Communications in Combinatorics and Optimization
Vol. ??, No. ?? (????), pp. 1-11
DOI:10.22049/CCO.2023.27338.1238

CCO
Commun. Comb. Optim.

Research Article

1-Edge contraction: Total vertex stress and confluence number

J. Shiny[1,*], J. Kok[2]

[1]Mathematic Research Center, Mary Matha Arts and Science College, Mananthavady, Kerala, India.
shinyjoseph314@gmail.com

[2]Independant Mathematics Researcher, City of Tshwane, South Africa & Visiting Faculty at CHRIST (Deemed to be a University), Bangalore, India.
jacotype@gmail.com

Received: 18 July 2021; Accepted: 22 March 2023
Published Online: 10 April 2023

Abstract: This paper introduces certain relations between 1-edge contraction and the total vertex stress and the confluence number of a graph. A main result states that if a graph G with $\zeta(G) = k \geq 2$ has an edge v_iv_j and a ζ-set $\mathcal{C}_G$ such that $v_i, v_j \in \mathcal{C}_G$ then, $\zeta(G/v_iv_j) = k - 1$. In general, either $\mathcal{S}(G/e_i) \leq \mathcal{S}(G/e_j)$ or $\mathcal{S}(G/e_j) \leq \mathcal{S}(G/e_i)$ is true. This observation leads to an investigation into the question: for which edge(s) e_i will $\mathcal{S}(G/e_i) = \max\{\mathcal{S}(G/e_j) : e_j \in E(G)\}$ and for which edge(s) will $\mathcal{S}(G/e_j) = \min\{\mathcal{S}(G/e_\ell) : e_\ell \in E(G)\}$?

Keywords: edge contraction, confluence number, total vertex stress.

AMS Subject classification: 05C012, 05C30, 05C38

1. Introduction

Unless stated otherwise, initial graphs will be finite, undirected and connected simple graphs. The *path* (*cycle*, *complete graph*, respectively) of order n is denoted by P_n (C_n, K_n, respectively). Contracting an edge $e_i = v_jv_k$ in G means to merge vertices v_j and v_k into a new vertex say, $v_{j/k}$ such that $N(v_{j/k}) = (N(v_j) \cup N(v_k)) \backslash \{v_j, v_k\}$. It can result in multiple edges. Convention is to eliminate all edges except one, in each of the multiple edge clusters. The edge contraction operation is denoted by G/v_jv_k or G/e_i. For clarity of other graph notation and concepts, see [1, 5, 11].
Recall that for a non-complete graph G, a non-empty subset $\mathcal{X} \subseteq V(G)$ is said to be

* *Corresponding Author*

a confluence set if for every unordered pair $\{u, v\}$ of distinct vertices (if such exist) in $V(G)\backslash\mathcal{X}$ for which $d_G(u, v) \geq 2$ there exists at least one shortest uv-path with at least one internal vertex, $w \in \mathcal{X}$. Such internal vertex is called a *confluence vertex* of G. For a complete graph the convention is that $\mathcal{X} = \emptyset$.
A minimal confluence set $\mathcal{X}$ has no proper subset which is a confluence set of G. The *confluence number* of G denoted by $\zeta(G)$ is the minimum cardinality of some minimal confluence set. Put differently, the confluence number of a graph G is the number of vertices in the smallest confluence set of G. A minimal confluence set with the minimum cardinality over all minimal confluence sets is simply called a *minimum confluence set* or a ζ-set. A minimum confluence set is denoted by $\mathcal{C}_G$ or if the context is clear by $\mathcal{C}$. From the definition of a ζ-set, it follows that the "closest" distinct confluence vertices in a ζ-set are at distance $1 \leq d(v_i, v_j) \leq 3$. The notions of confluence number and confluence sets play an important role in the study of *centrality and betweenness* in graphs. See a recent short survey with comprehensive references in [2].
By convention, we have that for a complete graph, $\zeta(K_n) = 0$. A lollipop graph $L^{\boxtimes}(m, n)$, $m \geq 3$, $n \geq 1$ is obtained from a complete graph K_m and a path P_n by joining one end-vertex of the path with an edge (or bridge) to one vertex of K_m. Clearly $\zeta(L^{\boxtimes}(m, 1)) = 1$. If the edge v_iu_1, $v_i \in V(K_m)$ is the bridge then, either $\{v_i\}$ or $\{u_1\}$ is a minimum confluence set. Also, $L^{\boxtimes}(m, 1)/v_iu_1$ is complete. So by convention $\zeta(L^{\boxtimes}(m, 1)/v_iu_1) = 0 = \zeta(L^{\boxtimes}(m, 1)) - 1$. This trivial reduction in the value of $\zeta(G)$, if possible, finds important consideration in real life applications. Edge contraction plays an important role in the study of topological properties and indices of graphs in the field of mathematical chemistry. See [3, 4]. Applications could possibly arise in the study of biological mutations as well.
Alfonso Shimbel introduced the notion of *vertex stress* in a graph G denoted by $\mathcal{S}_G(v)$, $v \in V(G)$ (see [9]). Recall that the vertex stress of vertex $v \in V(G)$ is the number of times v is contained as an internal vertex in all shortest paths between all pairs of distinct vertices in $V(G)\backslash\{v\}$. Formally stated, $\mathcal{S}_G(v) = \sum\limits_{u\neq w\neq v\neq u} \sigma(v)$ with $\sigma(v)$ the number of shortest paths between vertices u, w which contain v as an internal vertex. The total stress of a graph G is defined by $\mathcal{S}(G) = \sum_{v\in V(G)} \mathcal{S}_G(v)$. Also see [10] in respect of *stress regular graphs*.
We observe that in general either $\mathcal{S}(G/e_i) \leq \mathcal{S}(G/e_j)$ or $\mathcal{S}(G/e_j) \leq \mathcal{S}(G/e_i)$ for any two edges $e_i, e_j \in E(G)$ .This observation leads to an investigation into the question: for which edge(s) e_i will $\mathcal{S}(G/e_i) = max\{\mathcal{S}(G/e_j) : e_j \in E(G)\}$ and for which edge(s) will $\mathcal{S}(G/e_j) = min\{\mathcal{S}(G/e_\ell) : e_\ell \in E(G)\}$? Note that at least one ζ-set must be found to obtain $\zeta(G)$. Also, vertex stress $\mathcal{S}_G(v)$, $\forall\ v \in V(G)$ must be determined to obtain $\mathcal{S}(G)$. In certain applications other considerations such as social political, social demographic or global economical factors may dictate that the reduction in total vertex stress due to edge contraction must be minimized or maximized. In the final analysis, the sound principle of *studying mathematics for the sake of mathematics* serves as motivation for this research. For the purpose of this paper the aforesaid narrows the study down to mainly, the relations between edge contraction and the

confluence number and the total vertex stress in graphs.

Theorem 1. *Let G be a graph. Then $\zeta(G/e_i) \leq \zeta(G)$ for any edge e_i in G.*

Proof. Consider any shortest path P_{G/e_i} in G/e_i where $e_i = v_iv_j$. After contraction of e_i label the resultant vertex $v_{i/j}$. If $v_{i/j} \notin P_{G/e_i}$, then P_{G/e_i} is a path in G and hence there is a confluence vertex corresponding to P_{G/e_i}. If $v_{i/j} \in P_{G/e_i}$, then there is a path P in G such that either $v_i \in P$ or $v_j \in P$ or $v_i, v_j \notin P$ or e_i is an edge of P. The aforesaid is valid despite the fact that a new shortest path (or paths) may result in respect of some pair (or pairs) of vertices. If e_i is an edge of a path of length 2 in G , then that path reduced to an edge in G/e_i. Hence, a vertex $v_i \in \mathcal{C}_G$ is either required and is internal to some shortest path (or paths) in G/e_i or it is not required anymore. Therefore $\zeta(G/e_i) \leq \zeta(G)$. □

The next result is an immediate consequence of Theorem 1.

Corollary 1. *If there exists an edge v_iv_j in G such that both v_i and v_j are not in any ζ-set of G, then $\zeta(G/v_iv_j) = \zeta(G)$.*

Note that the corollary does not only exclude that the vertices v_i, v_j must not be in any identical ζ-set. For some families of graphs $\mathcal{F}$ the contraction of any edge e_i of $G \in \mathcal{F}$ yields $G/e_i \in \mathcal{F}$. Such graphs are said to be *family equivalent* or $/$-equivalent graphs under the $/$-operation. Cycle and paths serve as examples of such families.

Theorem 2. *For a graph G which is $/$-equivalent, $\mathcal{S}(G/e_i) \leq \mathcal{S}(G)$ for any edge e_i in G.*

Proof. If G is $/$-equivalent, it implies that $G, G/e_i \in \mathcal{F}$ for some $\mathcal{F}$. Let edge $e_i = v_jv_k \in E(G)$. After contraction of e_i label the resultant vertex $v_{j/k}$. Since the adjacency of each v_j is well-defined in G the parameter $\mathcal{S}_G(v_j)$ is a well-defined non-decreasing function $f(n)$ (n the order of G). Furthermore the order of G/e_i is 1 less than the order of G. Therefore $\mathcal{S}_{G/e_i}(v_{j/k}) \leq \mathcal{S}_G(v_j) + \mathcal{S}_G(v_k)$. By the definition of total vertex stress of a graph the result is settled. □

The next corollary is an immediate consequence of Theorem 2.

Corollary 2. *In a graph G which is $/$-equivalent and any edge e_j in G, there exists at least one vertex $v_i \in V(G) \cap V(G/e_j)$ such that $\mathcal{S}_{G/e_j}(v_i) \leq \mathcal{S}_G(v_i)$.*

2. 1-Edge contraction: Confluence number

This section begins with a main result.

Theorem 3. *If a graph G with $\zeta(G) = k \geq 2$ has an edge v_iv_j and a ζ-set $\mathcal{C}_G$ such that $v_i, v_j \in \mathcal{C}_G$, then $\zeta(G/v_iv_j) = k - 1$.*

Proof. Let $\zeta(G) = k \geq 2$ and G has an edge v_iv_j and a ζ-set $\mathcal{C}_G$ such that $v_i, v_j \in \mathcal{C}_G$. Consider the graph G/v_iv_j. Let the contracted edge constitute the vertex $v_{i/j}$. We have

$$N_{G/v_iv_j}[v_{i/j}] = (N_G(v_i)\backslash\{v_j\}) \cup (N_G(v_j)\backslash\{v_i\}) \cup \{v_{i/j}\}.$$

Note that all shortest paths in G which contained the internal confluence vertex (or vertices) v_i or v_j, now has the vertex $v_{i/j}$ as an internal confluence vertex in G/v_iv_j. So $v_{i/j}$ is a necessary and sufficient confluence vertex for all such shortest paths in G/v_iv_j. Furthermore, all shortest paths in G which do not contain either v_i or v_j remain shortest paths in G/v_iv_j. Hence, $(\mathcal{C}_G\backslash\{v_i, v_j\}) \cup \{v_{i/j}\}$ is a ζ-set of G/v_iv_j and $\zeta(G/v_iv_j) = k - 1$. □

Clearly Theorem 3 holds in respect of G/v_iv_j. As technique of proof it is convenient (and permissible) to refer to either, the "disappearance" of vertex v_i or, the "disappearance" of vertex v_j.
Example 1. Consider the path $P_6 = v_1v_2v_3v_4v_5v_6$. It is easy to verify that the sets $\{v_1, v_4\}$, $\{v_2, v_4\}$, $\{v_2, v_5\}$, $\{v_3, v_4\}$, $\{v_3, v_5\}$ and $\{v_3, v_6\}$ are all the possible ζ-sets of P_6. Theorem 3 applies to the existence of the ζ-set, $\{v_3, v_4\}$.
Note that because cycles are /-equivalent, any edge $e_i \in E(C_5)$ yields $\zeta(C_5/e_i) = 1 = 2 - 1 = \zeta(C_5) - 1$. This observation prompts the next theorem.
Recall that if a graph G does not have an induced subgraph H, then G is said to be H-free.

Theorem 4. *If a C_5-free graph G with $\zeta(G) = k \geq 2$ has $\zeta(G/v_iv_j) = k - 1$ for some edge v_iv_j, then G has an edge v_kv_l which is not necessarily distinct from edge v_iv_j, and G has a ζ-set $\mathcal{C}_G$ such that $v_k, v_l \in \mathcal{C}_G$. Hence, by Theorem 3 it follows that $\zeta(G/v_kv_l) = k - 1$.*

Proof. Let G be a C_5-free graph with $\zeta(G) = k \geq 2$ and $\zeta(G/v_iv_j) = k - 1$ for some edge v_iv_j. If the edge v_iv_j has its vertices $v_i, v_j \in \mathcal{C}_G$ for some ζ-set the result is settled. Assume that for all confluence sets $\mathcal{C}_G$, all pairs of "closest" confluence vertices $v_k, v_t \in \mathcal{C}_G$ has distance, $2 \leq d(v_k, v_t) \leq 3$. Consider any such shortest v_kv_t-path. Note that without loss of generality, the cases say, either $v_i = v_k$, $v_j \neq v_l$ or $v_i \neq v_k$, $v_j \neq v_l$ is implicitly permissible in Cases 1 and 2 below.

Case 1. Assume $d(v_k, v_t) = 2$. Consider a shortest path $v_kv_lv_t$ and contract the edge v_lv_t. Hence, $d_{G/v_lv_t}(v_k, v_{l/t}) = 1$.
Subcase 1.1. If the "disappearance" of vertex v_t results in a reduced confluence number, it implies that vertex v_l can substitute vertex v_t as a confluence vertex. The aforesaid is true because contracting the edge v_lv_t has a commutative interpretation i.e. either it is stated that vertex v_l "disappeared" or it is stated that vertex v_t "disappeared". Hence, $G/v_lv_t \cong G/v_tv_l$. Thus, it implies that both, $v_k, v_l \in \mathcal{C}$ for some smallest confluence set is valid. It means that an edge v_kv_l does exist in G such that,

$\zeta(G/v_kv_l) = k-1$.
Subcase 1.2. If the 'disappearance' of vertex v_t does not reduce the confluence number and this remains valid for all possible pairs over all possible smallest confluence sets, it implies that either $\zeta(G) < k$ in the first instance or $\zeta(G/v_iv_j) = k$. These possible contradictions yield Subcase 1(a) as the only possibility. This settles the result.

Case 2. Assume $d(v_k, v_t) = 3$. Consider the shortest path $v_kv_lv_sv_t$ and contract the edge v_sv_t. Hence, $d_{G/v_sv_t}(v_k, v_{s/t}) = 2$ which implies that a path $v_kv_lv_{s/t}$ exists. Furthermore, either $\zeta(G/v_sv_t) = k \Rightarrow \zeta(G/v_iv_j) = k$ which is a contradiction, or the vertex v_l can substitute vertex v_t as a confluence vertex. Hence, the edge v_kv_l exists in G with $v_k, v_l \in \mathcal{C}_G$. This settles the result. □

Corollary 3. *If a graph G with $|E(G)| \geq 1$ has $\zeta(G/v_iv_j) = k-1$ for any edge $v_iv_j \in E(G)$, then each ζ-set of G/v_iv_j is a subset of some ζ-set of G.*

Proof. The corollary is a direct consequence of the proof of Subcase 1.2 together relaxing the C_5-free condition. Furthermore, the vertex $v_{i/j}$ may be argued to be, either vertex v_i or vertex v_j. □

The following theorem is an immediate consequence of Theorems 3 and 4.

Theorem 5. *Let G be a C_5-free graph with $\zeta(G) = k \geq 2$. Then $\zeta(G/v_rv_s) = k-1$ for some edge v_rv_s if and only if G has a ζ-set $\mathcal{C}_G$ and an edge v_iv_j such that $v_i, v_j \in \mathcal{C}_G$.*

Theorem 5 finds illustrative application P_6. Let $\mathfrak{Z}_G = \{$all ζ-sets of $G\}$. From Example 1, it follows that,

$$\mathfrak{Z}_{P_6} = \{\{v_1, v_4\}, \{v_2, v_4\}, \{v_2, v_5\}, \{v_3, v_4\}, \{v_3, v_5\}, \{v_3, v_6\}\}.$$

Since $\zeta(P_6/v_1v_2) = 1 = \zeta(P_6) - 1$ the edge v_3v_4 exists and $\{v_3, v_4\}$ is a ζ-set of P_6 and so $v_3, v_4 \in \mathcal{C}_{P_6}$. We observe that each vertex v_i, $i = 1, 2, 3, 4, 5, 6$ is in some ζ-set of P_6. This prompts a corollary which is equivalent to Corollary 1.

Corollary 4. *If a graph G with $\zeta(G) = k \geq 2$ has $\zeta(G/v_iv_j) = k-1$ for some edge v_iv_j then each $v_k \in V(G)$ is in some ζ-set of G.*

Corollary 4 finds illustrative application in C_5. If a graph G with $\zeta(G) = k \geq 2$ has $\zeta(G/v_iv_j) = k-1$ for some edge v_iv_j, then G is said to be, 1^e-*tractable*. Note that some graphs H with $\zeta(H) = 1$ are 1^e-tractable as well. However, complete graphs are not 1^e-tractable. Some well known graph families will be discussed. For sake of convention to be used, we then recall the definition of each family of graphs.
(a) Double Star S_{k_1,k_2}, $k_1 \geq k_2 \geq 1$ is obtained by relabeling the two vertices of path P_2 as v_0, u_0 respectively whereafter, pendant vertices v_i, $i = 1, 2, 3, \ldots, k_1$ are

attached to v_0 and pendant vertices u_j, $j = 1, 2, 3, \dots, k_2$ are attached to u_0.
(b) A wheel graph W_n, $n \geq 3$ is obtained from a cycle C_n ($v_i's$ called rim vertices) by adding a central vertex v_0 together with the edges (or spokes) v_0v_i, $1 \leq i \leq n$.
(c) A helm graph H_n is obtained from a wheel graph W_n by attaching a pendant vertex (or leaf) u_i to v_i, $\forall\ i$.
(d) A flower graph Fl_n is obtained from a helm graph H_n by adding the edges v_0u_i, $\forall\ i$.
(e) A gear graph G_n is obtained from a wheel graph W_n by inserting a vertex u_i on the edge v_iv_{i+1}, $\forall\ i$ and $n + 1 \equiv 1$.
(f) A sunlet graph $S_n^{\ominus}$, $n \geq 3$ is obtained by taking cycle C_n together the isolated vertices u_i, $1 \leq i \leq n$ and adding the pendant edges v_iu_i.
(g) A sun graph $S_n^{\boxtimes}$, $n \geq 3$ is obtained by taking the complete graph K_n on the vertices $v_1, v_2, v_3, \dots, v_n$ together the isolated vertices u_i, $1 \leq i \leq n$ and adding the edges v_iu_i, u_iv_{i+1} and $n + 1 \equiv 1$. A sun graph has a boundary cycle denoted by $C^b(S_n^{\boxtimes}) = v_1u_1v_2u_2v_3u_3 \cdots u_nv_1$.

Proposition 1. *(a) A path P_n is 1^e-tractable if and only if $n \equiv 0 \pmod 3$.*
(b) A cycle C_n is 1^e-tractable if and only if $n \in \{4, 5, 7 + 3i \mid i = 0, 1, 2, \dots\}$.
(c) A double star S_{k_1,k_2} is 1^e-tractable if $k_1 \geq 2$, $k_2 \geq 2$.
(d) For $n = 3$, $n \geq 5$ a wheel graph is not 1^e-tractable. However W_4 is 1^e-tractable.
(e) A helm graph H_n is 1^e-tractable for $n \geq 3$.
(f) A flower graph Fl_n is not 1^e-tractable for $n \geq 3$.
(g) For $n \geq 5$ and odd a gear graph G_n is 1^e-tractable. The gear graph G_3 and gear graphs G_n, $n \geq 4$ and even are not 1^e-tractable.
(h) For $n \geq 3$ and odd a sunlet graph $S_n^{\ominus}$ is 1^e-tractable. For $n \geq 4$ and even a sunlet graph $S_n^{\ominus}$ is not 1^e-tractable.
(i) $S_3^{\boxtimes}$ is not 1^e-tractable. For $n \geq 4$ a sun graph $S_n^{\boxtimes}$ is 1^e-tractable.

Proof. (a) Since P_1, P_2 are complete, they are not 1^e-tractable. The path P_3 for which $\zeta(P_3) = 1$ is 1^e-tractable because contracting any edge results in a complete graph P_2 for which $\zeta(P_2) = 0$. Furthermore, it is known from [6] that $\zeta(P_n) = \lfloor \frac{n}{3} \rfloor$, $n \geq 3$. That settles the result '1^e-tractable if $n \equiv 0 \pmod 3$'. The converse follows from an easy contradiction by applying Theorem 3.

(b) The fact that $\zeta(C_3) = 0$, $\zeta(C_4) = 1$, $\zeta(C_5) = 2$ yields the result for the exceptions. Furthermore, it is known from [6] that $\zeta(C_n) = \lceil \frac{n}{3} \rceil$, $n \geq 5$. That settles the result '1^e-tractable if $n = 7 + 3i$, $i = 0, 1, 2, \dots$'. The converse follows from an easy contradiction provided the exceptions are acknowledged.

(c) If $\min\{k_1, k_2\} = 2$, then $\{v_0, u_0\}$ is a ζ- set of S_{k_1,k_2}. If $k_1 \geq 3$ and $k_2 \geq 3$, then $\mathcal{C}_{S_{k_1,k_2}} = \{v_0, u_0\}$ is the unique ζ- set of S_{k_1,k_2}. Since v_0u_0 is an edge of the double star, S_{k_1,k_2} is 1^e-tractable if $\min\{k_1, k_2\} \geq 2$ by Theorem 3.

(d) Since $\zeta(W_3) = 0$, it is not 1^e-tractable. Since $\zeta(W_n) = 1$, $n \geq 5$ and the contraction of any edge does not yield a complete graph, such wheel graphs are not

1^e-tractable. However, contracting a rim edge of W_4 yield the complete graph K_4. Therefore, W_4 is 1^e-tractable.

(e) It is known from [6] that $\zeta(H_n) = \lceil \frac{n}{2} \rceil + 1$ and that $v_0 \in \mathcal{C}$. Since some rim vertex $v_i \in \mathcal{C}$ and the edge v_0v_i exists, the result follows from Theorem 3.

(f) It is known from [6] that $\zeta(Fl_n) = 1$ and contracting any edge does not yield a complete graph thus the result.

(g) Part 1. The inner-area enclosed by the cycle $C'_{2n} = v_1u_1v_2u_2 \cdots v_nu_nv_1$ can be partitioned into n planar areas, each enclosed by a C_4. For all pairs v_i, v_j it is necessary and sufficient that $v_0 \in \zeta$-set. Let $n \geq 5$ be odd. Without loss of generality, an optimal minimal confluence set is given by $X_1 = \{v_0, u_1, u_3, \ldots, u_{n-2}, u_{n-1}\}$ or $X_2 = \{v_0, u_1, u_3, \ldots, u_{n-2}, v_n\}$ or $X_3 = \{v_0, u_1, u_3, \ldots, u_{n-2}, u_n\}$. It follows that a gear graph G_n does not have a parametric unique ζ-set for n is odd (see [7]). Since the edge v_0v_n exists, the result follows by Theorem 3.
Part 2. For G_3 and up to isomorphism the ζ-set $\{u_1, v_3\}$ is unique. Since $u_1v_3 \notin E(G_3)$, the gear graph G_3 is not 1^e-tractable. For $n \geq 4$ and even, reasoning similar to that in Part 1 show that up to isomorphism the ζ-set $X_1 = \{v_0, u_1, u_3, \ldots, u_{n-2}, u_{n-1}\}$ is unique. It follows that since no edge v_0u_i can exists that for $n \geq 4$ and even the gear graphs G_n are not 1^e-tractable.

(h) Part 1. It follows easily that up to isomorphism the sets $X_1 = \{v_1, v_3, v_5, \ldots, v_{n-2}, v_n\}$ and $X_2 = \{v_1, v_3, v_5, \ldots, v_{n-2}, v_{n-1}\}$ are the only distinguishable ζ-sets. Since $v_1, v_n \in X_1$ and edge v_1v_n exists the result follows from Theorem 3.
Part 2. It is known from [7] that up to isomorphism, the set
$X_1 = \{v_1, v_3, v_5, \ldots, v_{n-2}\}$ is the unique ζ-set. Since no pair of distinct vertices $v_i, v_j \in X_1$ exist such that edge $v_iv_j \in E(S_n^{\ominus})$, the result follows from Theorem 5.

(i) Only the boundary cycle of a sun graph in the cycle $C^b(S_n^{\boxtimes})$ requires consideration because of the existence of the clique K_n.
Part 1. For $S_3^{\boxtimes}$ and up to isomorphism the ζ-set $\{v_1, u_2\}$ is unique. Since $v_1u_2 \notin E(S_3^{\boxtimes})$ the result is immediate.
Part 2. It is easy to verify that for $n \geq 4$ any ζ-set contains a pair of distinct vertices v_i, v_j. Furthermore, $v_iv_j \in E(S_n^{\boxtimes})$. Hence, the result. $\square$

3. 1-Edge contraction: Total vertex stress

Recall that the total vertex stress in a graph G is given by $\mathcal{S}(G) = \sum\limits_{v \in V(G)} \mathcal{S}_G(v)$.
Note that an edge $v_iv_j \in E(G)$ can be such that for some ζ-set of G: (i) exactly one vertex say, v_i belongs to $\mathcal{C}_G$ or (ii) both v_i, v_j belong to $\mathcal{C}_G$ or (iii) $v_i, v_j \notin \mathcal{C}_G$.

Consider a lollipop graph $L^{\boxtimes}(m,1)$, $m \geq 3$ as mentioned in Section 1. It is trivial that $\mathcal{S}(L^{\boxtimes}(m,1)) = m-1$. It is also obvious that $\mathcal{S}(L^{\boxtimes}(m,1)/v_iu_1) = 0$, $\mathcal{S}(L^{\boxtimes}(m,1)/v_iv_j) = m-2$. So the maximum and minimum reduction in total vertex stress due to some edge contraction is respectively, $n-1$ and 1. We observe that for a graph G the values $\mathcal{S}(G/v_iv_j)$ and $\mathcal{S}(G/v_kv_l)$ may differ. Therefore an edge $e_i \in E(G)$ exists which yields $\mathcal{S}_{max}(G/e_i)$ and an edge $e_j \in E(G)$ exists which yields $\mathcal{S}_{min}(G/e_j)$. It implies that $\mathcal{S}_{min}(G/e_j) \leq \mathcal{S}_{max}(G/e_i)$.
Let $\Xi_{max}(G/e_j) = \mathcal{S}(G) - \mathcal{S}_{min}(G/e_j)$ and $\Xi_{min}(G/e_i) = \mathcal{S}(G) - \mathcal{S}_{max}(G/e_i)$, $e_i, e_j \in E(G)$ denote these respective reductions. A graph G for which $\Xi_{max}(G/e_j) = \Xi_{min}(G/e_i)$ is said to be *stress-stable* or *stable in respect of stress*. Put differently, G is stress-stable if and only if $\mathcal{S}(G) - \mathcal{S}(G/e_i) = \Xi(G/e_i) = constant$, for all $e_i \in E(G)$. It is trivial that if G is complete then G is stress-stable. Recall a result from [8].

Proposition 2. *[8] The total vertex stress in a path P_n, $n \geq 1$ is given by* $\mathcal{S}(P_n) = \frac{n(n-1)(n-2)}{6}$.

Theorem 6. *A path P_n, $n \geq 1$ is stress-stable.*

Proof. Since P_1, P_2 are complete, the statement holds. It follows that for any edge of P_n, $n \geq 3$ the graph operation i.e. edge contraction yields a reduction in total vertex stress equal to $\mathcal{S}(P_n) - \mathcal{S}(P_{n-1}) = \frac{(n-1)(n-2)}{2}$. Our interest lies in $n \geq 3$ since $n = 1, 2$ have been accounted for. For any $n \geq 3$ the contraction of any edge yields another path of order $n-1$. It implies that the function $f(n) = \frac{(n-1)(n-2)}{2}$ is independant of the selected edge $e_i = v_iv_{i+1}$, $1 \leq i \leq n-1$. Therefore $\Xi_{max}(P_n/e_i) = \Xi_{min}(P_n/e_j)$, $e_i, e_j \in E(P_n)$. □

From [10] we recall.

Theorem 7. *[10] The vertex stress of any vertex in a cycle C_{2n}, $n \geq 2$ is* $\mathcal{S}_{C_{2n}}(v) = \frac{n(n-1)}{2}$.

Theorem 8. *[10] The vertex stress of any vertex in a cycle C_{2n+1}, $n \geq 1$ is* $\mathcal{S}_{C_{2n+1}}(v) = \frac{n(n-1)}{2}$.

Theorems 7 and 8 with the definition of total vertex stress imply the next corollary.

Corollary 5. *(a) The total vertex stress of a cycle C_{2n}, $n \geq 2$ is* $\mathcal{S}(C_{2n}) = \frac{2n^2(n-1)}{2}$.
(b) The total vertex stress of a cycle C_{2n+1}, $n \geq 1$ is $\mathcal{S}(C_{2n+1}) = \frac{n(2n+1)(n-1)}{2}$.

Theorem 9. *A cycle C_m, $m \geq 3$ is stress-stable.*

Proof. Case 1. Let $m = 2n$, $n = 2, 3, \ldots$. It follows that for any edge of C_{2n}, $n \geq 1$ the graph operation i.e. edge contraction yields a reduction in total vertex stress equal to $\mathcal{S}(C_{2n}) - \mathcal{S}(C_{2n-1}) = \frac{(n-1)(5n-2)}{2}$. For any $m \geq 3$ the contraction of any edge yields another cycle of order $m-1$. It implies that the function $f(n) \mapsto \frac{(n-1)(5n-2)}{2}$ is independant of the selected edge $e_i = v_iv_{i+1}$, $1 \leq i \leq n(= \frac{m}{2})$. Therefore $\Xi_{max}(C_m/e_i) = \Xi_{min}(C_m/e_j)$, $e_i, e_j \in E(C_m)$. □

Observation 1. If the graphs $G/v_iv_j \cong G/v_kv_l$ for any two distinct edges $v_iv_j, v_kv_l \in E(G)$ then $\mathcal{S}(G/v_iv_j) = \mathcal{S}(G/v_kv_l)$. Therefore $\mathcal{S}(G) - \mathcal{S}(G/v_iv_j) = \mathcal{S}(G) - \mathcal{S}(G/v_kv_l)$, is some constant. This implies that $\Xi_{max}(G/v_iv_j) = \Xi_{min}(G/v_kv_l)$, $v_iv_j, v_kv_l \in E(G)$. So G is stress-stable.

Recall the useful definition of the total vertex stress induced by a vertex on a graph G.

Definition 1. Let $V(G) = \{v_i : 1 \leq i \leq n\}$ and for the ordered vertex pair (v_i, v_j) let there be $k_G(i,j)$ distinct shortest paths of length $l_G(i,j)$ from v_i to v_j. Then, define $\mathfrak{s}_G(v_i)$ as $\sum\limits_{j=1, j\neq i}^{n} k_G(i,j)(\ell_G(i,j)-1)$.

Lemma 1. *Let G be a graph with at least one leaf (pendant vertex). Let v_i and v_iu_j be a leaf and the pendant edge, respectively. Then $\mathcal{S}(G/v_iu_j) = \mathcal{S}(G) - \mathfrak{s}_G(v_i)$.*

Proof. Since vertex v_i is not internal to any shortest path in G, the result yields. □

4. Conclusion

In the paper, the effect of 1-edge contraction was studied at an introductory level. From Theorem 2, our experimental investigation suggests that $\mathcal{S}(G/e_i) \leq \mathcal{S}(G)$ for many graphs which are not /-equivalent. For a sunlet graph with $n \geq 5$ and odd, it is easy to verify that $\mathcal{S}(S_n^{\ominus}/e_i) > \mathcal{S}(S_n^{\ominus})$ if $e_i = v_iv_{i+1}$.

Problem 1. Under which conditions other than family equivalence, does the result $\mathcal{S}(G/e_i) \leq \mathcal{S}(G)$ hold? Alternatively, under which conditions, does $\mathcal{S}(G/e_i) > \mathcal{S}(G)$ hold?

It is obvious that for any non-complete graph which is not 1^e-tractable, there exists a minimum k number of edges say, set $\mathcal{Y} \subset E(G)$ for which, if all were contracted, then $\mathcal{S}(G/\mathcal{Y}) \leq \mathcal{S}(G)$. Such graph G is said to be $k^e_{>1}$-tractable. This is based on the next lemma.

Lemma 2. *A connected graph $G \ncong K_n$, $n \geq 3$ has a set of edges $X = \{e_i : e_i \in E(G)\}$, $1 \leq |X| \leq |E(G)| - 1$ such that $\zeta(G/X) < \zeta(G) - 1$.*

Proof. Contract any $|E(G)| - 1$ edge of G to yield P_2. Since P_2 is complete, the result $\zeta(G/X) < \zeta(G) - 1$ can be achieved with a set of edges say, X, $1 \leq |X| \leq |E(G)| - 1$. □

Based on Lemma 2 the study of $k^e_{\geq 1}$-tractable graphs remains open.

Problem 2. Characterize stress-stable graphs.
Conjecture 1. A graph G for which $\deg_G(v_i) + \deg_G(v_j) = \deg_G(v_k) + \deg_G(v_l)$ for each pair of edges v_iv_j, v_kv_l is stress-stable.

Conjecture 2. A graph G of order $n \geq 4$ for which $\mathcal{S}_G(v_i) + \mathcal{S}_G(v_j) = \mathcal{S}_G(v_k) + \mathcal{S}_G(v_l)$ for each pair of edges v_iv_j, v_kv_l is stress-stable.

Motivation. The result is true for all complete graphs. Consider non-complete graphs G of order $n \geq 4$. For convenience, a v_iv_j-path will be viewed as 'from v_i to v_j'. It permits the convenient view that an edge e_k on a v_iv_j-path has a natural *departure vertex* and an *arrival vertex* without implying orientation in G (directed graph). The number of shortest paths in G which depart from v_i through v_j say $\ell(v_j)$ will reduce the total vertex stress by $\ell(v_j)$ in G/v_iv_j. Similarly, the number of shortest paths in G which depart from v_j through v_i say $\ell(v_i)$ will reduce the total vertex stress by $\ell(v_i)$ in G/v_jv_i. All shortest paths in G which have vertices v_i, v_j as internal vertices will reduce the total vertex stress by $\ell(v_i) + \ell(v_j)$ in G/v_jv_i. The total reduction in total vertex stress is given by $2[\ell(v_i) + \ell(v_j)]$. Since $\mathcal{S}_G(v_i) + \mathcal{S}_G(v_j) = \mathcal{S}_G(v_k) + \mathcal{S}_G(v_l)$ for each pair of edges v_iv_j, v_kv_l it follows that $2[\ell(v_i) + \ell(v_j)] = 2[\ell(v_k) + \ell(v_l)]$ for each pair of edges v_iv_j, v_kv_l.
The outstanding case which requires investigation to settle the result is the shortening of some paths in G vis-a-vis in G/v_iv_j.

Problem 3. For which graphs does it hold true that, if

$$\deg_G(v_1) \leq \deg_G(v_2) \leq \cdots \leq \deg_G(v_n)$$

then

$$\mathcal{S}_G(v_1) \leq \mathcal{S}_G(v_2) \leq \cdots \leq \mathcal{S}_G(v_n)?$$

Motivation. Case 1. The result is trivially true for all complete graphs because $\deg_{K_n}(v_i) = n - 1$ and $\mathcal{S}_{K_n}(v_i) = 0$ for all i.
Case 2. An *almost-complete* graph K_n^{-1}, $n \geq 3$ is obtained by deleting exactly one edge from K_n. Let $V(K_n^{-1}) = \{v_1, v_2, v_3, \ldots, v_n\}$. Assume without loss of generality that the edge v_1v_n was deleted. It follows easily that $\mathcal{S}_{K_n^{-1}}(v_1) = \mathcal{S}_{K_n^{-1}}(v_n) = 0$ and $\mathcal{S}_{K_n^{-1}}(v_i) = 1$, $i = 2, 3, 4, \ldots, n-1$. Therefore the result holds for all almost-complete graphs.
Case 3. Let $G \notin \{K_n, K_n^{-1}\}$ be a graph of order $n \geq 3$ with clique number $\omega(G) \leq 3$. Suppose there exist a pair of distinct vertices v_i, v_j such that $\deg_G(v_i) < \deg_G(v_j)$ and $\mathcal{S}_G(v_i) > \mathcal{S}_G(v_j)$. Since, by definition $\mathcal{S}_G(v_l)$ is the number of times v_l, $1 \leq l \leq n$

is an internal vertex on a shortest path in G it implies that $\deg_G(v_i) \leq 2\mathcal{S}_G(v_i)$ and $\deg_G(v_j) \leq 2\mathcal{S}_G(v_j)$. Hence, $\deg_G(v_j) - \deg_G(v_i) \geq 0$ and so $2\mathcal{S}_G(v_j) - 2\mathcal{S}_G(v_i) \geq 0$ or, $\mathcal{S}_G(v_j) - \mathcal{S}_G(v_i) \geq 0$. The immediate aforesaid implies that $\mathcal{S}_G(v_j) \geq \mathcal{S}_G(v_i)$. The latter is a contradiction. Thus by immediate induction it follows that if

$$\deg_G(v_1) \leq \deg_G(v_2) \leq \cdots \leq \deg_G(v_n)$$

then

$$\mathcal{S}_G(v_1) \leq \mathcal{S}_G(v_2) \leq \cdots \leq \mathcal{S}_G(v_n).$$

Note that vertex deletion may result in a disconnected graph. The requirement is that the initial graph must be a finite, undirected and connected simple graph. The characterization of graphs G for which $\zeta(G - v) = \zeta(G) - 1$ remains open.

Conflict of interest:

The authors declare there is no conflict of interest in respect of this research.

References

[1] J.A. Bondy and U.S.R Murty, *Graph Theory with Applications*, Macmillan Press London, 1976.

[2] M. Changat, P.G. Narasimha-Shenoi, and G. Seethakuttyamma, *Betweenness in graphs: a short survey on shortest and induced path betweenness*, AKCE Int. J. Graphs Comb. **16** (2019), no. 1, 96–109.

[3] A.A. Dobrynin, *Wiener index of subdivisions of a tree*, Siberian Electron. Math. Reports **16** (2019), 1581–1586.

[4] ———, *On the Wiener index of the forest induced by contraction of edges in a tree*, MATCH Commun. Math. Comp. Chem. **16** (2021), no. 2, 321–326.

[5] F. Harary, *Graph Theory*, Addison-Wesley, 1969.

[6] J. Kok and J. Shiny, *On parametric equivalence, isomorphism and uniqueness: Cycle related graphs*, Open J. Discrete Appl. Math. **4** (2021), no. 1, 45–51.

[7] ———, *On parametric equivalent, isomorphic and unique sets*, Open J. Discrete Appl. Math. **4** (2021), no. 1, 19–24.

[8] J. Kok, J. Shiny, and V. Ajitha, *Total vertex stress alteration in cycle related graphs*, Matematichki Bilten **44** (2020), no. 2, 149–162.

[9] A. Shimbel, *Structural parameters of communication networks*, The Bulletin of Mathematical Biophysics **1** (1953), no. 4, 501–507.

[10] J. Shiny and V. Ajitha, *Stress regular graphs*, Malaya J. Mat. **8** (2020), no. 3, 1152–1154.

[11] D.B. West, *Introduction to Graph Theory*, Prentice-Hall Upper Saddle River, 1996.