Communications in Combinatorics and Optimization
Vol. 9, No. 2 (2024), pp. 329-338
https://doi.org/10.22049/cco.2023.27862.1373

CCO
Commun. Comb. Optim.

Research Article

A new construction for μ-way Steiner trades

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Received: 2 June 2022; Accepted: 14 February 2023
Published Online: 18 February 2023

Abstract: A μ-way (v, k, t) trade T of volume m consists of μ pairwise disjoint collections $T_1, \ldots, T_\mu$, each of m blocks of size k such that for every t-subset of a v-set V, the number of blocks containing this t-subset is the same in each T_i for $1 \leq i \leq \mu$. If any t-subset of the v-set V occurs at most once in each T_i for $1 \leq i \leq \mu$, then T is called a μ-way (v, k, t) Steiner trade. In 2016, it was proved that there exists a 3-way $(v, k, 2)$ Steiner trade of volume m when $12(k-1) \leq m$ for each k. Here we improve the lower bound to $8(k-1)$ for even k, by using a recursive construction.

Keywords: 3-way $(v, k, 2)$ Steiner trade; 1-solely balanced set; block design

AMS Subject classification: 05B30; 05B05

1. Introduction and preliminaries

Given a set of v treatments V, let k and t be two positive integers such that $t < k < v$. A (v, k, t) trade $T = \{T_1, T_2\}$ of volume m consists of two disjoint collections T_1 and T_2, each one containing m k-subsets of V, called blocks, such that every t-subset of V is contained in the same number of blocks in T_1 and T_2. A (v, k, t) trade is called (v, k, t) Steiner trade if any t-subset of V occurs at most once in $T_1(T_2)$. A (v, k, t) trade is also a (v, k, t') trade, for all $0 < t' < t$. In a (v, k, t) trade, both collections of blocks must cover the same set of elements. This set of elements is called the foundation of the trade and is denoted by found(T).

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The concept of trade was first introduced by Hedayat (1960) in the paper [9]. Hedayat and Li applied the method of trade-off and trades to construct BIBDs with repeated blocks (1979-1980). Later, Steiner trades were used and renamed by Milici and Quattrocchi (1986) as DMB (disjoint and mutually balanced). However, in 1916, Cole and Gumming used a concept that is $(v, 3, 2)$ trade of volumes 4 and 6. Determining the existence and non-existence of (v, k, t) trades of certain volumes is the most important problem in this field. Many papers have dealt with this problem (for example, see [3, 6, 10, 12, 14]).
Trades have various applications in combinatorial design theory. For instance, in the problems related to the structure of block designs, the method of constructing block designs, non-isomorphic block designs, block designs with repeated blocks, and determining defining set and intersection problem in block designs. The intersection problem for two and three block designs has been extensively investigated (see for example [1, 4, 13, 15, 19]). Now, we give an example of a connection between trades and intersection problem.

Example 1. Construct an $S(2, 4, 13)$ design (V, B) with $V = Z_{10} \cap \{a, b, c\}$. All blocks are listed below.

0	0	0	0	1	1	1	2	2	3	3	4	5
1	2	4	6	2	5	7	3	6	4	7	8	9
3	8	5	a	4	6	b	5	7	6	8	9	a
9	c	7	b	a	8	c	b	9	c	a	b	c

Now, consider the permutation $(7, b, c, 6)$ on V and construct the new block design (V, B'). All blocks of the new design are listed below.

0	0	0	0	1	1	1	2	2	3	3	4	5
1	2	4	7	2	5	b	3	7	4	b	8	9
3	8	5	a	4	7	c	5	b	7	8	9	a
9	6	b	c	a	8	6	c	9	6	a	c	6

Two block designs have two common blocks. Note that removing common blocks of the two block designs results in a trade of volume $13 - 2 = 11$.

Recently, a generalization of the concept of trade has been defined in [16] as follows.

Definition 1. A *μ-way (v, k, t) trade* of volume m consists of μ pairwise disjoint collections $T_1, \ldots, T_\mu$ each of m blocks, such that for every t-subset of a v-set V, the number of blocks containing this t-subset is the same in each T_i for $1 \leq i \leq \mu$. In other words, any set $T = \{T_i, T_j\}$ for $i \neq j$ is a (v, k, t) trade of volume m.

Definition 2. A μ-way (v, k, t) trade is a *μ-way (v, k, t) Steiner trade* if any t-subset of found(T) occurs at most once in every T_j for $j \geq 1$.

Each T_i contains m blocks $B_{1i},\ B_{2i},\dots,B_{mi}$, where B_{ij} denotes ith block of T_j. A type of μ-way (v,k,t) Steiner trade with an additional property has an important role in constructing the Steiner trade is defined as follows.

Definition 3. Let $T=\{T_1,\dots,T_\mu\}$ be a μ-way (v,k,t) Steiner trade. It is called a *μ-way t-solely balanced set* if there exist no blocks B_{ij} and B_{ab} such that $|B_{ij}\cap B_{ab}|>t$ for $1\le j<b\le\mu$. In other words, T_j and T_b $(1\le j<b\le\mu)$ contain no common $(t+1)-$subset.

Theorem 1. *([16]) The following statements hold.*
(i) Let $T=\{T_1,\dots,T_\mu\}$ be a μ-way (v,k,t) trade of volume m. Based on T, a μ-way $(v+\mu,k+1,t+1)$ trade T^* of volume μm can be constructed.
(ii) If T is also a μ-way t-solely balanced, then a μ-way $(v+\mu,k+1,t+1)$ Steiner trade T^* can be constructed.

We need some notation. Let T and T^* be two μ-way (v,k,t) trades of volume m. We consider $T+T^*=\{T_1\cup T_1^*,\dots,T_\mu\cup T_\mu^*\}$. It is easy to see that $T+T^*$ is a μ-way (v,k,t) trade. If T and T^* are Steiner trades and found(T) $\cap$ found(T*) $=\phi$, then $T+T^*$ is also a Steiner trade. So the problem of determining the set of all possible volume sizes of a μ-way (v,k,t) trade is one of the most important problems in combinatorial subjects. Some papers have dealt with this problem (for instance, see [2, 5, 16]). Let $S_{3s}(t,k)$ denote the set of all possible volume sizes of a 3-way (v,k,t) Steiner trade. The values of $S_{2s}(2,k)$ have been completely specified by Gray and Ramsay [7, 8] and Khodkar and Hoffman [11]. We also have the following.

Theorem 2. *[17, 18] $m\in S_{3s}(2,k)$ for all $m\ge 12(k-1)$ and each k.*

In this paper,by giving a new construction and some new results, we improve the lower bound to $8(k-1)$ for even k.

1.1. Construction (Recursive)

In this section, we introduce a construction that is used in the main results. The 1-solely balanced sets are the main bases for this construction.

Construction 3.

- Consider three $(k-1)\times(k-1)$ tables A, B, and C, for even k with r common rows.
- Construct three 4-way 1-solely balanced sets S_A, S_B, and S_C with block size $k-1$ from tables A, B, and C respectively. The blocks of S_{1A} are the rows of the table A, the blocks of S_{2A} are the columns of the table A, the blocks S_{3A} are the main diagonals of the table A, and the blocks of S_{4A} are the secondary diagonals of table A. In other words, the collections of each 4-way 1-solely balanced set have the following frame:
  Collection one: $\{(i,j)|\ 0\le j\le k-2\},\ 0\le i\le k-2$.
  Collection two:$\{(i-j,j)|\ 0\le j\le k-2\},\ 0\le i\le k-2$.

Collection three: Rename (i,j) as $v_{i+j(k-1)}$, for each $0 \leq i \leq k-2$ consider the blocks $\{v_j |\ i(k-1) \leq j \leq i(k-1)+k-2\}$.
Collection four: $\{(j-i,j)|0 \leq j \leq k-2\},\ 0 \leq i \leq k-2$.

We observe that the second collection $\{(i-j,j)|0 \leq j \leq k-2\},\ 0 \leq i \leq k-2$, and the last collection $\{(j-i,j)|0 \leq j \leq k-2\},\ 0 \leq i \leq k-2$ for even $k-1$ have the repeated pair. For $i=0$, the pair $((-j,j),(j,j))$ for $j=0$ and $\frac{k-1}{2}$ is repeated. Therefore, we can apply this method only for even k.

- Construct three 4-way Steiner trades with block size k from 1-solely balanced sets by applying Theorem 1.
- Remove the second collection from each of them and obtain three 3-way Steiner trades (In each case, the suitable collection will be removed).
- Add these three 3-way Steiner trades and remove the common blocks.

In the following example, we construct a 3-way $(29,6,2)$ Steiner trade of volume 59 by using Construction 3. The details of Construction 3 can be observed in the following example.

Example 2.
Steps one and two of Construction 3:
The 4-way 1-solely balanced sets S_A, S_B, and S_C are obtained from the tables A, B, and C, respectively. The blocks of S_{1A} are the rows of the table A, the blocks of S_{2A} are the columns of the table A, the blocks S_{3A} are the main diagonals of A, and the blocks of S_{4A} are the secondary diagonals of A. Consider the following three tables.

A :

1	2	3	4	5
a	b	c	d	e
f	g	h	i	j
k	l	m	n	o
p	q	r	s	t

B :

1	2	3	4	5
a'	b'	c'	d'	e'
f'	g'	h'	i'	j'
k'	l'	m'	n'	o'
p'	q'	r'	s'	t'

C :

1	2	3	4	5
a''	b''	c''	d''	e''
f''	g''	h''	i''	j''
k''	l''	m''	n''	o''
p''	q''	r''	s''	t''

Now consider the following 4-way 1-solely balanced sets:

S_A :

S_{1A}	S_{2A}	S_{3A}	S_{4A}
12345	$1afkp$	$1bhnt$	$1eimq$
$abcde$	$2gblq$	$2ciop$	$2ajnr$
$fghij$	$3chmr$	$3djkq$	$3bfos$
$klmno$	$4dins$	$4eflr$	$4cgkt$
$pqrst$	$5ejot$	$5agms$	$5dhlp$

S_B :

S_{1B}	S_{2B}	S_{3B}	S_{4B}
12345	$1a'f'k'p'$	$1'b'h'n't'$	$1e'i'm'q'$
$a'b'c'd'e'$	$2g'b'l'q'$	$2c'i'o'p'$	$2a'j'n'r'$
$f'g'h'i'j'$	$3c'h'm'r'$	$3d'j'k'q'$	$3b'f'o's'$
$k'l'm'n'o'$	$4d'i'n's'$	$4e'f'l'r'$	$4c'g'k't'$
$p'q'r's't'$	$5e'j'o't'$	$5a'g'm's'$	$5d'h'l'p'$

S_C :

S_{1C}	S_{2C}	S_{3C}	S_{4C}
12345	$1a''f''k''p''$	$1b''h''n''t''$	$1e''i''m''q''$
$a''b''c''d''e''$	$2g''b''l''q''$	$2c''i''o''p''$	$2a''j''n''r''$
$f''g''h''i''j''$	$3c''h''m''r''$	$3d''j''k''q''$	$3b''f''o''s''$
$k''l''m''n''o''$	$4d''i''n''s''$	$4e''f''l''r''$	$4c''g''k''t''$
$p''q''r''s''t''$	$5e''j''o''t''$	$5a''g''m''s''$	$5d''h''l''p''$

Step three of Construction 3:
Now, we construct three 4-way $(29, 6, 2)$ Steiner trades T_A, T_B, and T_c of volume 20, respectively, from 4-way 1-solely balanced sets S_A, S_B, and S_C by applying Theorem 1.

T_A :

T_{1A}	T_{2A}	T_{3A}	T_{4A}
$\tilde{x}12345$	$\tilde{x}1afkp$	$\tilde{x}1bhnt$	$\tilde{x}1eimq$
$\tilde{x}abcde$	$\tilde{x}2gblq$	$\tilde{x}2ciop$	$\tilde{x}2ajnr$
$\tilde{x}fghij$	$\tilde{x}3chmr$	$\tilde{x}3djkq$	$\tilde{x}3bfos$
$\tilde{x}klmno$	$\tilde{x}4dins$	$\tilde{x}4eflr$	$\tilde{x}4cgkt$
$\tilde{x}pqrst$	$\tilde{x}5ejot$	$\tilde{x}5agms$	$\tilde{x}5dhlp$
$\tilde{y}1afkp$	$\tilde{y}1bhnt$	$\tilde{y}1eimq$	$\tilde{y}12345$
$\tilde{y}2gblq$	$\tilde{y}2ciop$	$\tilde{y}2ajnr$	$\tilde{y}abcde$
$\tilde{y}3chmr$	$\tilde{y}3djkq$	$\tilde{y}3bfos$	$\tilde{y}fghij$
$\tilde{y}4dins$	$\tilde{y}4eflr$	$\tilde{y}4cgkt$	$\tilde{y}klmno$
$\tilde{y}5ejot$	$\tilde{y}5agms$	$\tilde{y}5dhlp$	$\tilde{y}pqrst$
$\tilde{z}1bhnt$	$\tilde{z}1eimq$	$\tilde{z}12345$	$\tilde{z}1afkp$
$\tilde{z}2ciop$	$\tilde{z}2ajnr$	$\tilde{z}abcde$	$\tilde{z}2gblq$
$\tilde{z}3djkq$	$\tilde{z}3bfos$	$\tilde{z}fgijh$	$\tilde{z}3chmr$
$\tilde{z}4eflr$	$\tilde{z}4cgkt$	$\tilde{z}klmno$	$\tilde{z}4dins$
$\tilde{z}5agms$	$\tilde{z}5dhlp$	$\tilde{z}pqrst$	$\tilde{z}5ejot$
$\tilde{w}1eimq$	$\tilde{w}12345$	$\tilde{w}1afkp$	$\tilde{w}1bhnt$
$\tilde{w}2ajnr$	$\tilde{w}abcde$	$\tilde{w}2gblq$	$\tilde{w}2ciop$
$\tilde{w}3bfos$	$\tilde{w}fghij$	$\tilde{w}3chmr$	$\tilde{w}3djkq$
$\tilde{w}4cgkt$	$\tilde{w}klmno$	$\tilde{w}4dins$	$\tilde{w}4eflr$
$\tilde{w}5dhlp$	$\tilde{w}pqrst$	$\tilde{w}5ejot$	$\tilde{w}5agms$

T_B :

T_{1B}	T_{2B}	T_{3B}	T_{4B}
$\hat{y}12345$	$\hat{y}1afkp$	$\hat{y}1bhnt$	$\hat{y}1eimq$
$\hat{y}a'b'c'd'e'$	$\hat{y}2g'b'l'q'$	$\hat{y}2c'i'o'p'$	$\hat{y}2a'j'n'r'$
$\hat{y}f'g'h'i'j'$	$\hat{y}3c'h'm'r'$	$\hat{y}3d'j'k'q'$	$\hat{y}3b'f'o's'$
$\hat{y}k'l'm'n'o'$	$\hat{y}4d'i'n's'$	$\hat{y}4e'f'l'r'$	$\hat{y}4c'g'k't'$
$\hat{y}p'q'r's't'$	$\hat{y}5e'j'o't'$	$\hat{y}5a'g'm's'$	$\hat{y}5d'h'l'p'$
$\hat{x}1a'f'k'p'$	$\hat{x}1b'h'n't'$	$\hat{x}1e'i'm'q'$	**$\hat{x}12345$**
$\hat{x}2g'b'l'q'$	$\hat{x}2c'i'o'p'$	$\hat{x}2a'j'n'r'$	$\hat{x}a'b'c'd'e'$
$\hat{x}3c'h'm'r'$	$\hat{x}3d'j'k'q'$	$\hat{x}3b'f'o's'$	$\hat{x}f'g'h'i'j'$
$\hat{x}4d'i'n's'$	$\hat{x}4e'f'l'r'$	$\hat{x}4c'g'k't'$	$\hat{x}k'l'm'n'o'$
$\hat{x}5ejot$	$\hat{x}5\mathrm{agms}$	$\hat{x}5\mathrm{dhlp}$	$\hat{x}\mathrm{pqrst}$
$\hat{w}1b'h'n't'$	$\hat{w}1e'i'm'q'$	$\hat{w}12345$	$\hat{w}1a'f'k'p'$
$\hat{w}2c'i'o'p'$	$\hat{w}2a'j'n'r'$	$\hat{w}a'b'c'd'e'$	$\hat{w}2g'b'l'q'$
$\hat{w}3d'j'k'q'$	$\hat{w}3b'f'o's'$	$\hat{w}fg'i'j'h'$	$\hat{w}3c'h'm'r'$
$\hat{w}4e'f'l'r'$	$\hat{w}4c'g'k't'$	$\hat{w}k'l'm'n'o'$	$\hat{w}4d'i'n's'$
$\hat{w}5a'g'm's'$	$\hat{w}5d'h'l'p'$	$\hat{w}p'q'r's't'$	$\hat{w}5e'j'o't'$
$\hat{z}1e'i'm'q'$	$\hat{z}12345$	$\hat{z}1a'f'k'p'$	$\hat{z}1b'h'n't'$
$\hat{z}2a'j'n'r'$	$\hat{z}a'b'c'd'e'$	$\hat{z}2g'b'l'q'$	$\hat{z}2c'i'o'p'$
$\hat{z}3b'f'o's'$	$\hat{z}f'g'h'i'j'$	$\hat{z}3c'h'm'r'$	$\hat{z}3d'j'k'q'$
$\hat{z}4c'g'k't'$	$\hat{z}k'l'm'n'o'$	$\hat{z}4d'i'n's'$	$\hat{z}4e'f'l'r'$
$\hat{z}5d'h'l'p'$	$\hat{z}p'q'r's't'$	$\hat{z}5e'j'o't'$	$\hat{z}5a'g'm's'$

T_C :

T_{1C}	T_{2C}	T_{3C}	T_{4C}
$\grave{z}12345$	$\grave{z}1a''f''k''p''$	$\grave{z}1b''h''n''t''$	$\grave{z}1e''i''m''q''$
$\grave{z}a''b''c''d''e''$	$\grave{z}2g''b''l''q''$	$\grave{z}2c''i''o''p''$	$\grave{z}2a''j''n''r''$
$\grave{z}f''g''h''i''j''$	$\grave{z}3c''h''m''r''$	$\grave{z}3d''j''k''q''$	$\grave{z}3b''f''o''s''$
$\grave{z}k''l''m''n''o''$	$\grave{z}4d''i''n''s''$	$\grave{z}4e''f''l''r''$	$\grave{z}4c''g''k''t''$
$\grave{z}p''q''r''s''t''$	$\grave{z}5e''j''o''t''$	$\grave{z}5a''g''m''s''$	$\grave{z}5d''h''l''p''$
$\grave{w}1a''f''k''p''$	$\grave{w}1b''h''n''t''$	$\grave{w}1e''i''m''q''$	$\grave{w}12345$
$\grave{w}2g''b''l''q''$	$\grave{w}2c''i''o''p''$	$\grave{w}2a''j''n''r''$	$\grave{w}a''b''c''d''e''$
$\grave{w}3c''h''m''r''$	$\grave{w}3d''j''k''q''$	$\grave{w}3b''f''o''s''$	$\grave{w}f''g''h''i''j''$
$\grave{w}4d''i''n''s''$	$\grave{w}4e''f''l''r''$	$\grave{w}4c''g''k''t''$	$\grave{w}k''l''m''n''o''$
$\grave{w}5e''j''o''t''$	$\grave{w}5a''g''m''s''$	$\grave{w}5d''h''l''p''$	$\grave{w}p''q''r''s''t''$
$\grave{x}1b''h''n''t''$	$\grave{x}1e''i''m''q''$	**$\grave{x}12345$**	$\grave{x}1a''f''k''p''$
$\grave{x}2c''i''o''p''$	$\grave{x}2a''j''n''r''$	$\grave{x}a''b''c''d''e''$	$\grave{x}2g''b''l''q''$
$\grave{x}3d''j''k''q''$	$\grave{x}3b''f''o''s''$	$\grave{x}f''g''i''j''h''$	$\grave{x}3c''h''m''r''$
$\grave{x}4e''f''l''r''$	$\grave{x}4c''g''k''t''$	$\grave{x}k''l''m''n''o''$	$\grave{x}4d''i''n''s''$
$\grave{x}5a''g''m''s''$	$\grave{x}5d''h''l''p''$	$\grave{x}p''q''r''s''t''$	$\grave{x}5e''j''o''t''$
$\grave{y}1e''i''m''q''$	$\grave{y}12345$	$\grave{y}1a''f''k''p''$	$\grave{y}1b''h''n''t''$
$\grave{y}2a''j''n''r''$	$\grave{y}a''b''c''d''e''$	$\grave{y}2g''b''l''q''$	$\grave{y}2c''i''o''p''$
$\grave{y}3b''f''o''s''$	$\grave{y}f''g''h''i''j''$	$\grave{y}3c''h''m''r''$	$\grave{y}3d''j''k''q''$
$\grave{y}4c''g''k''t''$	$\grave{y}k''l''m''n''o''$	$\grave{y}4d''i''n''s''$	$\grave{y}4e''f''l''r''$
$\grave{y}5d''h''l''p''$	$\grave{y}p''q''r''s''t''$	$\grave{y}5e''j''o''t''$	$\grave{y}5a''g''m''s''$

Now, we have three 4-way $(29, 6, 2)$ Steiner trades T_A, T_B, and T_C, of volume 20 . By adding three trades we obtain one 4-way Steiner trade of volume 60.

Step four of Construction 3:
Take $\tilde{x} = \hat{x} = \dot{x}$. Then there exists the common block $\mathbf{\tilde{x}12345}$ in these trades. This block is not in the second collection of the trades T_A, T_B, and T_C. Remove T_{2A}, T_{2B}, and T_{2C} from 4-way $(29, 6, 2)$ Steiner trades T_A, T_B, and T_C and Obtain three 3-way $(29, 6, 2)$ Steiner trades T'_A, T'_B, and T'_C of volume 20.
Step five of Construction 3:
We now add three 3-way $(29, 6, 2)$ Steiner trades T'_A, T'_B, and T'_C, of volume 20 and remove the common block $\mathbf{\tilde{x}12345}$. By adding these trades, we have a 3-way $(29, 6, 2)$ Steiner trade of volume $20 + 20 + 20 - 1 = 60 - 1 = 12 \times 5 - 1$.

By using the above construction, we obtain some theorems that are used in the main results.

2. Main Results

In [17], it was proved that there exists a 3-way $(v, k, 2)$ Steiner trade of volume m when $m \geq 12(k-1)$ for $k \geq 15$. In [18], it was shown that it is correct also for $k \leq 14$. In the following theorem, we improve the lower bound from $12(k-1)$ to $11(k-1)$, when k is even. In this theorem, we apply Construction 3.

Theorem 4. *There exists a 3-way $(v, k, 2)$ Steiner trade of volume $12(k-1) - r$ for $r \in \{0, \ldots, k-2\}$ with block size k for even k.*

Proof. The first step of Construction 3: Take three $(k-1) \times (k-1)$ matrices A, B, and C with r common rows.
The second step of Construction 3: Construct three 4-way 1-solely balanced sets S_A, S_B, and S_C of volume $k-1$ as shown in Example 2.
The third step of Construction 3: Now apply Theorem 1 to construct three 4-way Steiner trades of volume $4(k-1)$ with block size k denoted by T_A, T_B, and T_c.
In this proof, take $\tilde{x}$=$\hat{x} = \bar{x}$.

T_A :

T_{1A}	T_{2A}	T_{3A}	T_{4A}
$\mathbf{\tilde{x}S_{1A}}$	$\tilde{x}S_{2A}$	$\tilde{x}S_{3A}$	$\tilde{x}S_{4A}$
$\tilde{y}S_{2A}$	$\tilde{y}S_{3A}$	$\tilde{y}S_{4A}$	$\tilde{y}S_{1A}$
$\tilde{z}S_{3A}$	$\tilde{z}S_{4A}$	$\tilde{z}S_{1A}$	$\tilde{z}S_{2A}$
$\tilde{w}S_{4A}$	$\tilde{w}S_{1A}$	$\tilde{w}S_{2A}$	$\tilde{w}S_{3A}$

T_B :

T_{1B}	T_{2B}	T_{3B}	T_{4B}
$\hat{y}S_{1B}$	$\hat{y}S_{2B}$	$\hat{y}S_{3B}$	$\hat{y}S_{4B}$
$\tilde{x}S_{2B}$	$\tilde{x}S_{3B}$	$\tilde{x}S_{4B}$	$\boldsymbol{\tilde{x}S_{1B}}$
$\hat{w}S_{3B}$	$\hat{w}S_{4B}$	$\hat{w}S_{1B}$	$\hat{w}S_{2B}$
$\hat{z}S_{4B}$	$\hat{z}S_{1B}$	$\hat{z}S_{2B}$	$\hat{z}S_{3B}$

T_C :

T_{1C}	T_{2C}	T_{3C}	T_{4C}
$\bar{z}S_{1C}$	$\bar{z}S_{2C}$	$\bar{z}S_{3C}$	$\bar{z}S_{4C}$
$\bar{w}S_{2C}$	$\bar{w}S_{3C}$	$\bar{w}S_{4C}$	$\bar{w}S_{1C}$
$\tilde{x}S_{3C}$	$\tilde{x}S_{4C}$	$\boldsymbol{\tilde{x}S_{1C}}$	$\tilde{x}S_{2C}$
$\bar{y}S_{4C}$	$\bar{y}S_{1C}$	$\bar{y}S_{2C}$	$\bar{y}S_{3C}$

The fourth step of Construction 3: The bold parts of trades contain the r common blocks. Remove, the second collection from each 4-way Steiner trade and obtain three 3-way Steiner trades.
The fifth step of Construction 3: These trades have r common blocks in the different collections.
Therefore, $T:\ (T_1, T_2, T_3) = (T_{1A}, T_{3A}, T_{4A}) + (T_{1B}, T_{3B}, T_{4B}) + (T_{1C}, T_{3C}, T_{4C})$ is a 3-way $(v, k, 2)$ Steiner trade of volume $12(k-1) - r$ for $r \in \{0, \ldots, k-2\}$. The r common blocks are removed when the trades are added. □

In the following theorem, we improve the lower bound and prove that $m \in S_{3s}(2, k)$ for all $m \geq 10(k-1) + 1$ when k is even. In this theorem, we apply Construction 3.

Theorem 5. *There exists a 3-way $(v, k, 2)$ Steiner trade of volume $11(k-1) - r$ for $r \in \{0, \ldots, k-2\}$ with block size k for even k.*

Proof. The first step of Construction 3: Take three $(k-1) \times (k-1)$ matrices A, B, and C with r common rows.
The second step of Construction 3: Construct two 4-way 1-solely balanced sets S_A, S_B of volume $k-1$ and one 3-way 1-solely balanced set S_C of volume $k-1$ as shown in Example 2.
The third step of Construction 3: Now apply Theorem 1 to construct two 4-way Steiner trades of volume $4(k-1)$ with block size k denoted T_A and T_B, and construct one 3-way Steiner trade of volume $3(k-1)$ with block size k denoted T_C. In this proof, take $\tilde{x} = \hat{x} = \bar{x}$.

T_A :

T_{1A}	T_{2A}	T_{3A}	T_{4A}
$\boldsymbol{\tilde{x}S_{1A}}$	$\tilde{x}S_{2A}$	$\tilde{x}S_{3A}$	$\tilde{x}S_{4A}$
$\tilde{y}S_{2A}$	$\tilde{y}S_{3A}$	$\tilde{y}S_{4A}$	$\tilde{y}S_{1A}$
$\tilde{z}S_{3A}$	$\tilde{z}S_{4A}$	$\tilde{z}S_{1A}$	$\tilde{z}S_{2A}$
$\tilde{w}S_{4A}$	$\tilde{w}S_{1A}$	$\tilde{w}S_{2A}$	$\tilde{w}S_{3A}$

T_B :

T_{1B}	T_{2B}	T_{3B}	T_{4B}
$\hat{y}S_{1B}$	$\hat{y}S_{2B}$	$\hat{y}S_{3B}$	$\hat{y}S_{4B}$
$\tilde{x}S_{2B}$	$\tilde{x}S_{3B}$	$\tilde{x}S_{4B}$	$\boldsymbol{\tilde{x}S_{1B}}$
$\hat{w}S_{3B}$	$\hat{w}S_{4B}$	$\hat{w}S_{1B}$	$\hat{w}S_{2B}$
$\hat{z}S_{4B}$	$\hat{z}S_{1B}$	$\hat{z}S_{2B}$	$\hat{z}S_{3B}$

T_C :

T_{1C}	T_{2C}	T_{3C}
$\bar{z}S_{1C}$	$\bar{z}S_{2C}$	$\bar{z}S_{3C}$
$\tilde{x}S_{3C}$	$\boldsymbol{\tilde{x}S_{1C}}$	$\tilde{x}S_{2C}$
$\bar{y}S_{1C}$	$\bar{y}S_{2C}$	$\bar{y}S_{3C}$

The fourth step of Construction 3: The bold parts of trades contain the r common blocks. Now remove the third collection from T_A, and T_B.
The fifth step of Construction 3: We obtain three 3-way Steiner trades. These trades have r common blocks in the different collections. Therefore, $T:\ (T_1, T_2, T_3) = (T_{1A}, T_{2A}, T_{4A}) + (T_{1B}, T_{2B}, T_{4B}) + (T_{1C}, T_{2C}, T_{3C})$ is a 3-way $(v, k, 2)$ Steiner trade of volume $11(k-1) - r$ for $r \in \{0, \ldots, k-2\}$. □

In what follows, we improve the lower bound and prove that $m \in S_{3s}(2, k)$ for all $m \geq 9(k-1) + 1$ when k is even. In this theorem, we apply Construction 3.

Theorem 6. *There exists a 3-way* $(v, k, 2)$ *Steiner trade of volume* $10(k-1)-r$ *for* $r \in \{0, \ldots, k-2\}$ *with block size* k *for even* k.

Proof. The first step of Construction 3: Take three $(k-1) \times (k-1)$ matrices A, B, and C with r common rows.
The second step of Construction 3: Construct two 3-way 1-solely balanced sets S_A, S_C of volume $k-1$ and one 4-way 1-solely balanced set S_B of volume $k-1$ as shown in Example 2.
The third step of Construction 3: Now apply Theorem 1 to construct two 3-way Steiner trade of volume $3(k-1)$ with block size k denoted T_A and T_C, and construct one 4-way Steiner trade of volume $4(k-1)$ with block size k denoted T_B. In this proof, take $\tilde{x} = \hat{x} = \bar{x}$.

T_A :

T_{1A}	T_{2A}	T_{3A}
$\boldsymbol{\tilde{x}S_{1A}}$	$\tilde{x}S_{2A}$	$\tilde{x}S_{3A}$
$\tilde{y}S_{2A}$	$\tilde{y}S_{3A}$	$\tilde{y}S_{4A}$
$\tilde{z}S_{3A}$	$\tilde{z}S_{4A}$	$\tilde{z}S_{1A}$

T_B :

T_{1B}	T_{2B}	T_{3B}	T_{4B}
$\hat{y}S_{1B}$	$\hat{y}S_{2B}$	$\hat{y}S_{3B}$	$\hat{y}S_{4B}$
$\tilde{x}S_{2B}$	$\tilde{x}S_{3B}$	$\tilde{x}S_{4B}$	$\boldsymbol{\tilde{x}S_{1B}}$
$\hat{w}S_{3B}$	$\hat{w}S_{4B}$	$\hat{w}S_{1B}$	$\hat{w}S_{2B}$
$\hat{z}S_{4B}$	$\hat{z}S_{1B}$	$\hat{z}S_{2B}$	$\hat{z}S_{3B}$

T_C :

T_{1C}	T_{2C}	T_{3C}
$\bar{z}S_{1C}$	$\bar{z}S_{2C}$	$\bar{z}S_{3C}$
$\tilde{x}S_{3C}$	$\boldsymbol{\tilde{x}}S_{1C}$	$\tilde{x}S_{2C}$
$\bar{y}S_{1C}$	$\bar{y}S_{2C}$	$\bar{y}S_{3C}$

The fourth step of Construction 3: The bold part of trades contains the r common blocks. Now remove the first collection of T_B.
The fifth step of Construction 3: We obtain three 3-way Steiner trades. These trades have r common blocks in the different collections. Therefore, $T : (T_1, T_2, T_3) = (T_{1A}, T_{2A}, T_{3A}) + (T_{2B}, T_{3B}, T_{4B}) + (T_{1C}, T_{2C}, T_{3C})$ is a 3-way $(v, k, 2)$ Steiner trade of volume $10(k-1)-r$ for $r \in \{0, \ldots, k-2\}$. □

Theorem 7. *There exists a 3-way* $(v, k, 2)$ *Steiner trade of volume* $9(k-1)-r$ *for* $r \in \{0, \ldots, k-2\}$ *with block size* k *for even* k.

Proof. The first step of Construction 3: Take three $(k-1) \times (k-1)$ matrices A, B, and C with r common rows.
The second step of Construction 3: Construct three 3-way 1-solely balanced sets S_A, S_B, and S_C of volume $k-1$ as shown in Example 2.
The third step of Construction 3: Now apply Theorem 1 to construct two 3-way Steiner trade of volume $3(k-1)$ with block size k denoted T_A and T_C, and construct one 4-way Steiner trade of volume $4(k-1)$ with block size k denoted T_B. In this proof, take $\tilde{x} = \hat{x} = \bar{x}$.

T_A :

T_{1A}	T_{2A}	T_{3A}
$\boldsymbol{\tilde{x}S_{1A}}$	$\tilde{x}S_{2A}$	$\tilde{x}S_{3A}$
$\tilde{y}S_{2A}$	$\tilde{y}S_{3A}$	$\tilde{y}S_{4A}$
$\tilde{z}S_{3A}$	$\tilde{z}S_{4A}$	$\tilde{z}S_{1A}$

T_B :

T_{1B}	T_{2B}	T_{3B}
$\hat{y}S_{1B}$	$\hat{y}S_{2B}$	$\hat{y}S_{3B}$
$\tilde{x}S_{2B}$	$\tilde{x}S_{3B}$	$\boldsymbol{\tilde{x}S_{1B}}$
$\hat{w}S_{3B}$	$\hat{w}S_{1B}$	$\hat{w}S_{2B}$
$\hat{z}S_{1B}$	$\hat{z}S_{2B}$	$\hat{z}S_{3B}$

T_C :

T_{1C}	T_{2C}	T_{3C}
$\bar{z}S_{1C}$	$\bar{z}S_{2C}$	$\bar{z}S_{3C}$
$\tilde{x}S_{3C}$	$\boldsymbol{\tilde{x}}S_{1C}$	$\tilde{x}S_{2C}$
$\bar{y}S_{1C}$	$\bar{y}S_{2C}$	$\bar{y}S_{3C}$

The fourth step of Construction 3: The bold part of trades contains the r common blocks. Now remove the first collection of T_B.

The fifth step of Construction 3: We obtain three 3-way Steiner trades. These trades have r common blocks in the different collections. Therefore, $T : (T_1, T_2, T_3) = (T_{1A}, T_{2A}, T_{3A}) + (T_{2B}, T_{3B}, T_{4B}) + (T_{1C}, T_{2C}, T_{3C})$ is a 3-way $(v, k, 2)$ Steiner trade of volume $9(k-1) - r$ for $r \in \{0, \ldots, k-2\}$. □

Main Theorem:

Theorem 8. *There exists a 3-way $(v, k, 2)$ Steiner trade of volume m for $m \geq 8(k-1)$ when $k \geq 4$ is even.*

Proof. For $k = 4$, the authors already proved in [17] that $S_{3s}(2, 4) = N \setminus \{1, 2, 3, 4, 5, 6, 7\}$, and it is concluded from Theorems 4, 5, 6, and 7 for $k \geq 6$. □

Conflict of interest. The authors declare that they have no conflict of interest.

Data Availability. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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