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Research Article

Face-Magic Labelings of Some Grid-Related Graphs

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Abstract: A type $(1,1,1)$ face-magic labeling of a planar graph $G = (V, E, F)$ is a bijection from $V \cup E \cup F$ to the set of labels $\{1, 2, \ldots, |V| + |E| + |F|\}$ such that the weight of every n-sided face of G is equal to the same fixed constant. The weight of a face $\mathcal{F} \in F$ is equal to the sum of the labels of the vertices, edges, and face that determine $\mathcal{F}$. It is known that the grid graph $P_m \square P_n$ admits a type $(1,1,1)$ face-magic labeling, but the proof in the literature is quite lengthy. We give a simple proof of this result and show two more infinite families of gridded graphs admit type $(1,1,1)$ face-magic labelings.

Keywords: type (a, b, c) face-magic graph labeling, edge-magic total graph labeling.

AMS Subject classification: 05C78

1. Introduction

Let $G = (V, E, F)$ be a planar graph. A labeling of G is an assignment of integers, or sometimes elements of a group, to a subset of $V \cup E \cup F$. In 1983, Lih introduced the following labeling of the vertices, edges, and faces of a planar graph [5]. Let $\ell : V \cup E \cup F \to \{1, 2, \ldots, |V| + |E| + |F|\}$ be a bijection and define $N(\mathcal{F})$ as the set of vertices and edges that comprise the face $\mathcal{F} \in F$. The weight of each face $\mathcal{F} \in F$ is defined as

$$w(\mathcal{F}) = \ell(\mathcal{F}) + \sum_{x \in N(\mathcal{F})} \ell(x).$$

If the weight of every n-sided face is equal to the same fixed constant $\mu(n)$, then we say ℓ is a type $(1,1,1)$ face-magic labeling of G. We refer to $\mu(n)$ as the *magic constant* and to G as a *face-magic graph of type* $(1,1,1)$.
Lih's definition has become a subcase of a broader family known as type (a, b, c) face-magic labelings. We refer the reader to Gallian's dynamic survey for details and recently updated results involving these and other types of graph labelings [4].

A somewhat related labeling is as follows. Let $G = (V, E)$ be a graph and $f : V \cup E \to \{1, 2, \ldots, |V| + |E|\}$ a bijection. If there exists a constant k such that

$$f(x) + f(xy) + f(y) = k$$

for every edge $\{x, y\} \in E$, then f is an *edge-magic total labeling* (EMT) of G and G is an EMT graph. The author generalized this labeling and showed how it can be used as a tool for producing face-magic labelings of many types in [3].
It is well known that paths and cycles admit EMT labelings. Figueroa-Centeno, Ichishima, and Muntaner-Batle showed the following in [2].

Theorem 1 (Figueroa-Centeno, et al. [2]). *Let G be a bipartite or tripartite edge-magic total graph and m be any odd integer. Then mG, the graph of m disjoint copies of G, is also an edge-magic total graph.*

We will use this result to construct face-magic labelings of type $(1, 1, 1)$ for some grid-related graphs in the sections that follow.

2. A new proof of an old result

The planar grid graph $G = (V, E, F) \cong P_m \Box P_n$ is the graph with

$$V(G) = \{(i, j) : 1 \le i \le m, 1 \le j \le n\},$$

and

$$\begin{aligned} E(G) &= \{\{(i, j), (i, j + 1)\} : 1 \le i \le m, 1 \le j \le n - 1\} \\ &\cup \{\{(i, j), (i + 1, j)\} : 1 \le i \le m - 1, 1 \le j \le n\} \end{aligned}$$

Assuming its natural embedding in the plane, we associate each of the $(m-1)(n-1)$ 4-sided faces $\mathcal{F}_{i,j}$ with the vertex (i, j) in its upper left corner. The exterior $(2m+2n-4)$-sided face is denoted $\mathcal{F}_\infty$. In total, G contains $4mn - 2m - 2n + 2$ vertices, edges, and faces. Bača proved the following result in 1992. We give a much shorter proof using edge-magic total labelings.

Theorem 2 (Bača [1]). *The planar grid graph $P_m \Box P_n$ is a type $(1, 1, 1)$ face-magic graph for all $m, n \ge 2$ and $m + n \ne 4$*

Proof. Let $G = (V, E, F) \cong P_m \Box P_n$ be embedded in the plane as its namesake suggests and f be an edge-magic total labeling of $H \cong (2m-1)P_n$ with magic constant k. Such a labeling exists by Theorem 1. For convenience, let $t = |V| + |E| + |F| = 4mn - 2m - 2n + 2$. Write $V(H) = \{v^i_j : 1 \le i \le 2m - 1, 1 \le j \le n\}$ where $(v^i_1, v^i_2, \ldots, v^i_n)$ is the i^{th} copy of P_n. Therefore, $f : V(H) \cup E(H) \to \{1, 2, \ldots, t - 1\}$, is a bijection and

$$f(v^i_j) + f(\{v^i_j, v^i_{j+1}\}) + f(v^i_{j+1}) = k$$

for $1 \le i \le 2m-1$, and $1 \le j \le n-1$.
We claim the following labeling, $\ell : V \cup E \cup F \to \{1, 2, \dots, t\}$ is a type $(1,1,1)$ face-magic labeling of G. Let $\ell(\mathcal{F}_\infty) = t$ and

$$\begin{aligned}
&\ell(i,j) = f(v_j^i) \text{ for } 1 \le i \le m, 1 \le j \le n,\\
&\ell(\{(i,j),(i,j+1)\}) = f(\{v_j^i, v_{j+1}^i\}) \text{ for } 1 \le i \le m, 1 \le j \le n-1,\\
&\ell(\{(i,j),(i+1,j)\}) = f(v_i^{m+i}) \text{ for } 1 \le i \le m-1, 1 \le j \le n, \text{and}\\
&\ell(\mathcal{F}_{i,j}) = f(v_i^{m+i}, v_{i+1}^{m+i}) \text{ for } 1 \le i \le m-1, 1 \le j \le n-1.
\end{aligned}$$

Obviously ℓ is a bijection, and since $2(m+n)-4 > 4$, the weight of the exterior face is irrelevant, so we need only check the weight of each 4-sided face. Let $\mathcal{F}_{i,j} \in F$ be given. Then

$$\begin{aligned}
&w(\mathcal{F}_{i,j}) = \ell(i,j) + \ell(\{(i,j),(i,j+1)\}) + \ell(i,j+1)\\
&+ \ell(\{(i,j),(i+1,j)\}) + \ell(\mathcal{F}_{i,j}) + \ell(\{(i,j+1),(i+1,j+1)\})\\
&+ \ell(i+1,j) + \ell(\{(i+1,j),(i+1,j+1)\}) + \ell(i+1,j+1)\\
&= 3k.
\end{aligned}$$

Notice that the sum of the three terms in each of the first three lines in the above sum equal k. Since k is a constant, the proof is complete. □

3. Tent graphs

For positive integers m and n where $m, n \ge 2$, we define the $m \times n$ *tent graph* $T(m,n)$ as follows. Let $G \cong T(m,n)$ and

$$\begin{aligned}
V(G) &= \{(i,j) : 1 \le i \le m, 1 \le j \le n\} \cup \{p\},\\
E(G) &= \{\{p,(1,j)\} : 1 \le j \le n\}\\
&\cup \{\{(i,j),(i,j+1)\} : 1 \le i \le m, 1 \le j \le n-1\}\\
&\cup \{\{(i,j),(i+1,j)\} : 1 \le i \le m-1, 1 \le j \le n\},\\
F(G) &= \{\mathcal{F}_{i,j} : 1 \le i \le m-1, 1 \le j \le n-1\}\\
&\cup \{\mathcal{F}_{p,j} : 1 \le j \le n-1\}\\
&\cup \{\mathcal{F}_\infty\}.
\end{aligned}$$

$T(m,n)$ is the graph obtained by adding a single vertex p to the Cartesian product $P_m \square P_n$ and joining p with every vertex in the top row. See Figure 1 for a depiction of the graph. The Mongolian tent graph is a similar graph, but contains only 4-sided faces [4].

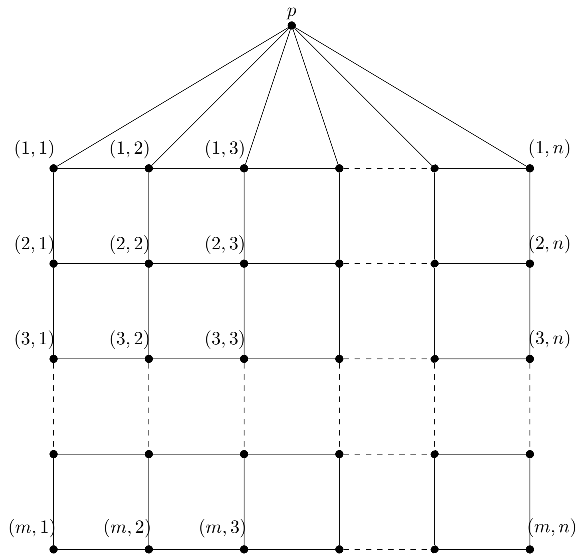


Figure 1. **The $m \times n$ tent graph, $T(m,n)$**

We associate each of the $(m-1)(n-1)$ 4-sided faces $\mathcal{F}_{i,j}$ with the vertex (i,j) in its upper left corner and each of the $n-1$ 3-sided faces $\mathcal{F}_{p,j}$ with the vertex in its lower left corner, $(1,j)$. The exterior $(2m+n-1)$-sided face is denoted $\mathcal{F}_\infty$. In total, G contains $4mn-2m+2$ vertices, edges, and faces.

Theorem 3. *The $m \times n$ tent graph $T(m,n)$ is a type $(1,1,1)$ face-magic graph for all $m,n \geq 2$.*

Proof. Assume the natural embedding of $G = (V,E,F) \cong T(m,n)$ and apply the labeling ℓ from the proof of Theorem 2 to the subgraph $H \cong P_m \square P_n$ of G. For convenience, let $t = 4mn-2m-2n+2$. Thus far, the exterior face has been labeled as well as the vertices, edges, and faces forming the 4-sided faces using the integers $\{1,2,\dots,t\}$. Furthermore, the weight of every 4-sided face is equal to some constant $3k'$.
We need only label the vertex p, edges in the set $\{\{p,(1,j)\} : 1 \leq j \leq n\}$, and the faces in the set $\{\mathcal{F}_{p,j} : 1 \leq j \leq n-1\}$. Let f be an edge-magic total labeling of the path $P_n \cong (v_1,v_2,\dots,v_n)$ with magic constant k and define

$$\ell(\{p,(1,j)\}) = f(v_j) + t \text{ for } 1 \leq j \leq n,$$
$$\ell(\mathcal{F}_{p,j}) = f(\{v_j,v_{j+1}\}) + t \text{ for } 1 \leq j \leq n-1, \text{ and}$$
$$\ell(p) = 2n + t.$$

Clearly, $\ell : V \cup E \cup F \to \{1,2,\dots,4mn-2m+2\}$ is a bijection, so we need only check the weights. Let $\mathcal{F} \in F$. If $\mathcal{F}$ has 4 sides, we have already established that

$w(\mathcal{F}) = 3k'$. If $\mathcal{F} = \mathcal{F}_\infty$, then it is the only $(2m+n-1)$-sided face so its weight is irrelevant. Finally, if $\mathcal{F}$ has 3 sides, then $\mathcal{F} = \mathcal{F}_{p,j}$ for some $1 \le j \le n-1$ and we have

$$\begin{aligned} w(\mathcal{F}) &= \ell(1,j) + \ell(\{(1,j),(1,j+1)\}) + \ell(1,j+1) \\ &+ \ell(\{p,(1,j)\}) + \ell(\mathcal{F}_{p,j}) + \ell(\{p,(1,j+1)\}) + \ell(p) \\ &= k' + f(v_j) + f(\{v_j, v_{j+1}\}) + f(v_{j+1}) + 2n + 4t \\ &= k' + k + 2n + 4t \\ &= k' + 16mn - 8m - 6n + 4. \end{aligned}$$

Since every s-sided face has the same weight for $s \in \{3, 4, 2m+n-1\}$, the proof is complete. □

Wrapping the tent graph around a cylinder and adding edges that connect the boundary vertices forms a graph we call a *circular tent graph*. For positive integers $m \ge 2$ and $n \ge 3$, define the $m \times n$ *circular tent graph* $CT(m,n)$ as follows. Let $G \cong CT(m,n)$ and

$$V(G) = \{(i,j) : 1 \le i \le m, 1 \le j \le n\} \cup \{p\},$$

$$\begin{aligned} E(G) &= \{\{p,(1,j)\} : 1 \le j \le n\} \\ &\cup \{\{(i,j),(i,j+1)\} : 1 \le i \le m, 1 \le j \le n\} \\ &\cup \{\{(i,j),(i+1,j)\} : 1 \le i \le m-1, 1 \le j \le n\}, \end{aligned}$$

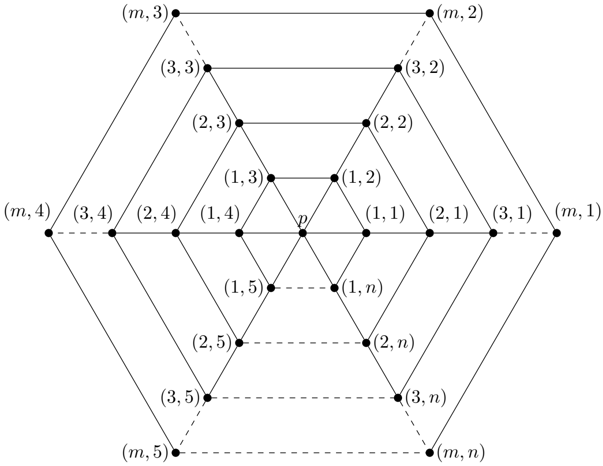
$$\begin{aligned} F(G) &= \{\mathcal{F}_{i,j} : 1 \le i \le m-1, 1 \le j \le n\} \\ &\cup \{\mathcal{F}_{p,j} : 1 \le j \le n\} \\ &\cup \{\mathcal{F}_\infty\}, \end{aligned}$$

with arithmetic taken modulo n where appropriate. Simply, $CT(m,n)$ is the graph obtained from the Cartesian product $P_m \square C_n$ by adding a single vertex p and joining it with every vertex in the innermost layer isomorphic to C_n. In total, $CT(m,n)$ contains $4mn+2$ vertices, edges, and faces. See Figure 2 for a rendition of $CT(m,n)$. Notice that $CT(1,n)$ is isomorphic to the wheel graph $W_n \cong C_n + K_1$.

Theorem 4. *The $m \times n$ circular tent graph $CT(m,n)$ is a type $(1,1,1)$ face-magic graph for all $m \ge 1$ and $n \ge 5$.*

Proof. Assume the embedding of $G = (V,E,F) \cong CT(m,n)$ as in Figure 2 and let f be an edge-magic total labeling of $H \cong (2m-1)C_n$ with magic constant k. Such a labeling exists by Theorem 1. Write $V(H) = \{v_j^i : 1 \le i \le 2m-1, 1 \le j \le n\}$ where $(v_1^i, v_2^i, \ldots, v_n^i, v_1^i)$ is the i^{th} copy of C_n. Observe that $f : V(H) \cup E(H) \to \{1, 2, \ldots, 4mn - 2n\}$ is a bijection with the property

$$f(v_j^i) + f(\{v_j^i, v_{j+1}^i\}) + f(v_{j+1}^i) = k$$



tent.pdf

Figure 2. **The $m \times n$ circular tent graph $CT(m, n)$**

for $1 \leq i \leq 2m - 1$ and $1 \leq j \leq n$ (with arithmetic performed modulo n where appropriate). Similarly, let f' be an edge-magic total labeling of $H' = C_n \cong (v_1, v_2, \ldots, v_n, v_1)$ with magic constant k'. Note that $f' : V(H') \cup E(H') \to \{1, 2, \ldots, 2n\}$ is a bijection with the property

$$f'(v_i) + f'(\{v_i, v_{i+1}\}) + f'(v_{i+1}) = k'$$

for $i = 1, 2, \ldots, n$ (with arithmetic performed modulo n).
We claim the following labeling, $\ell : V \cup E \cup F \to \{1, 2, \ldots, 4mn + 2\}$ is a type $(1, 1, 1)$ face-magic labeling of G. Let $\ell(p) = 4mn + 1$ and $\ell(\mathcal{F}_\infty) = 4mn + 2$. Then define

$$\begin{aligned}
&\ell(i, j) = f(v_j^i) \text{ and} \\
&\ell(\{(i, j), (i, j + 1)\}) = f(\{v_j^i, v_{j+1}^i\}) \text{ for } 1 \leq i \leq m, 1 \leq j \leq n, \\
&\ell(\{(i, j), (i + 1, j)\}) = f(v_i^{m+i}) \text{ and} \\
&\ell(\mathcal{F}_{i,j}) = f(v_i^{m+i}, v_{i+1}^{m+i}) \text{ for } 1 \leq i \leq m - 1, 1 \leq j \leq n, \text{ and} \\
&\ell(\{p, (1, j)\}) = f'(v_j) + 4mn - 2n, \\
&\ell(\mathcal{F}_{p,j}) = f'(\{v_j, v_{j+1}\}) + 4mn - 2n, \\
&\ell(\{p, (1, j + 1)\}) = f'(v_{j+1}) + 4mn - 2n \text{ for } 1 \leq j \leq n.
\end{aligned}$$

Obviously ℓ is a bijection, and since $n > 4$, the weight of the exterior face is irrelevant, so we need only check the weights of the 3- or 4-sided faces. Let $\mathcal{F}_{i,j} \in F$ be given. If $i = p$ and $1 \leq j \leq n$, then $\mathcal{F}_{i,j}$ is a 3-sided face and

$$\begin{aligned}
w(\mathcal{F}_{p,j}) &= \ell(1, j) + \ell(\{(1, j), (1, j + 1)\}) + \ell(1, j + 1) \\
&+ \ell(\{p, (1, j)\}) + \ell(\mathcal{F}_{p,j}) + \ell(\{p, (1, j + 1)\}) + \ell(p) \\
&= k + k' + 16mn - 6n + 1.
\end{aligned}$$

On the other hand, if $1 \le i \le m-1$ and $1 \le j \le n$, then $\mathcal{F}_{i,j}$ is a 4-sided face and

$$\begin{aligned}
w(\mathcal{F}_{i,j}) &= \ell(i,j) + \ell(\{(i,j),(i,j+1)\}) + \ell(i,j+1) \\
&+ \ell(\{(i,j),(i+1,j)\}) + \ell(\mathcal{F}_{i,j}) + \ell(\{(i,j+1),(i+1,j+1)\}) \\
&+ \ell(i+1,j) + \ell(\{(i+1,j),(i+1,j+1)\}) + \ell(i+1,j+1) \\
&= 3k.
\end{aligned}$$

Since the weight is constant in both cases, the proof is complete. □

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