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Research Article

The crossing numbers of join product of four graphs on six vertices with discrete graphs

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Abstract: The main aim of the paper is to give the crossing number of the join product $G^* + D_n$ for the graph G^* isomorphic to 4-regular graph on six vertices except for two distinct edges with no common vertex such that two remaining vertices are still adjacent, and where D_n consists of n isolated vertices. The proofs are done with possibility of an existence of a separating cycle in some particular drawing of the investigated graph G^* and also with the help of well-known exact values for crossing numbers of join products of two subgraphs H_k of G^* with discrete graphs.

Keywords: graph, good drawing, crossing number, join product, separating cycle

AMS Subject classification: 05C10, 05C38

1. Introduction

The issue of reducing the number of crossings on edges of simple graphs is interesting in a lot of areas. Probably one of the most popular areas is the implementation of the VLSI layout because it caused a significant revolution in circuit design and thus had a strong effect on parallel calculations. Crossing numbers have also been studied to improve the readability of hierarchical structures and automated graphs. The visualized graph should be easy to read and understand. For the sake of clarity of graphic drawings, some reduction of an edge crossing is probably the most important. Note that examining number of crossings of simple graphs is an NP-complete problem by Garey and Johnson [8].

The *crossing number* $\mathrm{cr}(G)$ of a simple graph G with the vertex set $V(G)$ and the edge set $E(G)$ is the minimum possible number of edge crossings in a drawing of G in the plane (for the definition of a *drawing* see Klešč [19]). It is easy to see that a drawing with minimum number of crossings (an optimal drawing) is always a *good*

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drawing, meaning that no edge crosses itself, no two edges cross more than once, and no two edges incident with the same vertex cross. Let D be a good drawing of the graph G. We denote the number of crossings in D by $\mathrm{cr}_D(G)$. Let G_i and G_j be edge-disjoint subgraphs of G. We denote the number of crossings between edges of G_i and edges of G_j by $\mathrm{cr}_D(G_i, G_j)$, and the number of crossings among edges of G_i in D by $\mathrm{cr}_D(G_i)$. It is easy to see that for any three mutually edge-disjoint subgraphs G_i, G_j, and G_k of G, the following equations hold:

$$\mathrm{cr}_D(G_i \cup G_j) = \mathrm{cr}_D(G_i) + \mathrm{cr}_D(G_j) + \mathrm{cr}_D(G_i, G_j)\,,$$

$$\mathrm{cr}_D(G_i \cup G_j, G_k) = \mathrm{cr}_D(G_i, G_k) + \mathrm{cr}_D(G_j, G_k)\,.$$

Throughout this paper, some parts of proofs will be based on Kleitman's result [16] on crossing numbers for some complete bipartite graphs $K_{m,n}$ on $m+n$ vertices with a partition $V(K_{m,n}) = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$ containing an edge between every pair of vertices from V_1 and V_2 of sizes m and n, respectively. He showed that

$$\mathrm{cr}(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor, \quad \text{if} \quad \min\{m,n\} \le 6. \tag{1}$$

For an overview of several exact values of crossing numbers for specific graphs or some families of graphs, see Clancy et al. [5]. The main goal of this survey is to summarize all such published results for crossing numbers along with references also in an effort to give priority to the author who published the first result. Chapter 4 is devoted to the issue of crossing numbers of join product with all simple graphs of order at most six mainly due to unknown values of $\mathrm{cr}(K_{m,n})$ for both m, n more than six in (1). The join product of two graphs G_i and G_j, denoted $G_i + G_j$, is obtained from vertex-disjoint copies of G_i and G_j by adding all edges between $V(G_i)$ and $V(G_j)$. For $|V(G_i)| = m$ and $|V(G_j)| = n$, the edge set of $G_i + G_j$ is the union of the disjoint edge sets of the graphs G_i, G_j, and the complete bipartite graph $K_{m,n}$. Let D_n denote the *discrete graph* (sometimes called *empty graph*) on n vertices. The exact values for crossing numbers of $G + D_n$ for all graphs G of order at most four are given by Klešč and Schrötter [21], and also for some connected graphs G of order five and six [1–3, 6, 7, 10–15, 17–20, 22, 23, 26–29, 31–34]. The aim of this paper is to extend known results concerning this topic to new connected graphs. Note also that $\mathrm{cr}(G + D_n)$ are known only for some disconnected graphs G, see [24, 25, 30].
For this purpose, we present a new technique regarding the use of knowledge from the subgraphs whose values of crossing numbers are already known. It is appropriate to combine this idea with possibility of an existence of a separating cycle in some particular drawing of investigated graph. In a good drawing D of some graph G, we say that a cycle C *separates* two different vertices of the subgraph $G \setminus C$ if they are contained in different components of $\mathbb{R}^2 \setminus C$. This considered cycle C is said to be a *separating* cycle of the graph G in D.
In Section 2 we refer the graph G_1 on six vertices and eight edges isomorphic to the complete bipartite graph $K_{2,4}$ for which the crossing number of $G_1 + D_n$ was obtained

by Ho [14]. The crossing numbers of $G_k + D_n$ for two other graphs G_k, $k = 2, 3$ on six vertices are given in Corollaries 1 and 2 by adding new edges to the graph G_1. Section 3 is devoted to the graph $G^* = (V(G^*), E(G^*))$ isomorphic to 4-regular graph on six vertices except for two distinct edges with no common vertex such that two remaining vertices are still adjacent. Many possible drawings of the graph G^* are partially solved using redrawings of two edges of G^* in Figures 5 and 6(a) together with well-known exact values of $\mathrm{cr}(H_k + D_n)$ for two subgraphs H_k of G^* presented in Theorems 2 and 3.

In Figure 1, the edges of $K_{6,n}$ cross each other $6\binom{\lceil \frac{n}{2} \rceil}{2} + 6\binom{\lfloor \frac{n}{2} \rfloor}{2} = 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ times and each subgraph T^i crosses edges of G^* twice. Thus, $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2n$ crossings appear among edges of the graph $G^* + D_n$ in this drawing. The main goal of our paper is to show that the crossing number of $G^* + D_n$ is equal to this upper bound.

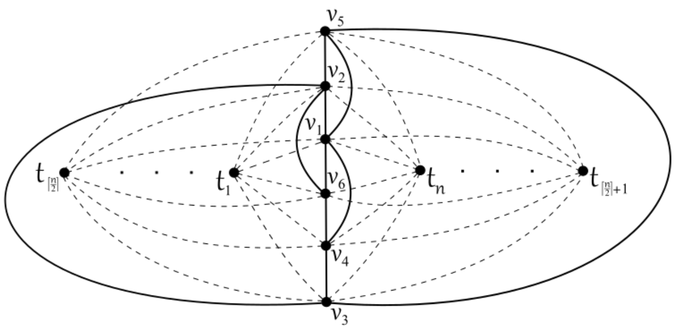

Figure 1. **The good drawing of $G^* + D_n$ with $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2n$ crossings**

The crossing number of $G^* + D_n$ equal to $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2n$ will be determined in Theorem 4 with the proof that is strongly based on Lemma 2. This lemma in a very special form could also be used to establish crossing numbers of other graphs. In the proofs of the paper, we will often use the term "region" also in nonplanar subdrawings. In this case, crossings are considered to be vertices of the "map".

2. Three Graphs G_1, G_2 and G_3

Let G_1 be the graph isomorphic to the complete bipartite graph $K_{2,4}$. The crossing numbers of the join products of $K_{2,4}$ with the discrete graphs D_n have been well-known by Ho [14]. Let G_2 be the graph obtained from the planar drawing of G_1 in Figure 2 by adding the edge v_1v_2, i.e., $G_2 = G_1 \cup \{v_1v_2\}$. Similarly, let $G_3 = G_1 \cup \{v_1v_2, v_3v_4\}$.

Theorem 1 ([14]). *If $n \geq 1$, then* $\mathrm{cr}(G_1 + D_n) = 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2n$.

Due to Theorem 1, the good drawing of $G_1 + D_n$ in Figure 3 is optimal. We can add

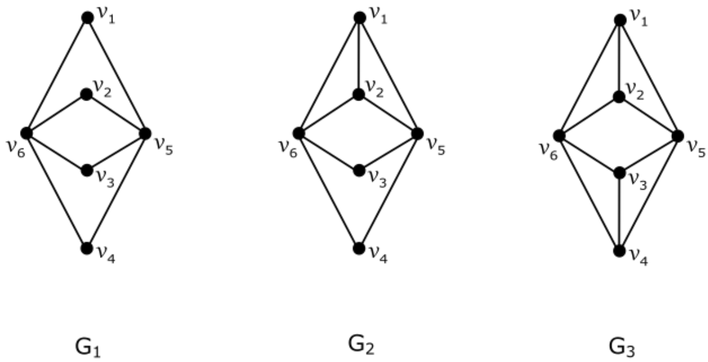

Figure 2. **Planar drawings of three graphs G_1, G_2 and G_3**

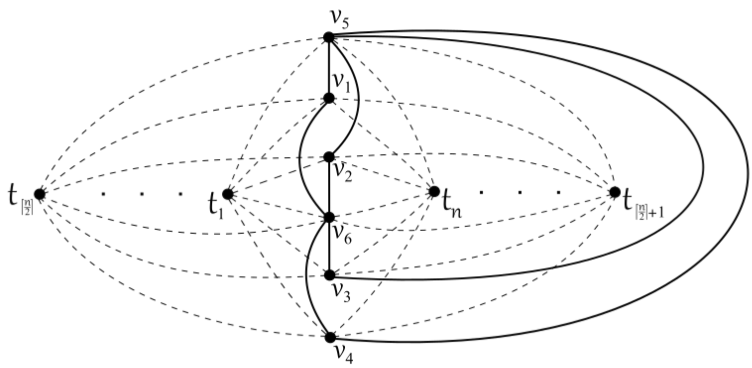

Figure 3. **The good drawing of $G_1 + D_n$ with $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2n$ crossings**

both edges v_1v_2 and v_3v_4 into this drawing without additional crossings, and therefore, the drawings of $G_2 + D_n$ and $G_3 + D_n$ with exactly $6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2n$ crossings are obtained. On the other hand, $G_1 + D_n$ is a subgraph of $G_2 + D_n$ that is a subgraph of $G_3 + D_n$, and therefore, $\mathrm{cr}(G_3 + D_n) \geq \mathrm{cr}(G_2 + D_n) \geq \mathrm{cr}(G_1 + D_n)$. Thus, the next results are obvious.

Corollary 1. *If $n \geq 1$, then $\mathrm{cr}(G_2 + D_n) = 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2n$.*

Corollary 2. *If $n \geq 1$, then $\mathrm{cr}(G_3 + D_n) = 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2n$.*

3. The Crossing Number of $G^* + D_n$

The join product $G^* + D_n$ (sometimes used notation $G^* + nK_1$) consists of one copy of the graph G^* and n vertices $t_1, \dots, t_n$, and any vertex t_i is adjacent to every vertex of the graph G^*. We denote by T^i the subgraph induced by six edges incident with the fixed vertex t_i, which yields that

$$G^* + D_n = G^* \cup K_{6,n} = G^* \cup \left(\bigcup_{i=1}^{n} T^i \right). \tag{2}$$

We consider a good drawing D of $G^* + D_n$. By the *rotation* $\mathrm{rot}_D(t_i)$ of a vertex t_i in D we understand the cyclic permutation that records the (cyclic) counterclockwise order in which edges leave t_i, as defined by Hernández-Vélez et al. [9] or Woodall [35]. We use the notation (123456) if the counter-clockwise order of edges incident with the fixed vertex t_i is t_iv_1, t_iv_2, t_iv_3, t_iv_4, t_iv_5 and t_iv_6. We recall that rotation is a cyclic permutation. In the given drawing D, it is highly desirable to separate n subgraphs T^i into three mutually disjoint subsets depending on how many times edges of G^* could be crossed by T^i in D. Let us denote by R_D and S_D the set of subgraphs for which $\mathrm{cr}_D(G^*, T^i) = 0$ and $\mathrm{cr}_D(G^*, T^i) = 1$, respectively. Edges of G^* are crossed by each remaining subgraph T^i at least twice in D. Note that if D is a good drawing of $G^* + D_n$ with the empty set $R_D \cup S_D$, then $\sum_{i=1}^{n} \mathrm{cr}_D(G^*, T^i) \geq 2n$ enforces at least $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2n$ crossings in D provided by

$$\mathrm{cr}_D(G^* + D_n) = \mathrm{cr}_D(K_{6,n}) + \mathrm{cr}_D(G^*, K_{6,n}) + \mathrm{cr}_D(G^*) \geq 6 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2n.$$

According to the expected result of the main Theorem 4, this leads to a consideration of the nonempty set $R_D \cup S_D$ in all good drawings of $G^* + D_n$.
Let us discuss all possible drawings of G^* induced by D with the degree sequence $(3, 3, 3, 3, 4, 4)$. There are exactly two vertices of degree 3 adjacent with both vertices of degree 4. The graph G^* contains a cycle C_4 induced on four remaining vertices of degree 4, 4, 3, and 3 as a subgraph (for brevity, we write $C_4(G^*)$), and let v_1, v_2, v_3, and v_4 be their vertex notation in the appropriate order of the cycle $C_4(G^*)$. In the rest of the paper, suppose also that $\deg(v_5) = 3$ and $\deg(v_6) = 3$ if $v_4v_5, v_3v_6 \notin E(G^*)$. Let H_1 be the graph consisting of two leaves adjacent with two opposite vertices of one 4-cycle. Let H_2 be the graph consisting of one 3-cycle and three leaves of which exactly two are adjacent with the same vertex of such 3-cycle. See also their drawings in Figure 4. The crossing numbers of the join products of H_1 and H_2 with the discrete graphs D_n have been well-known by Berežný and Staš [2], and Staš [32], respectively.

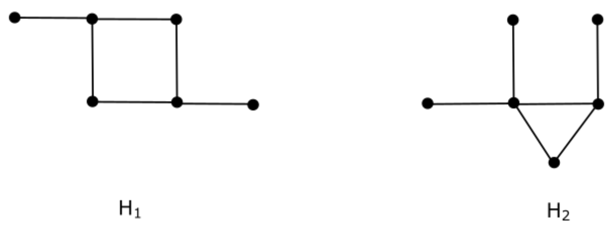

Figure 4. Two graphs H_k on six vertices with well-known values of $\mathrm{cr}(H_k + D_n)$, where $k = 1, 2$

Theorem 2 ([2], Theorem 3.1). *If $n \geq 1$, then* $\mathrm{cr}(H_1 + D_n) = 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor$.

Theorem 3 ([32], Theorem 1). *If $n \geq 1$, then* $\mathrm{cr}(H_2 + D_n) = 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor$.

In Figure 5(a), we can redraw a crossing of two edges of $C_4(G^*)$ to get a new drawing of G^* induced by D (with vertex notation in a different order) with fewer edge crossings. The redrawing of $C_4(G^*)$ in Figure 5(b) produces a drawing of the graph G_3. Both considered redrawings of the cycle $C_4(G^*)$ allow us to suppose that edges of $C_4(G^*)$ do not cross each other in all discussed good drawings of G^*.

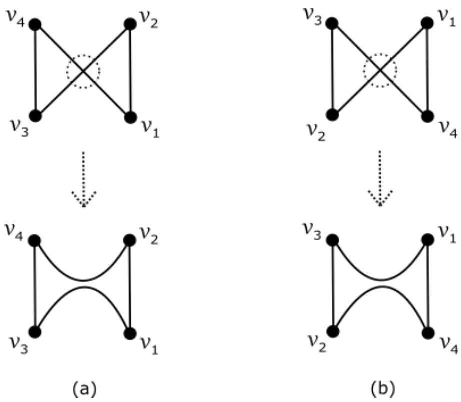

Figure 5. Elimination of a crossing on edges of $C_4(G^*)$. **(a)**: redrawing of the graph G^* with vertex notation in a different order; **(b)**: redrawing of the cycle $C_4(G^*)$ which causes a drawing of new graph G_3

In an effort to reach less than $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2n$ crossings, two remaining vertices v_5 and v_6 of the graph G^* must be placed in the same region of the considered good subdrawing $D(C_4(G^*))$ because removing all edges of $C_4(G^*)$ results a good drawing of $H_1 + D_n$ with at least $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor$ crossings. Note that $C_4(G^*)$ would be a separating cycle in this drawing and thus crossed at least n times through all subgraphs T^i. Besides that the subdrawing $D(G^*)$ cannot be planar if at least one of the sets R_D and S_D should be nonempty. Clearly, $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor + n + 1$ is at least as much as our expected result of $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2n$. Based on that, in the rest of the paper, let both vertices v_5 and v_6 be placed in common outer region of $D(C_4(G^*))$.

The edge v_1v_5 cannot cross the edge v_3v_4 of the cycle $C_4(G^*)$ in any optimal drawing of $G^* + D_n$, otherwise, we also obtain a new drawing of G^* induced by D (with vertex notation in a different order) with fewer edge crossings by redrawing of subgraph in Figure 6(a).

Of course, the same holds for pair of edges v_3v_5 and v_1v_4. Removing all edges of the separating cycle $v_2v_3v_5v_2$ in Figure 6(b) produces a good drawing which includes $H_1 + D_n$ as a subgraph. Consequently, the result of Theorem 2 implies at least $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor + n + 1 + 1$ crossings on edges of $G^* + D_n$. Finally, considering the

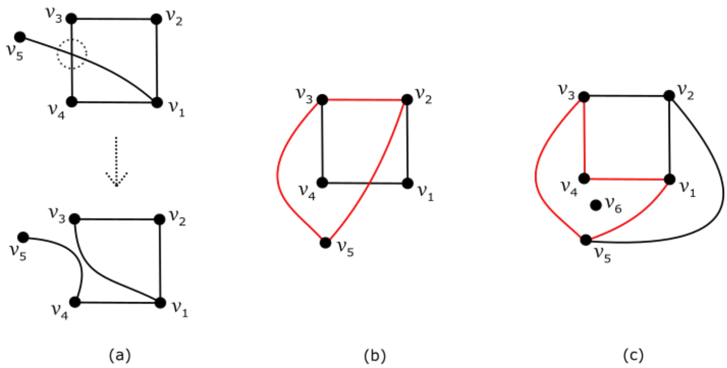

Figure 6. **Elimination of a crossing and two separating cycles on edges of G^***

planar subdrawing $D(G^* \setminus v_6)$, the vertex v_6 is placed in the quadrangular region of $D(G^* \setminus v_6)$ with four vertices v_1, v_3, v_4 and v_5 of G^* on its boundary provided by we assume the nonempty set $R_D \cup S_D$. Again using the idea of a separating cycle, if we consider subdrawing of the graph G^* presented in Figure 6(c) then the well-known value of $\mathrm{cr}(H_2 + D_n)$ enforces at least $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor + n + 1$ crossings on edges of $G^* + D_n$. Due to symmetry of the graph G^*, the same discussion with respect to the vertex v_6 of G^* offers same results. The proof of Lemma 1 can be omitted based on all the above observations.

Lemma 1. *In any optimal drawing D of the join product $G^* + D_n$, edges of $C_4(G^*)$ do not cross each other. Moreover, if both vertices v_5 and v_6 are placed in common region of $D(C_4(G^*))$ then the drawing of G^* induced by D is isomorphic to one of the two drawings depicted in Figure 7.*

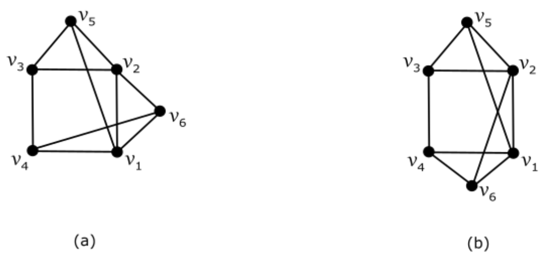

Figure 7. Two considered nonplanar drawings of the graph G^* with a possibility of obtaining a subgraph T^i by which edges of G^* can be crossed at most once

In the proof of Theorem 4, the following lemma related to some restricted subdrawings of the graph $G^* + D_n$ is also required.

Lemma 2. *For $n \geq 2$, let D be a good drawing of $G^* + D_n$ in which for some i, $i \in \{1, \ldots, n\}$, and for all $j = 1, \ldots, n$, $j \neq i$, $\mathrm{cr}_D(G^* \cup T^i, T^j) \geq 5$. If $\mathrm{cr}_D(G^* \cup T^i, T^j) > 5$ for p different subgraphs T^j, then D has at least $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 4\lfloor \frac{n}{2} \rfloor + p + \mathrm{cr}_D(G^* \cup T^i)$ crossings.*

Proof. Assume, without loss of generality, that the edges of $F^n = G^* \cup T^n$ are crossed in D at least five times by the edges of every subgraph T^j, $j = 1, \ldots, n-1$, and that p of the subgraphs T^j cross the edges of F^n more than five times. As $G^* + D_n = K_{6,n-1} \cup F^n$, we have

$$\begin{aligned}
\mathrm{cr}_D(G^* + D_n) &= \mathrm{cr}_D\,(K_{6,n-1}) + \mathrm{cr}_D(K_{6,n-1}, F^n) + \mathrm{cr}_D(F^n) \\
&\geq 6\left\lfloor \tfrac{n-1}{2} \right\rfloor \left\lfloor \tfrac{n-2}{2} \right\rfloor + 5(n-1) + p + \mathrm{cr}_D(G^* \cup T^n) \\
&\geq 6\left\lfloor \tfrac{n}{2} \right\rfloor \left\lfloor \tfrac{n-1}{2} \right\rfloor + 4\left\lfloor \tfrac{n}{2} \right\rfloor + p + \mathrm{cr}_D(G^*, T^n) + \mathrm{cr}_D(G^*).
\end{aligned}$$

□

Note that the last estimate used in the proof of Lemma 2 does not offer at least $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2n$ crossings only for n odd with $p + \mathrm{cr}_D(G^* \cup T^n) \leq 1$. In the following, we are able to compute the exact values of crossing numbers of the join products of the graph G^* with both discrete graphs D_1 and D_2 using the algorithm located on the website `http://crossings.uos.de/`. This algorithm can find the crossing numbers of small undirected graphs. It uses an ILP formulation, based on Kuratowski subgraphs, and solves it via branch-and-cut-and-price. The system also generates verifiable formal proofs, as described by Chimani and Wiedera [4]. Unfortunately, the capacity of this system is restricted.

Lemma 3. $\mathrm{cr}(G^* + D_1) = 2$ *and* $\mathrm{cr}(G^* + D_2) = 4$.

Theorem 4. *If $n \geq 1$, then $\mathrm{cr}(G^* + D_n) = 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2n$.*

Proof. By Lemma 3, the result is true for $n = 1$ and $n = 2$. The drawing in Figure 1 shows that $\mathrm{cr}(G^* + D_n) \leq 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2n$. To prove the reverse inequality by induction on n, suppose now that there is a drawing D of $G^* + D_n$ with

$$\mathrm{cr}_D(G^* + D_n) < 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2n \quad \text{for some } n \geq 3, \tag{3}$$

and let

$$\mathrm{cr}(G^* + D_m) = 6\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor + 2m \quad \text{for any positive integer } m < n. \tag{4}$$

For easier reading, if $r = |R_D|$ and $s = |S_D|$, then the assumption (3) together with $\mathrm{cr}_D(K_{6,n}) \geq 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ using (1) imply the following relation with respect to edge crossings of G^* in D:

$$\mathrm{cr}_D(G^*) + \sum_{T^i \in R_D} \mathrm{cr}_D(G^*, T^i) + \sum_{T^i \in S_D} \mathrm{cr}_D(G^*, T^i) + \sum_{T^i \notin R_D \cup S_D} \mathrm{cr}_D(G^*, T^i) < 2n,$$

i.e.,

$$\mathrm{cr}_D(G^*) + 0r + 1s + 2(n - r - s) < 2n. \tag{5}$$

The obtained inequality (5) forces $r + s \geq 1$, and so there is at least one subgraph T^i by which edges of G^* are crossed at most once in D. Lemma 1 together with the assumption (3) offer only two possible nonplanar subdrawings of the graph G^* shown in Figure 7. For both such subdrawings, the set R_D must be nonempty because there is no possibility to obtain a subgraph $T^i \in S_D$.
Let us first consider the subdrawing of G^* induced by D given in Figure 7(a). For any $T^i \in R_D$, the reader can easily see that the subgraph $F^i = G^* \cup T^i$ is uniquely represented by $\mathrm{rot}_D(t_i) = (143526)$. Since edges of F^i are crossed by each other subgraph T^j at least five times, Lemma 2 contradicts the assumption (3) in D. Now, assume the subdrawing of G^* induced by D given in Figure 7(b). In this case, the subgraph $F^i = G^* \cup T^i$ is represented by $\mathrm{rot}_D(t_i) = (164352)$. If there is some T^j such that $\mathrm{cr}_D(G^* \cup T^i, T^j) < 5$, then the vertex t_j must be placed in the pentagonal region of subdrawing $D(G^*)$ with two vertices v_3 and v_4 of G^* on its boundary, and $\mathrm{cr}_D(G^* \cup T^i, T^j) = 4$ enforces $\mathrm{cr}_D(T^i, T^j) = 0$. Thus, by fixing the subgraph $T^i \cup T^j$, we have

$$\mathrm{cr}_D(G^* + D_{n-2}) + \mathrm{cr}_D(T^i \cup T^j) + \mathrm{cr}_D(K_{6,n-2}, T^i \cup T^j) + \mathrm{cr}_D(G^*, T^i \cup T^j)$$

$$\geq 6\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 2(n-2) + 0 + 6(n-2) + 4 = 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2n,$$

where edges of $T^i \cup T^j$ are crossed by each other subgraph T^k at least six times using $\mathrm{cr}_D(K_{6,3}) \geq 6$ again thanks to (1). This subcase again confirms a contradiction with (3) in D.
We have shown that there are at least $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2n$ crossings in each good drawing D of $G^* + D_n$, and this completes the proof of Theorem 4. □

4. Conclusions

We expect that similar forms of discussions can be used to estimate unknown values of the crossing numbers of other graphs on six vertices with a much larger number of edges in the join products with discrete graphs, and also with paths and cycles.

Conflict of interest. The author declares that they have no conflict of interest.

Data Availability. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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