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Some algebraic properties of the subdivision graph of a graph

S. Morteza Mirafzal

Department of Mathematics, Faculty of Basic Sciences, Lorestan University, Khorramabad, Iran
mirafzal.m@lu.ac.ir

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Abstract: Let $G = (V, E)$ be a connected graph with the vertex-set V and the edge-set E. The subdivision graph $S(G)$ of the graph G is obtained from G by adding a vertex in the middle of every edge of G. In this paper, we investigate some properties of the graphs $S(G)$ and $L(S(G))$, where $L(S(G))$ is the line graph of $S(G)$. We will see that $S(G)$ and $L(S(G))$ inherit some properties of G. For instance, we show that if $G \ncong C_n$, then $Aut(G) \cong Aut(L(S(G)))$ (as abstract groups), where C_n is the cycle of order n.

Keywords: subdivision graph, line graph, connectivity, automorphism group, Hamiltonian graph

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Constructing new families of graphs from families of graphs which are in the hand, in some aspects, can be one of the important tasks in graph theory. For example, line graphs are basic graph transformations with various number of results about their properties in the literature. The line graph method is a very important technique for constructing a larger graph from a given graph. The graph constructed by the line graph method can easily obtain many desirable properties from the original graph, such as degree, diameter, connectivity, eulericity, hamiltonicity, and so forth. The method of line graph has been widely used in the designing of interconnection networks.

Another method is the power graph of a given graph [2]. Given two graphs G and H and a positive integer k, we say that G is the k-th power of H (and denote this by $G = H^k$) if the vertex-sets of G and H coincide and two distinct vertices are adjacent in G if and only if they are at distance at most k in H. The graph H is then called a k-th root of G. In the case $k = 2$, we say that G is the square of H and H is the square root of G. It is known that if the graph G is a 2-connected graph, then the graph G^2 is a panconnected graph [4].

Let $G = (V, E)$ be a connected graph with the vertex-set V and the edge-set E. The *subdivision* graph $S(G)$ of the graph G is obtained from G by putting a new vertex in the middle of every edge of G. It is easy to check that the graph $S(G)$ is isomorphic to the bipartite graph with the vertex-set $V_1 = V \cup E$ in which two vertices $v \in V$ and $e \in E$ are adjacent if and only if v is incident to the edge e in the graph G, that is, $e = \{v, w\}$ for some $w \in V$. Hence in the sequel, we call the latter graph as the subdivision of the graph G.

In this paper, we investigate some properties of the graphs $S(G)$ and $L(S(G))$, where $L(S(G))$ is the line graph of $S(G)$. We will see that $L(S(G))$ inherits some properties of G, in such a way, it is possible to construct from the graph G other larger graphs with some desired properties. For instance, we will see how we can construct a 3-regular Hamiltonian graph of order greater than given integer n. Also, we determine the automorphism group of $S(G)$. We will see that if $G \ncong C_n$, the cycle of order n, then $Aut(G) \cong Aut(S(G)) \cong Aut(L(S(G)))$ (as abstract groups). This fact helps us in constructing large graphs with some desired symmetry properties.

1. Preliminaries

In this paper, a graph $G = (V, E)$ is considered as a finite undirected simple graph where $V = V(G)$ is the vertex-set and $E = E(G)$ is the edge-set. For all the terminology and notation not defined here, we follow [1, 2, 5].

A (connected) bipartite graph is said to be *biregular* if all vertices on the same side of the bipartition have the same degree. Particularly, we refer to a bipartite graph with parts of size m and n as an (r, s)-bipartite biregular graph if the m vertices in the same part each has degree r and the n vertices in the same part each has degree s. A graph G of order $n > 2$ is called *pancyclic* if G contains a cycle of length l for each integer l with $3 \leq l \leq n$. A graph G of order $n > 2$ is called *panconnected* if for every two vertices u and v, there is a u-v path of length l for every integer l with $d(u, v) \leq l \leq n - 1$. Note that if a graph G is panconnected, then G is pancyclic. A graph G of order $n > 2$ is called *Hamilton-connected* if for any pair of distinct vertices u and v, there is a Hamilton u-v path, namely, there is a u-v path of length $n - 1$. It is clear that if G is a panconnected graph then G is a Hamilton-connected graph. If $n > 2$, then the graph K_n, the complete graph on n vertices, is a panconnected graph.

A vertex cut of the graph G is a subset U of V such that the subgraph G-U which is induced by the set V-U is either trivial or not connected. The *connectivity* $\kappa(G)$ of a nontrivial connected graph G is the minimum cardinality of all vertex cuts of G. If we denote by $\delta(G)$ the minimum degree of G, then $\kappa(G) \leq \delta(G)$. A graph G is called *k-connected* (for a positive integer k) if $|V(G)| > k$ and $G - X$ is connected for every subset $X \subset V(G)$ with $|X| < k$. It is trivial that if a positive integer m is such that $m \leq \kappa(G)$, then G is an m-connected graph.

Theorem 1. *[3] If G is a 2-connected graph, then G^2 is Hamilton-connected.*

By Theorem 1, and [4] we can deduce the following result.

Theorem 2. *The square of a graph G is panconnected whenever G is a 2-connected graph.*

2. Main Results

Definition 1. Let $G = (V, E)$ be a connected graph. The subdivision graph $S(G)$ of G is a bipartite graph with the vertex set $V \cup E$ in which vertices $v \in V$ and $e \in E$ are adjacent if and only if the vertex v is incident on the edge e in the graph G. In other words, each edge $e = \{u, v\} \in E$ is deleted and replaced by two edges $\{u, w\}$ and $\{w, v\}$ with the new vertex $w = w_e$.

The graphs Q_3 (cube) and $S(Q_3)$ are depicted in Figure 1.

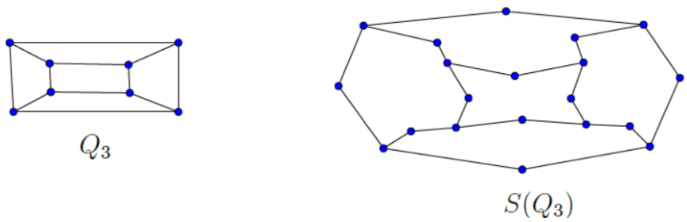

Figure 1. Q_3 **and** $S(Q_3)$

Example 1. Definition 1, implies that if G is the cycle C_4 then the graph $S(G)$ is the cycle C_8. In fact if $n \geq 3$ is an integer and $G = C_n$, then $S(G)$ is a connected 2-regular graph with $2n$ vertices, and hence $S(G)$ is isomorphic with the cycle C_{2n}.

Note Let $G = (V, E)$ be a graph and $e = \{x, y\}$ be an edge of G. In the sequel we write $e = xy$ instead of $e = \{x, y\}$.

Let $G = (V, E)$ be a connected graph. Definition 1, follows that if v is a vertex of degree k in the graph G, then $\deg(v)$ in the graph $S(G)$ is k whereas the degree of each $e \in E$ in $S(G)$ is 2. Hence, if G is a k regular graph, then $S(G)$ is a $(k, 2)$-bipartite biregular graph.
Let $G = (V, E)$ be a graph. If $P : v_1, v_2, \dots, v_m$ be a path (walk) in G, then $Q : v_1, v_1v_2, v_2, \dots, v_{m-1}, v_{m-1}v_m, v_m$ is a path (walk) in $S(G)$. In other words, corresponding to any path (walk) in G there is a path (walk) in $S(G)$. Nothing this observation, we can easily obtain the following result.

Proposition 1. *If $G = (V, E)$ is a connected graph, then $S(G)$ is a connected graph.*

Proof. The proof is straightforward. □

Let G be a graph and $D(G)$ be its diameter. We can easily deduce the following result.

Corollary 1. *Let $G = (V, E)$ be a connected graph, then we have $D(S(G)) \leq 2D(G) + 2$.*

The obtained bound in Corollary 1, is sharp, that is, there are graphs G with diameter D such that the diameter of the graph $S(G)$ is $2D + 2$. In fact if $n \geq 4$ and $G = K_n$, the complete graph of order n which is a graph with diameter 1, then it is easy to check that $D(S(G)) = 4 = 2 \times 1 + 2$.

Theorem 3. *If $G = (V, E)$ is a 2-connected graph, then the graph $S(G)$ is a 2-connected graph.*

Proof. We know that a graph X is a 2-connected graph if and only if the graph $X - w$ is a connected graph for every vertex w in the graph G. Hence we show that the graph $S(G) - w$ is a connected graph for every vertex w in the graph $S(G)$. Let w be a vertex in the graph $S(G)$. We consider two cases.

Case 1. $w \in V$.
We show that if v_1 and v_2 are vertices in the graph $S(G) - w$, then there is a path from v_1 to v_2 in the graph $S(G) - w$. There are three cases, namely, (i) $v_1, v_2 \in V - w$, (ii) $v_1 \in V - w$ and $v_2 \in E$, (iii) $v_1, v_2 \in E$.

(i) Let $v_1, v_2 \in V - w$. Since the graph G is a 2-connected graph, then there is a path P between v_1 and v_2 in the graph $G - w$. We now can easily check that there is a path from v_1 to v_2 in the graph $S(G) - w$.

(ii) Let $v_1 \in V - w$ and $v_2 \in E$. Let $v_2 = \{x, y\} = xy$, where $x, y \in V$. It is clear that $x \neq w$ or $y \neq w$. Let $x \neq w$. Then by the case (i), there is a path P from the vertex v_1 to the vertex x in the graph $G - w$. Now it is not hard to check that there is a path from v_1 to v_2 in the graph $S(G) - w$.

(iii) Let $v_1, v_2 \in E$. Let $v_1 = xy$ and $v_2 = uv$, where $x, y, u, v \in V$. We can assume that $x \neq w$ and $u \neq w$. Then, from the case (i) it follows that there is a path P from the vertex x to the vertex u in the graph $G - w$. We now can deduce that there is a path from x to u in the graph $S(G) - w$.

Case 2. $w \in E$.
Let $w = xy$ where $x, y \in V$. We show that if v_1 and v_2 are vertices in the graph $S(G) - w$, then there is a path from v_1 to v_2 in the graph $S(G) - w$. There are three cases, namely, (i) $v_1, v_2 \in V$, (ii) $v_1 \in V$ and $v_2 \in E - w$, (iii) $v_1, v_2 \in E - w$.

(i) Let $v_1, v_2 \in V$. Since the graph G is a 2-connected graph, then $\lambda(G) = \lambda \geq 2$, where $\lambda(G)$ is the edge connectivity of G. Note that in fact we have $\lambda(G) \geq \kappa(G)$ [2]. Hence we can construct a path P from v_1 to v_2 in the graph G such that P does not contain the edge $w = xy$. Thus, by a similar method which has been seen in the

proof of the Case 1 we can conclude that there is a path from v_1 to v_2 in the graph $S(G) - w$.
The proof for the cases (ii) and (iii) are similar to the case (i) of Case 2 and cases (ii) and (iii) of cases 1. □

Remark 1. Note that since $\delta(S(G))$, the minimum degree of the graph $S(G)$ is 2, then we have $\kappa(S(G)) \leq 2$. Hence if the graph $G = (V, E)$ is a 2-connected graph, then by Theorem 3, the connectivity of the graph $S(G)$ is maximal.

Let $G = (V, E)$ be a graph. The *line graph* $L(G)$ of the graph G is constructed by taking the edges of G as vertices of $L(G)$, and joining two vertices in $L(G)$ whenever the corresponding edges in G have a common vertex. If $e = xy$ is a vertex in $L(G)$, then $\deg_{L(G)} e = \deg(x) + \deg(y) - 2$. Hence if G is a connected k-regular graph, then $L(G)$ is a connected $2k - 2$-regular graph. Thus the valency of the line graph of the k-regular graph G is 'almost' twofold of the valency of the graph G.
From Theorem 3, some interesting results follow. Consider the graph $S^2(G)$, the square of the subdivision graph of the graph G. It is not hard to check that $S^2(G)$ is graph with the vertex-set $V \cup E$ and the edge set $E \cup E(L(G)) \cup E(S(G))$. Note that two vertices $v, w \in V$ are adjacent in $S^2(G)$ if and only if they are at distance 2 in $S(G)$, that is, there is an edge between v and w in the graph G, hence v and w are adjacent in G. By a similar reason, two vertices $e_1, e_2 \in E$ are adjacent in $S^2(G)$ if and only if they are adjacent in the graph $L(G)$. In the literature sometimes the graph $S^2(G)$ is referred to as the *total* graph of G, denoted by $T(G)$ [6]. Now, from Theorem 3, and Theorem 2, the following interesting result follows.

Corollary 2. *If the graph G is a 2-connected graph, then the total graph $T(G)$ is a panconnected graph.*

Let $G = (V, E)$ be a connected k-regular graph of order n with $k \geq 2$. We now want to investigate some properties of the graph $L(S(G))$, the line graph of the graph $S(G)$. The graph $LS = L(S(G))$ has some interesting properties. It is clear that the order of LS is $kn = 2|E(G)|$ which is greeter than the orders of the graphs G and $L(G)$. If $w = \{v, e\}$, $v \in V, e \in E$ is a vertex in the graph LS, then for its degree we have $\deg(w) = \deg(v) + \deg(e) - 2 = k + 2 - 2 = k$. Hence LS is a k-regular graph. In other words, the valency of the graph LS is equal to the valency of the graph G. By Proposition 1, the graph $S(G)$ is a connected graph, hence the graph $LS = L(S(G))$ is a k-regular connected graph. Therefore by a recursive method, we can construct connected k-regular graphs of orders greeter than any given integer i, from any connected k-regular graph X in the hand. Moreover if the graph X is a 2-connected graph, then by Theorem 3, the constructed graphs by our method are 2-connected.
If k is an even integer, then the degree of each vertex in the graph $S(G)$ is an even integer (namely, 2 or k), thus the graph $S(G)$ is Eulerian [2]. Therefore, if k is an

even integer and G is a connected k-regular graph, then the graph $LS = L(S(G))$ is a k-regular Hamiltonian graph.

Some symmetry properties of the graph $S(G)$

The group of all permutations of a set V is denoted by S$ym(V)$ or just S$ym(n)$ when $|V| = n$. A *permutation group* Γ on V is a subgroup of S$ym(V)$. In this case we say that Γ acts on V. If G is a graph with vertex-set V, then we can view each automorphism of G as a permutation of V, and so $Aut(G)$ is a permutation group on V. Let the group Γ act on V, we say that Γ is *transitive* (or Γ acts *transitively* on V) if there is just one orbit. This means that given any two element u and v of V, there is an element β of G such that $\beta(u) = v$. The graph G is called *vertex-transitive* if $Aut(G)$ acts transitively on $V(G)$. The action of $Aut(G)$ on $V(G)$ induces an action on $E(G)$ by the rule $\beta\{x, y\} = \{\beta(x), \beta(y)\}$, where $\beta \in Aut(G)$, and G is called *edge-transitive* if this action is transitive. The graph G is called *symmetric* (*arc-transitive*), if for all vertices u, v, x, y of G such that u and v are adjacent, and x and y are adjacent, there is an automorphism α such that $\alpha(u) = x$, and $\alpha(v) = y$. It is clear that a symmetric graph is vertex-transitive and edge-transitive.
One of the problems concerning a graph $G = (V, E)$ is the determination of its automorphism group. Although in most situations it is difficult to determine the automorphism group of a graph G, there are various papers concerning this matter in the literature, and some of the recent works include [9, 11–17].
Let $G = (V, E)$ be a connected graph and $S = S(G)$. In this subsection we show that the graph S inherits some symmetry properties from the graph G. Let α be an automorphism of the graph G. This automorphism induces an automorphism f_α of the graph S. In fact if we define the mapping f_α: $V(S) \to V(S)$ by this rule,

$$f_\alpha(w) = \begin{cases} \alpha(w) & \text{if } w \in V \\ \{\alpha(x), \alpha(y)\} & \text{if } w \in E, w = \{x, y\},\ x, y \in V \end{cases} \tag{1}$$

then f_α is an automorphism of the graph S with this property, $f_\alpha(V) = V,\ f_\alpha(E) = E$. Note that if $e = \{v, \{v, w\}\}$ is an edge of the graph S, then $f_\alpha(e) = \{\alpha(v), \{\alpha(v), \alpha(w)\}\}$, is an edge of graph S. It is easy to see that if $H = \{f_\alpha \mid \alpha \in Aut(G)\}$, then H is isomorphic to the group $Aut(G)$. Hence $Aut(S(G))$ contains a subgroup isomorphic to the group $Aut(G)$.

Proposition 2. *Let $G = (V, E)$ be a connected graph. If G is an arc-transitive graph, then the graph $L = L(S(G))$ is a vertex-transitive graph.*

Proof. It is clear that if a graph X is edge-transitive, then its line graph $L(X)$ is a vertex-transitive graph. Thus, we show that $S(G)$ is an edge-transitive graph. Let $e_1 = \{x, xy\}$ and $e_2 = \{u, uv\}$ be edges in $S(G)$. Since the graph G is an arc-transitive graph, then there is an automorphism $\alpha \in Aut(G)$ such that $\alpha(x) = u$ and $\alpha(y) = v$.

Then, from (1) we have;

$$f_\alpha(e_1) = \{\alpha(x), \alpha(x)\alpha(y)\} = \{u, uv\} = e_2$$

Since f_α is an automorphism of the graph $S(G)$, then $S(G)$ is an edge-transitive graph. □

Let $G = (V, E)$ be a connected graph of order n. If the degree of every vertex in G is 2, then G is the cycle C_n. Thus by Proposition 1, $S(G)$ is a 2-regular connected graph of order $2n$ and hence $S(G)$ is the cycle C_{2n}. We know that $Aut(C_n) \cong \mathbb{D}_n$, where $\mathbb{D}_n$ is the dihedral group of order $2n$. Therefore $Aut(S(G)) \cong \mathbb{D}_{4n} \ncong Aut(G)$. We show that this is the only exceptional case, that is, if $G \ncong C_n$, then $Aut(S(G)) \cong Aut(G)$. In the sequel, we need the following result.

Lemma 1. *Let $G = (U \cup W, E)$, $U \cap W = \emptyset$ be a connected bipartite graph. If f is an automorphism of the graph G, then $f(U) = U$ and $f(W) = W$, or $f(U) = W$ and $f(W) = U$.*

Proof. Automorphisms of G preserve the distance between vertices and since two vertices are in the same part if and only if they are at even distance from each other, the result follows. □

We have the following definition due to Sabidussi [18].

Definition 2. Let $G = (V, E)$ be a graph with the vertex-set V and the edge-set E. Let $N(v)$ denote the set of neighbors of the vertex v of G. We say that G is an irreducible graph if for every pair of distinct vertices $x, y \in V$ we have $N(x) \neq N(y)$.

From Definition 2, it follows that the cycle C_n, $n \neq 4$, is irreducible, but the complete bipartite graph $K_{m,n}$ is not irreducible, when $(m, n) \neq (1, 1)$. It is easy to check that if G is a simple connected graph of order $n \geq 3$, then its subdivision $S(G)$ is irreducible.

Lemma 2. *Let $G = (U \cup W, E)$, $U \cap W = \emptyset$ be a bipartite irreducible graph. If f is an automorphism of G such that $f(u) = u$ for every $u \in U$, then f is the identity automorphism of G.*

Proof. Let $w \in W$ be an arbitrary vertex. Since f is an automorphism of the graph G, then for the set $N(w) = \{u \mid u \in U, \{u, w\} \in E(G)\}$, we have $f(N(w)) = \{f(u) \mid u \in U, \{f(u), f(w)\} \in E(G)\} = N(f(w))$. On the other hand, since for every $u \in U$, $f(u) = u$, then we have $f(N(w)) = N(w)$, and therefore $N(f(w)) = N(w)$. Now since G is an irreducible graph we must have $f(w) = w$. Hence, for every vertex x in $V(G)$ we have $f(x) = x$ and thus f is the identity automorphism of the graph G. □

Theorem 4. *Let $G = (V, E)$ be a connected graph of order $n \geq 3$ such that $G \ncong C_n$. Then $Aut(S(G)) \cong Aut(G)$.*

Proof. We know that $S(G)$ is a connected bipartite graph with the vertex set $V_1 = V \cup E$ such that each of its vertices which is in E is of degree 2. Let f be an automorphism of the graph $S(G)$. Since $G \ncong C_n$, then then there is a vertex v in $V = V(G)$ such that $\deg(v) \neq 2$. We know that $\deg(f(v)) = \deg(v)$, thus $f(v) \notin E$. Therefore $f(v) \in V$, and hence by Lemma 1, we have $f(V) = V$. Let $\alpha = f_{|_V}$, be the restriction of the mapping f to V. Then α is a bijection of the set V. We assert that in fact α is an automorphism of the graph G. Let v and w be adjacent vertices in G. Then v and w, as vertices of $S(G)$, have a common neighbor in the graph $S(G)$, namely, the vertex $\{v, w\}$. Thus, $f(v) = \alpha(v)$ and $f(w) = \alpha(w)$ as vertices of $S(G)$ have a common neighbor in $S(G)$, and hence they are adjacent in the graph G. We now deduce that α is an automorphism of the graph G. Let f_α be the induced automorphism of α which is defined in (1). If we let $l = f^{-1}f_\alpha$, then l is an automorphism of the graph $S(G)$ such that $l(v) = v$ for every $v \in V$. Since $S(G)$ is an irreducible graph, hence from Lemma 2, it follows that $l = I$, the identity automorphism of $S(G)$, and hence $f = f_\alpha$. Therefore, $Aut(S(G)) \leq H = \{f_\alpha \mid \alpha \in Aut(G)\}$. It is clear that $H \leq Aut(S(G))$, hence we have $H = Aut(S(G))$. On the other hand $H \cong Aut(G)$, hence $Aut(S(G)) \cong Aut(G)$.

□

Let $G = (V, E)$ be a graph and $L(G)$ be its line graph. There is an important relation between $Aut(G)$ and $Aut(L(G)$. In fact, we have the following result [1, chapter 15].

Theorem 5. *Let G be a connected graph. The mapping $\theta : Aut(G) \to Aut(L(G)$ defined by the rule,*

$$\theta(g)\{u, v\} = \{g(u), g(v)\}, \ g \in Aut(G), \ \{u, v\} \in E(G),$$

is a group homomorphism and in fact we have;
(i) θ is a monomorphism provided $G \neq K_2$;
(ii) θ is an epimorphism provided G is not K_4, K_4 with one edge deleted, or K_4 with two adjacent edges deleted.

From Theorem 4, and Theorem 5, we have the following result.

Theorem 6. *Let $n \geq 5$ and $G = (V, E)$ be a connected graph such that $G \ncong C_n$. Then $Aut(L(S(G))) \cong Aut(G)$.*

Remark 2. Let $[n] = \{1, 2, \ldots, n\}$ and K_n be the complete graph with the vertex-set $[n]$. It is easy to see that $S(K_n)$ is the bipartite graph with the vertex-set $V = V_1 \cup V_2$, where $V_1 = \{v \ : \ v \subset [n], |v| = 1\}$ and $V_2 = \{v \ : \ v \subset [n], |v| = 2\}$, in which two vertices v and w are adjacent if and only if $v \subset w$ or $w \subset v$. Now consider the graph $G = L(S(K_n))$. This graph is the same as the graph which has been investigated in [11] under the name $L(n)$. In other words, we have $L(S(K_n)) \cong L(n)$. Figure 2. depicts the graph $L(4)$.

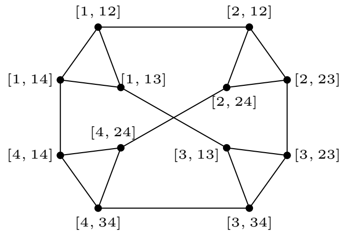

Figure 2. **The graph** $L(4)$

Note that in this figure $[i, ij]$ denotes the vertex $\{\{i\}, \{i, j\}\}$. The graph $L(n)$ has some interesting properties, for instance it is an integral graph with exactly five eigenvalues, -2,-1,0,$n-2$,$n-1$ [10]. Moreover, $L(n)$ is a Cayley graph if and only if n is a power of a prime integer [13]. An interesting property of the graph $S(K_n)$ has been appeared in [8] ($S(K_n)$ is (isomorphic to) the square root of the Johnson graph $J(n+1, 2)$, that is, $S^2(K_n)$ is isomorphic to the Johnson graph $J(n+1, 2)$). Concerning the automorphism group of the graph $L(n)$, since $Aut(K_n) \cong Sym([n])$, hence by Theorem 6, we have $Aut(L(n)) \cong Aut(L(S(K_n))) \cong Sym([n])$.

Hamiltonian properties of the graph $L(S(G))$

Let $G = (V, E)$ be a connected graph. We saw that if G is a k-regular graph with k is an even integer, then the graph $L(S(G))$ is a Hamiltonian graph. We now show that if G is Hamiltonian then $L(S(G))$ is a Hamiltonian graph. For our purpose we need the following result due to Harary and Nash-Williams [7].

Theorem 7. *Let G be a graph of order n with $n \geq 4$. Then the line graph $L(G)$ has a Hamiltonian cycle if and only if G has a closed trail which includes at least one vertex of each edge of G.*

Theorem 8. *Let $G = (V, E)$ be a connected graph of order n with $n \geq 3$. If G is a Hamiltonian graph, then $L(S(G))$ is a Hamiltonian graph.*

Proof. When $n = 3$, then $G = C_3$, the cycle of order 3, which is a Hamiltonian graph. Note that $S(G)$ is C_6, hence $L(S(G))$ is C_6, which is a Hamiltonian graph. Hence in the remainder of the proof, we assume that $n \geq 4$. Let $G = (V, E)$ be a Hamiltonian graph and $V = \{v_1, v_2, \dots, v_n\}$. Hence, G has a Hamiltonian cycle $C : w_1, w_2, \dots, w_n, w_1$, where $\{v_1, v_2, \dots, v_n\} = \{w_1, w_2, \dots, w_n\}$. Therefore $P : w_1, w_1w_2, w_2, w_2w_3, \dots, w_{n-1}, w_{n-1}w_n, w_n, w_nw_1, w_1$ is a closed trail (in fact is a cycle) in the graph $S(G)$. If $e = \{w_i, w_iw_j\}$ is an edge of the graph $S(G)$, then P clearly includes the vertex w_i of e. Therefore by Theorem 7, the graph $L(S(G))$ is a Hamiltonian graph. $\square$

We know that K_n is a Hamiltonian graph, hence by Theorem 8, we have the following result.

Corollary 3. *The graph $L(n) \cong L(S(K_n))$ is a Hamiltonian graph.*

3. Conclusion

In this paper, we investigated some properties of the graph $S(G)$, the subdivision of G, and its line graph $L(S(G))$. We saw in Theorem 3, Theorem 4, Theorem 6, and Theorem 8, that the graphs $S(G)$ and $L(S(G))$ inherit some desired properties of the graph G. In particular, $L(S(G))$ inherits Hamiltonicity and some symmetry properties from G.

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