of singleton sets V_3, V_4 and V_5 and $\pi_{1,2} = \{V_2, V_6, V_7, V_8\} \subset \pi_1$, which V_2 is in a coalition with each of singleton sets V_6, V_7 and V_8 , and then by an identical argument the same result holds. Note that for notational convenience, let $|\pi_{1,1}| = |\pi_{1,2}| = 4$. Moreover, an identical argument shows the same result holds when V_1 and V_2 do not form a coalition partner and we may assume that $|\pi_{1,1}| = |\pi_{1,2}| = 5$, or $|\pi_{1,1}| = 5$ and $|\pi_{1,2}| = 4$.

Case 2. We suppose that there are 7 singleton sets in the partition and exactly one set of π_1 must consists of three vertices. Without loss of generality, let $|V_1| = 3$ and $|V_j| = 1$ for $2 \leq j \leq 8$. Suppose S is a dominating set in the graph G_2 . Since $|S| \geq 3$, no two singleton sets form a coalition. Now every set V_j for $2 \leq j \leq 8$ must be in a coalition with V_1 . By Theorem 4, V_1 is a coalition partner of at most 4 singleton sets. Therefore, remain at least 4 singleton sets, neither of which can form a coalition with V_1 , and it is a contradiction. It follows that we cannot create a partition of order 8. Hence, $C(G_2) \leq 7$.

Now we show that there is a partition of order 7 for the graph G_2 . Let $\pi'_1 = \{V'_1 = \{2, 5, 8\}, V'_2 = \{3, 6, 10\}, V'_3 = \{1, 4, 7, 9\}\}$ be a domatic partition of a graph G_2 , where $d(G_2) = 3$. Since any partition of a non-singleton, minimal dominating set into two nonempty sets creates two non-dominating sets whose union forms a coalition, so we can partition each of the minimal dominating sets $V'_1 = \{2, 5, 8\}$ and $V'_2 = \{3, 6, 10\}$ into two sets such as $V'_{1,1} = \{2\}$, $V'_{1,2} = \{5, 8\}$, $V'_{2,1} = \{6\}$, $V'_{2,2} = \{3, 10\}$, which each of $V'_{1,1}$ and $V'_{1,2}$ form a coalition with each of $V'_{2,1}$ and $V'_{2,2}$, respectively. Now we create a partition π''_1 of sets and put the sets $V'_{1,1} = \{2\}$, $V'_{1,2} = \{5, 8\}$, $V'_{2,1} = \{6\}$ and $V'_{2,2} = \{3, 10\}$ in this partition. To obtain other sets of partition π''_1 , let $V''_3 = \{1, 4, 9\} \subset V'_3$ be a minimal dominating set contained in V'_3 . Then we can partition it into two non-dominating sets $V''_{3,1} = \{1\}$ and $V''_{3,2} = \{4, 9\}$, add these two sets to π''_1 . Now remains the singleton set $V' = \{7\}$, which is not a dominating set. The set U' forms a coalition with $V'_{2,2} = \{3, 10\}$, then we can add U' to π''_1 . Hence, $C(G_2) \geq 7$. We know that $C(G_2) \leq 7$, and so we have $C(G_2) = 7$. So we can create a maximal c -partition of G_2 of order 7 as follows: $\pi''_1 = \{V'_{1,1} = \{2\}, V'_{1,2} = \{5, 8\}, V''_{2,1} = \{6\}, V''_{2,2} = \{3, 10\}, V''_{3,1} = \{1\}, V''_{3,2} = \{4, 9\}, U' = \{7\}\}$.

Note that using the same approach we can obtain the coalition number for other graphs in this part.

- (iii) Let us obtain the coalition number of the graph G_{17} . From our previous discussions, similarly by the same argument, we can show there are no partitions of order 8, 9 and 10 for the graph G_{17} . Therefore, now we show a partition of order 7 does not exist for the graph G_{17} . We can suppose $\pi_2 = \{V_1, V_2, \dots, V_7\}$ is a partition with seven sets for G_{17} . We deduce the following cases.

Case 1. We may assume that there are exactly three sets of π_2 must contain six vertices. Without loss of generality, let $|V_1| = |V_2| = |V_3| = 2$ and $|V_j| = 1$ for $4 \leq j \leq 7$. From our previous discussions, no two singleton sets can form a coalition. Now every set V_j for $4 \leq j \leq 7$ must be in a coalition with at least one of V_1, V_2 , or V_3 . Let V_1 forms a coalition with each of V_2 and V_3 , and V_2 forms a coalition with V_3 . Then, by Theorem 4, each of V_1, V_2 and V_3 can be in at most two additional coalitions. Now we can assume that one of V_1, V_2 , or V_3 is in a coalition with exactly two sets of V_j for $4 \leq j \leq 7$ and each of the two other sets can be in a coalition with one set of V_j for $4 \leq j \leq 7$. Without loss of generality, suppose there are three collections such as $\pi_{2,1} = \{V_1, V_4, V_5\} \subset \pi_2$, which V_1 is in a coalition with each of singleton sets V_4 and V_5 , and $\pi_{2,2} = \{V_2, V_6\} \subset \pi_2$, which V_2 is in a coalition with singleton set V_6 , and $\pi_{2,3} = \{V_3, V_7\} \subset \pi_2$, which V_3 is in a coalition with singleton set V_7 such that $\cup_{j=1}^3 \pi_{2,j} = \pi_2$ and $\pi_{2,1} \cap \pi_{2,2} = \emptyset$, $\pi_{2,1} \cap \pi_{2,3} = \emptyset$ and $\pi_{2,2} \cap \pi_{2,3} = \emptyset$. Without loss of generality, suppose $\pi_{2,1} = \{V_1 = \{1, 3\}, V_4 = \{2\}, V_5 = \{5\}\}$, $\pi_{2,2} = \{V_2 = \{4, 10\}, V_6 = \{8\}\}$ and $\pi_{2,3} = \{V_3 = \{6, 7\}, V_7 = \{9\}\}$. It can be seen that, V_1 forms a coalition with each of V_2 and V_3 , and V_2 forms a coalition with V_3 , however, V_1 does not form a coalition with each of singleton sets V_4 and V_5 , and V_2 does not form a coalition with V_6 , and also V_2 is not in a coalition with singleton set V_7 . Therefore, there are no three collections such as $\pi_{2,1}, \pi_{2,2}$ and $\pi_{2,3}$ and it is a contradiction. Implying that, there is no a partition of order 7 and then $C(G_{17}) \leq 6$. Similarly, we can show when V_1, V_2 and V_3 be any three sets of vertices such that $|V_1| = |V_2| = |V_3| = 2$, and V_1 forms a coalition with each of V_2 and V_3 , and V_2 forms a coalition with V_3 , the same result holds. Note that if V_1 and V_2 do not form a coalition partner and we may assume $|\pi_{2,1}| = |\pi_{2,2}| = |\pi_{2,3}| = 3$, or $|\pi_{2,1}| = |\pi_{2,2}| = 3$ and $|\pi_{2,3}| = 2$, and then the same result holds.

With an identical argument we have the same result, when at least one pair of sets V_1, V_2 or V_3 does not form a coalition. Therefore, $C(G_{17}) \leq 6$.

Case 2. We may assume that there are 5 singleton sets and two sets of π_2 must contain three and two vertices. Without loss of generality, let $|V_1| = 3, |V_2| = 2$ and $|V_j| = 1$ for $3 \leq j \leq 7$. Since G_2 has no dominating set with less than three vertices, then no two singleton sets are in a coalition. Now every set V_j for $3 \leq j \leq 7$ must be in a coalition with at least one of V_1 or V_2 . Theorem 4 implies that, each of V_1 and V_2 is in a coalition with at most $\Delta(G) + 1 = 4$ sets. Let V_1 and V_2 form a coalition, then each of V_1 and V_2 can be in at most three additional coalitions. Without loss of generality, suppose there are two collections such as $\pi'_{2,1} = \{V_1, V_3, V_4, V_5\} \subset \pi_2$, which V_1 is in a coalition with each of singleton sets V_3, V_4 and V_5 , and $\pi'_{2,2} = \{V_2, V_6, V_7\} \subset \pi_2$, which V_2 is in a coalition with each of singleton sets V_6 and V_7 such that $\pi'_{2,1} \cup \pi'_{2,2} = \pi_2$ and $\pi'_{2,1} \cap \pi'_{2,2} = \emptyset$. Without loss of generality, let $\pi'_{2,1} = \{V_1 = \{1, 3, 4\}, V_3 = \{7\}, V_4 = \{10\}, V_5 = \{8\}\}$, and $\pi'_{2,2} = \{V_2 = \{2, 5\}, V_6 = \{6\}, V_7 = \{9\}\}$. It can be seen, V_1 and V_2 form a coalition, however, V_1 and V_2 do not form a coalition with V_5 and V_7 ,

respectively. Then, there is at least one singleton set in $\pi'_{2,1}$, which does not form coalition with V_1 , and also there is at least one singleton set in $\pi'_{2,2}$, which does not form a coalition with V_2 . Therefore, there are no two collections such as $\pi'_{2,1}$, $\pi'_{2,2}$ and it is a contradiction. Implying that, there is no a partition of order 7 and then $C(G_{17}) \leq 6$. Similarly, we can show when V_1 and V_2 be any two sets of vertices such that $|V_1| = 3$ and $|V_2| = 2$, and V_1 forms a coalition with V_2 , the same result holds. Hence, there is no a collection such as $\pi'_{2,1}$ and it is a contradiction. Implying that, there is no a partition of order 7 and then $C(G_{17}) \leq 6$. Note that if V_1 and V_2 form a coalition partner and we assume that $|\pi'_{2,1}| = 3$ and $|\pi'_{2,2}| = 4$, or $|\pi'_{2,1}| = |\pi'_{2,2}| = 4$, then an identical argument proves the same result holds.

If V_1 and V_2 do not form a coalition, then each of V_1 and V_2 is in coalition with at most $\Delta(G) + 1 = 4$ sets. Without loss of generality, assume $\pi'_{2,1} = \{V_1, V_3, V_4, V_5, V_6\} \subset \pi_2$, which V_1 is in a coalition with each of singleton sets V_3, V_4, V_5 and V_6 , and $\pi'_{2,2} = \{V_2, V_7\} \subset \pi_2$, which V_2 is in a coalition with singleton set V_7 such that $\pi'_{2,1} \cup \pi'_{2,2} = \pi_2$ and $\pi'_{2,1} \cap \pi'_{2,2} = \emptyset$. Without loss of generality, let $V_2 = \{3, 5\}$, $V_7 = \{9\}$. According to the all dominating sets of cardinality 4, which are listed in [2], may be seen that every pair of dominating sets of cardinality 4, which have three vertices in common, either whose union of a dominating set of cardinality 3 and a singleton set, or whose union of a non-dominating set of cardinality 3 and a singleton set. Now we consider the remaining vertices 1, 2, 4, 6, 7, 8, 10. Without loss of generality, assume $\pi'_{2,1} = \{V_1 = \{1, 2, 4\}, V_3 = \{6\}, V_4 = \{7\}, V_5 = \{8\}, V_6 = \{10\}\}$. It can be seen, V_1 forms a coalition with each of V_5 and V_6 , and does not form a coalition with each of V_3 and V_4 . Therefore, there is no collection such as $\pi'_{2,1}$ and it is a contradiction. Similarly, when V_1 is any non-dominating set of cardinality 3, we cannot create 4 coalitions of order 4, which have three vertices in common by remaining vertices 1, 2, 4, 6, 7, 8, 10. Hence, we cannot create a partition of order 7. Implying that, $C(G_{17}) \leq 6$. Note that an identical argument shows if $V_2 \cup V_7$ is any dominating set of cardinality 3, then the same result holds. Finally, to complete our argument when V_1 and V_2 do not form a coalition partner, we assume following subcases, and then the same result holds for the following subcases.

Subcase 2.1 $|\pi'_{2,1}| = |\pi'_{2,2}| = 5$.

Subcase 2.2 $|\pi'_{2,1}| = 5$ and $|\pi'_{2,2}| = 4$, or $|\pi'_{2,1}| = 4$ and $|\pi'_{2,2}| = 5$.

Subcase 2.3 $|\pi'_{2,1}| = 5$ and $|\pi'_{2,2}| = 3$, or $|\pi'_{2,1}| = 3$ and $|\pi'_{2,2}| = 5$.

Subcase 2.4 $|\pi'_{2,1}| = 2$, $|\pi'_{2,2}| = 5$.

Subcase 2.5 $|\pi'_{2,1}| = 4$, $|\pi'_{2,2}| = 4$.

Subcase 2.6 $|\pi'_{2,1}| = 4$ and $|\pi'_{2,2}| = 3$, or $|\pi'_{2,1}| = 3$ and $|\pi'_{2,2}| = 4$.

Case 3. We can suppose there are 6 singleton sets in the partition and exactly one set of π_2 must consists of four vertices. Without loss of generality, let

$|V_1| = 4$ and $|V_j| = 1$ for $2 \leq j \leq 7$. Since G_{17} has no dominating set with less than three vertices, no two singleton sets can be in a coalition. Now every set V_j for $2 \leq j \leq 7$ must be in a coalition with V_1 . By Theorem 4, V_1 is in a coalition with at most $\Delta(G) + 1 = 4$ sets. That means V_1 is a coalition partner of at most 4 singleton sets. Therefore, remain at least three singleton set, neither of which can form a coalition with V_1 , and it is a contradiction. Then, we cannot create a partition of order 7. Hence, $C(G_{17}) \leq 6$.

To complete the proof, we create a c -partition of order 6 for the graph G_{17} and we show that this partition is the maximal partition. We first create a domatic partition. We can suppose $\pi_2'' = \{V_1'' = \{1, 7, 3\}, V_2'' = \{2, 4, 5, 6, 8, 9, 10\}\}$ be a domatic partition of a graph G_{17} , where $d(G_{17}) = 2$. Since any partition of a non-singleton, minimal dominating set into two nonempty sets creates two non-dominating sets whose union forms a coalition, so we can partition the minimal dominating set $V_1 = \{1, 7, 3\}$ into two sets $V_{1,1}'' = \{1\}$ and $V_{1,2}'' = \{7, 3\}$, which form a coalition. Now we create a partition π_2''' of sets and put the sets $V_{1,1}'' = \{1\}$ and $V_{1,2}'' = \{7, 3\}$ in this partition. To obtain other sets of partition π_2''' , let $V_2''' = \{2, 8, 4\} \subset V_2''$ be a minimal dominating set contained in V_2'' . Then, we can partition it into two non-dominating sets $V_{2,1}''' = \{2\}$ and $V_{2,2}''' = \{8, 4\}$, add these two sets to π_2''' . The set $T = \{5, 6, 9, 10\}$ remains which is not a dominating set, else there are at least 3 disjoint dominating sets in G , a contradiction, because $d(G_{17}) = 2$. The set T forms a coalition with other sets, so we can add T to π_2''' . Thereby creating a c -partition of G_{17} of order at least 5, however, we can create a c -partition with order 6 because there are two sets $T_1 = \{5, 6\} \subset T$ and $T_2 = \{9, 10\} \subset T$ form a coalition with $V_{2,1}''' = \{2\}$. Therefore, we can remove the set T and add the sets $T_1 = \{5, 6\}$ and $T_2 = \{9, 10\}$ to π_2''' . So $C(G_{17}) \geq 6$. Furthermore, as we have seen before, we have $C(G_{17}) \leq 6$. Therefore, $C(G_{17}) = 6$. So we can establish a maximal c -partition of G of order 6 as follows:

$$\pi_2''' = \{V_{1,1}'' = \{1\}, V_{1,2}'' = \{7, 3\}, V_{2,1}''' = \{2\}, V_{2,2}''' = \{8, 4\}, T_1 = \{5, 6\}, T_2 = \{9, 10\}\}.$$

Using the argument used to obtain the coalition number of the graph G_{17} , we can obtain the coalition number for other graphs belong to this part. □

4. Conclusion

Study of the coalition number of regular graphs is a main subject in the coalition theory. In this paper, we have determined the coalition number of the cubic graphs of order at most 10. We observed that the coalition numbers of all these graph are 6, 7, or 8. We think that this is true for all cubic graphs of order at least 6. This raises us to propose the following open problem.

Problem 1. Is it true that for any cubic graphs G of order at least 6, $C(G) \in \{6, 7, 8\}$?

Recently, total coalition has introduced and investigated in [1]. Two disjoint sets $V_1, V_2 \subseteq V$ are called a total coalition in G , if neither V_1 and V_2 is a total dominating set of G but $V_1 \cup V_2$ is a total dominating set. A total coalition partition of G is a vertex partition $\pi = \{V_1, V_2, \dots, V_k\}$ such that no set of π is a total dominating set but each set $V_i \in \pi$ forms a total coalition with another set $V_j \in \pi$. The maximum cardinality of a total coalition partition of G is called the total coalition number of G , denoted by $TC(G)$. It is natural to study the total coalition number of regular graphs, especially, cubic graphs. So we close the paper with the following open problems.

Problem 2. What is the total coalition numbers of the cubic graphs of order more than four?

Problem 3. What are the sharp bounds for the total coalition numbers of k -regular graphs?

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