Communications in Combinatorics and Optimization
Vol. 9, No. 3 (2024), pp. 451-466
https://doi.org/10.22049/cco.2023.27973.1408

CCO
Commun. Comb. Optim.

Research Article

On the *rna* number of generalized Petersen graphs

Deepak Sehrawat* **and Bikash Bhattacharjya**†

Department of Mathematics, Indian Institute of Technology Guwahati, Guwahati, India
*deepakmath55555@iitg.ac.in
†b.bikash@iitg.ac.in

Received: 22 August 2022; Accepted: 10 February 2023
Published Online: 15 February 2023

Abstract: A signed graph (G, σ) is called a parity signed graph if there exists a bijective mapping $f \colon V(G) \to \{1, \ldots, |V(G)|\}$ such that for each edge uv in G, $f(u)$ and $f(v)$ have same parity if $\sigma(uv) = +1$, and opposite parity if $\sigma(uv) = -1$. The *rna* number $\sigma^-(G)$ of G is the least number of negative edges among all possible parity signed graphs over G. Equivalently, $\sigma^-(G)$ is the least size of an edge-cut of G that has nearly equal sides.

In this paper, we show that for the generalized Petersen graph $P_{n,k}$, $\sigma^-(P_{n,k})$ lies between 3 and n. Moreover, we determine the exact value of $\sigma^-(P_{n,k})$ for $k \in \{1, 2\}$. The *rna* numbers of some famous generalized Petersen graphs, namely, Petersen graph, Dürer graph, Möbius-Kantor graph, Dodecahedron, Desargues graph and Nauru graph are also computed. Recently, Acharya, Kureethara and Zaslavsky characterized the structure of those graphs whose *rna* number is 1. We use this characterization to show that the smallest order of a $(4n+1)$-regular graph having *rna* number 1 is $8n+6$. We also prove the smallest order of $(4n-1)$-regular graphs having *rna* number 1 is bounded above by $12n-2$. In particular, we show that the smallest order of a cubic graph having *rna* number 1 is 10.

Keywords: generalized Petersen graph, parity labeling, parity signed graph, *rna* number, edge-cut

AMS Subject classification: 05C78, 05C22, 05C40

1. Introduction

All graphs and signed graphs considered in this paper are simple, connected and undirected. For all the graph theoretic terms that are used in this paper but not defined, we refer the reader to [3]. Harary [4] was the first who introduced the concept

* *Corresponding Author*

of signed graphs in 1953. Signed graphs are a type of extension of graphs. In recent years, research on signed graphs has become one of the hot topics in graph theory. A signed graph (G, σ) consists of a graph G together with a function $\sigma\colon E(G) \to \{+1, -1\}$. Graph G is called the *underlying graph* of (G, σ) and σ a *signature* of (G, σ). An edge e in (G, σ) is called *positive* if $\sigma(e) = +1$, and *negative* otherwise.
Recently, Acharya and Kureethara [1] introduced a special type of signed graph called the parity signed graph. The concept of parity labeling (see Definition 1) is equivalent to a partition of the vertex set of a graph into two subsets A and B such that $||A| - |B|| \leq 1$. In [2], the authors characterized some families of parity signed graphs, namely, signed stars, bistars, cycles, paths and complete bipartite graphs.
The idea of the *rna* number of a parity signed graph was introduced by Acharya and Kureethara [1]. It is more appropriate to say the *rna* number of a graph, nevertheless, in accordance with its definition. The *rna* number of a graph G, denoted $\sigma^-(G)$, is the least number of negative egdes among all the possible parity signed graphs over G. It is equal to the least size of a cut whose sides are nearly equal. The *rna* number of some families of graphs such as stars, wheels, paths, cycles and complete graphs are already computed. For details, see [1, 2].
Acharya et al. in [2, Theorem 3.4] proved that for any natural number k, there exists a graph G such that $\sigma^-(G) = k$. This means no constant upper bound exists for the *rna* number in general. However, in [1, Proposition 2.8], authors proved that for a complete graph K_n on n vertices, $\sigma^-(K_n) = \lfloor \frac{n^2}{4} \rfloor$, where $n \geq 2$. Note that $\lfloor \frac{n^2}{4} \rfloor$ is a trivial upper bound for the *rna* number of graphs on n vertices. Very recently, Kang et al. [5] proved that $\sigma^-(G) \leq \lfloor \frac{m}{2} + \frac{n}{4} \rfloor$, where G is any graph on n vertices and m edges. Further, they proved that the bound $\lfloor \frac{m}{2} + \frac{n}{4} \rfloor$ is achieved by only three graphs, namely, $K_n, K_n - e$ and $K_n - \Delta$, where K_n denotes the complete graph on n vertices and Δ denotes a triangle.
The rest of the paper is organized as follows. In Section 2, we give necessary definitions and existing results related to parity signed graphs and the *rna* number. In Section 3, we discuss generalized Petersen graphs and their forbidden cuts of equal sides. In Section 4, we show that

(1) $\sigma^-(G)$ of the generalized Petersen graph $P_{n,k}$ lies between 3 and n, and these bounds are attained;

(2) for $n \geq 3$, $\sigma^-(P_{n,1}) = \begin{cases} 3 & \text{if } n = 3 \\ 4 & \text{if } n \text{ is even} \\ 5 & \text{if } n \text{ is odd and } n \geq 5; \end{cases}$

(3) for $n \geq 7$, $\sigma^-(P_{n,2}) = \begin{cases} 6 & \text{if } n \text{ is even} \\ 7 & \text{if } n \text{ is odd.} \end{cases}$

We further determine the *rna* numbers of the Petersen graph, Dürer graph, Möbius-Kantor graph, Dodecahedron, Desargues graph and the Nauru graph.
Acharya et al. [2, Theorem 3.5] characterized the structure of those graphs whose *rna* number is 1. More precisely, they proved that for any connected graph G, $\sigma^-(G) = 1$

if and only if G has a cut-edge joining two graphs whose orders differ by at most one. In Section 5, we apply this characterization to prove that the smallest order of a $(4n+1)$-regular graph of *rna* number 1 is $8n+6$, and that such a graph is unique. For the smallest order of $(4n-1)$-regular graphs having *rna* number 1, we obtain an upper bound of $12n-2$. Finally, we show that the smallest order of a cubic graph having *rna* number 1 is 10.

2. Preliminaries

A graph G is an ordered pair $(V(G), E(G))$, where $V(G)$ and $E(G)$ represent the vertex set and the edge set of G, respectively. By $|V(G)|$ and $|E(G)|$, we denote the order and the size of G, respectively. For a subset A of $V(G)$, the *induced subgraph* $G[A]$ is the subgraph of G whose vertex set is A and edge set consists of all edges of G having both end vertices in A. An edge e of a connected graph G is said to be a *cut-edge* of G if deletion of e results in a disconnected graph. An *edge-cut* (or a *cut*) in a connected graph is the set of all edges between a subset A and its complement. Such an edge-cut is denoted by $[A, A^c]$. The *size* of the cut $[A, A^c]$ is the number of edges in $[A, A^c]$. A cut is said to be an *even cut* or an *odd cut* according as its size is even or odd, respectively. The numbers $|A|$ and $|A^c|$ are called the sides of the cut $[A, A^c]$.
Now we give some necessary definitions and results.

Definition 1. [2] For a given graph G of order n and a bijective mapping $f\colon V(G) \to \{1, \ldots, n\}$, define $\sigma_f\colon E(G) \to \{+1, -1\}$ such that $\sigma_f(uv) = +1$ if $f(u)$ and $f(v)$ are of the same parity and $\sigma_f(uv) = -1$ if $f(u)$ and $f(v)$ are of different parity, where $uv \in E(G)$. We define Σ_f to be the signed graph (G, σ_f), which is called a *parity signed graph*.

A cycle in a signed graph is *positive* if the product of its edge signs is positive. A signed graph is said to be *balanced* if all of its cycles are positive. Every parity signed graph is balanced, see [2, Theorem 1].

Definition 2. [1] The *rna* number of a graph G, denoted $\sigma^-(G)$, is the least number of negative edges among all possible parity signed graphs over G.

Note that finding the least number of negative edges among all parity signed graphs over a graph G is equivalent of finding the least size of a cut of G with nearly equal sides [1, 2]. More precisely, if G is of even order, then $\sigma^-(G)$ is the least size of a cut of G whose sides are equal. If G is of odd order, then $\sigma^-(G)$ is the least size of a cut of G whose sides differ by exactly 1.
Now we mention the *rna* number of some well known graphs.
In [1], the authors proved that $\sigma^-(P_n) = 1$, where P_n is the path of order n. They also proved that if C_n is the cycle on n vertices, then $\sigma^-(C_n) = 2$. Further, they proved that if $K_{1,n}$ is the star on $n+1$ vertices and K_n is the complete graph on n

vertices, then $\sigma^-(K_{1,n}) = \lceil \frac{n}{2} \rceil$ and $\sigma^-(K_n) = \lfloor \frac{n^2}{4} \rfloor$. The authors in [2], proved that if W_n is the wheel on $n+1$ vertices, then $\sigma^-(W_n) = \lceil \frac{n+4}{2} \rceil$.
Throughout the paper, the notation $[n]$ represents the set $\{0, 1, \ldots, n-1\}$ for each $n \in \mathbb{N}$.

3. Generalized Petersen graphs and their forbidden cuts

Definition 3. Let n and k be positive integers such that $2 \leq 2k < n$. Then the *generalized Petersen graph* $P_{n,k}$ is the graph whose vertex set and edge set are given by $V(P_{n,k}) = \{u_i, v_i \colon i \in [n]\}$ and $E(P_{n,k}) = \{u_i u_{i+1}, v_i v_{i+k}, u_i v_i \colon i \in [n]\}$, respectively, where the subscripts are read modulo n.

The vertices in $\{u_i \colon i \in [n]\}$ and $\{v_i \colon i \in [n]\}$ are called the *outer vertices* and the *inner vertices*, respectively, of $P_{n,k}$. Sometimes, we also call the outer vertices and the inner vertices the *u-vertices* and *v-vertices*, respectively. The cycle formed by the u-vertices is called the *outer cycle* and it is denoted by C_o. The cycle(s) formed by the v-vertices is(are) called the *inner cycle(s)*. If $d := \gcd(n,k) > 1$, then $P_{n,k}$ has d inner cycles. These inner cycles are denoted by $C_1, \ldots, C_d$, where $v_i \in V(C_{i+1})$ for $i \in [d]$. If $d = 1$, then $P_{n,k}$ has only one inner cycle, and in this case the inner cycle is denoted by C_I. The edges of the form $u_i v_i$ are called the *spokes* of $P_{n,k}$. The vertices u_i and v_i are called the *partner* of each other for each $i \in [n]$.

Lemma 1. *If n is even and $n \geq 4$, then $P_{n,k}$ cannot have an odd cut of equal sides.*

Proof. Let $n = 2\ell$ for some $\ell \geq 2$. On the contrary, let $P_{2\ell,k}$ have an odd cut of equal sides. Hence there exists a subset A of $V(P_{2\ell,k})$ such that $|A| = 2\ell$ and $|[A, A^c]| = 2r+1$ for some positive integer r.
If $d_A(a)$ is the degree of the vertex a in $P_{2\ell,k}[A]$, then

$$\sum_{a \in A} d_A(a) = 3(2\ell) - (2r+1), \text{ an odd integer.}$$

This is a contradiction to the handshaking lemma. Therefore, no odd cut of equal sides is possible in $P_{2\ell,k}$. This completes the proof. □

Lemma 2. *If n is odd and $n \geq 5$, then $P_{n,k}$ cannot have an even cut of equal sides.*

Proof. Let $n = 2\ell+1$ for some $\ell \geq 2$. Let, if possible, $P_{2\ell+1,k}$ have an even cut of equal sides. Hence there exists $A \subseteq V(P_{2\ell+1,k})$ such that $|A| = 2\ell+1$ and $|[A, A^c]| = 2r$ for some positive integer r.
If $d_A(a)$ is the degree of the vertex a in $P_{2\ell+1,k}[A]$, then

$$\sum_{a \in A} d_A(a) = 3(2\ell+1) - 2r, \text{ an odd integer.}$$

This is a contradiction to the handshaking lemma. Therefore, no even cut of equal sides is possible in $P_{2\ell+1,k}$. This completes the proof. $\square$

4. The *rna* number of generalized Petersen graphs

According to Kang et al. [5], $\sigma^-(P_{n,k}) \leq 2n$. In this section, we prove that the *rna* number of $P_{n,k}$ is bounded above by n, improving the upper bound of Kang et al. [5] for the class of generalized Petersen graphs. Further, we compute $\sigma^-(P_{n,1})$ for $n \geq 3$, and $\sigma^-(P_{n,2})$ for $n \geq 6$.

The *edge-connectivity* $\kappa'(G)$ of a graph G is the least size of its cuts. For a connected graph G with minimum degree δ, it is well known that $1 \leq \kappa'(G) \leq \delta$. A simple but important result is the following.

Theorem 1. *If G is a graph of edge-connectivity k, then $\sigma^-(G) \geq k$.*

Proof. Since $\kappa'(G) = k$, no cut of G with nearly equal sides can have less than k edges. Hence $\sigma^-(G) \geq k$, as desired. $\square$

Theorem 2. *If $n \geq 3$ and $k \geq 1$, then $3 \leq \sigma^-(P_{n,k}) \leq n$.*

Proof. Since $\kappa'(P_{n,k}) = 3$, the lower bound follows from Theorem 1.
Define $f\colon V(P_{n,k}) \to \{1, \ldots, 2n\}$ such that $f(u_i) = 2i + 1$ and $f(v_i) = 2i + 2$ for $0 \leq i \leq n - 1$. The labeling f induces the parity signed graph $(P_{n,k}, \sigma_f)$. Note that all the spokes are negative in $(P_{n,k}, \sigma_f)$. Thus, the number of negative edges in $(P_{n,k}, \sigma_f)$ is n. Hence $\sigma^-(P_{n,k}) \leq n$. $\square$

The lower bound and the upper bound in Theorem 2 are attained. This holds true because in Theorem 5 and Example 1, we show that $\sigma^-(P_{3,1}) = 3$ and $\sigma^-(P_{5,2}) = 5$, respectively.

Lemma 3. *If $n \geq 4$ and $k \geq 1$, then $P_{n,k}$ cannot have a cut of size three of equal sides.*

Proof. We analyse two cases depending on whether n is odd or even.
Case 1. Let $n = 2\ell$ for $\ell \geq 2$. Let there exist a subset A of $V(P_{2\ell,k})$ such that $|A| = 2\ell$ and $|[A, A^c]| = 3$. Denote the degree of a vertex a in $P_{2\ell,k}[A]$ by $d_A(a)$. We have

$$\sum_{a \in A} d_A(a) = 3(2\ell) - 3, \text{ an odd integer.}$$

This shows that $P_{2\ell,k}[A]$ does not satisfy the handshaking lemma. Hence no such A is possible.
Case 2. Let $n = 2\ell + 1$ for $\ell \geq 2$. Let there exist a subset A of $V(P_{2\ell+1,k})$ such that $|A| = 2\ell + 1$ and $|[A, A^c]| = 3$. If A contains either all u-vertices or all v-vertices,

then all the spokes are in $[A, A^c]$. This is a contradiction to the assumption that $|[A, A^c]| = 3$. Therefore, A must contain u-vertices as well as v-vertices. Consequently $[A, A^c]$ contains at least two edges of C_o, since u-vertices lie in both A and A^c.
Now we consider two subcases.
Subcase 2(i). Let $\gcd(2\ell+1, k) = 1$. In this sub-case, $P_{2\ell+1,k}$ has exactly one inner cycle C_I. The condition that the v-vertices lie in both A and A^c forces $[A, A^c]$ to contain at least two edges of C_I. Thus we have $|[A, A^c]| \geq 4$, a contradiction to the assumption that $|[A, A^c]| = 3$.
Subcase 2(ii). Let $\gcd(2\ell+1, k) = d \geq 2$. In this sub-case, $P_{2\ell+1,k}$ has the inner cycles $C_1, \dots, C_d$. Note that all vertices of C_i lie entirely in A or in A^c. Otherwise, we get a contradiction on the size of $[A, A^c]$. Therefore, the vertices of at least one inner cycle do not lie in A, and the vertices of at least one inner cycle do not lie in A^c. Hence both A and A^c contain at least three u-vertices.
It is easy to see that if $|[A, A^c]| = 3$, then exactly two edges of $[A, A^c]$ must be edges of C_o, and the third edge must be a spoke. Let this spoke be u_jv_j for some $j \in \{0, 1, \dots, 2\ell\}$. Without loss of generality, let $u_j \in A$ and $v_j \in A^c$. Since exactly one spoke lies in $[A, A^c]$, the remaining u-vertices of A must have their partners in A. Thus the number of u-vertices and v-vertices in A are $\ell+1$ and ℓ, respectively. Also, $[A, A^c]$ has exactly two edges of C_o. Therefore, there exists a path of length ℓ induced by the u-vertices of A. Let the end vertices of this path be u_r and $u_{r+\ell}$ for some $r \in \{0, 1, \dots, 2\ell\}$. Consequently, the set of v-vertices in A is $\{v_r, v_{r+1}, \dots, v_{r+\ell}\} \setminus \{v_j\}$ and the subgraph induced by these v-vertices of A must be union of some inner cycle(s).
The condition $2k < 2\ell+1$, together with $\gcd(2\ell+1, k) = d \geq 2$, implies that $3 \leq k \leq \ell$. For $v_{r+\ell} \neq v_j$, consider the inner cycle $v_{r+\ell}v_{r+\ell+k} \dots v_{r+\ell-k}v_{r+\ell}$ containing the vertex $v_{r+\ell}$. We see that $v_{r+\ell+k} \in \{v_r, v_{r+1}, \dots, v_{r+\ell}\}$ only if $k \geq \ell+1$. As $k \leq \ell$, we conclude that all the vertices of the inner cycle containing $v_{r+\ell}$ do not lie in A. This is a contradiction to the fact that all vertices of C_i lie entirely in A or in A^c. If $v_{r+\ell} = v_j$, then we consider the inner cycle containing $v_{r+\ell-1}$ and get a similar contradiction. This completes this proof. $\square$

Theorem 3. *Let $n \geq 5$ and $k \geq 2$. If $\gcd(n, k) = 1$, then $5 \leq \sigma^-(P_{n,k}) \leq n$.*

Proof. The upper bound follows from Theorem 2.
For odd n, the lower bound follows from Lemma 3 and Lemma 2. Let n be an even integer. By Lemma 3, $\sigma^-(P_{n,k}) \geq 4$. Now we show that the *rna* number of $P_{n,k}$ cannot be 4.
Let, if possible, $P_{n,k}$ have a cut of size four of equal sides. Therefore, there exists a subset A of $V(P_{n,k})$ such that $|A| = n$ and $|[A, A^c]| = 4$. If A contains either all u-vertices or all v-vertices, then $[A, A^c]$ contains precisely n spokes of $P_{n,k}$, that is, $|[A, A^c]| = n \geq 5$, a contradiction to the fact that $|[A, A^c]| = 4$. Therefore, A must contain u-vertices as well as v-vertices.

Since $\gcd(n,k) = 1$, there is only one inner cycle in $P_{n,k}$. Note that $[A, A^c]$ must contain exactly two edges of C_o and two edges of C_I, because A (and also A^c) contains vertices of both C_o and C_I. Also, A contains as many u-vertices as v-vertices. Otherwise, $[A, A^c]$ will contain at least one spoke, giving $|[A, A^c]| \geq 5$. Further, the condition that $[A, A^c]$ contains exactly two edges of C_I enforces the v-vertices of A to induce a path of order $\frac{n}{2}$. Let this path be given by $P \coloneqq v_r v_{r+k} \dots v_{r+(\frac{n}{2}-1)k}$ for some $r \in \{0, \dots, n-1\}$.
Similarly, all u-vertices of A induce a path of order $\frac{n}{2}$. Let this path be $Q \coloneqq u_j u_{j+1} \dots u_{j+(\frac{n}{2}-1)}$ for some $j \in \{0, \dots, n-1\}$. Since $k \geq 2$, at least one vertex of Q cannot have its partner among the vertices of P. Hence at least two spokes belong to $[A, A^c]$. Therefore, $|[A, A^c]| \geq 6$, a contradiction. This completes the proof. □

The lower bound in Theorem 3 is attained, see Example 1.

Theorem 4. *Let n be an even integer and k be an odd integer such that $n \geq 8$ and $k \geq 3$. If $\gcd(n,k) = 1$, then $6 \leq \sigma^-(P_{n,k}) \leq n$.*

Proof. The result follows from Theorem 3 and Lemma 1. □

The lower bound in Theorem 4 is attained, see Example 3.

4.1. The *rna* number of $P_{n,1}$

In this section, we determine $\sigma^-(P_{n,1})$ for $n \geq 3$.

Theorem 5. *If $n \geq 3$, then*

$$\sigma^-(P_{n,1}) = \begin{cases} 3 & \text{if } n = 3 \\ 4 & \text{if } n \text{ is even} \\ 5 & \text{if } n \text{ is odd and } n \geq 5. \end{cases}$$

Proof. Clearly, $\sigma^-(P_{3,1}) \geq 3$. Define the mapping $f \colon V(P_{3,1}) \to \{1, \dots, 6\}$ such that $f(u_i) = 2i + 1$ and $f(v_i) = 2i + 2$ for $0 \leq i \leq 2$. Clearly, all the spokes of $P_{3,1}$ are negative and all the edges of the C_o and C_I are positive in $(P_{3,1}, \sigma_f)$. Thus $\sigma^-(P_{3,1}) = 3$.
For $n \geq 4$, we analyze two cases depending on whether n is even or odd.
Case 1. Let $n = 2\ell$ for $\ell \geq 2$. By Lemma 3, we have $\sigma^-(P_{2\ell,1}) \geq 4$. To show that $\sigma^-(P_{2\ell,1}) = 4$, we produce a parity signed $P_{2\ell,1}$ that contains exactly four negative edges. Define $A = \{u_i, v_i \colon i < \ell\}$. Assign even numbers to the vertices of A and odd numbers to the other vertices. Hence, every edge of $P_{2\ell,1}$ having both end vertices in A (or in A^c) is positive in $(P_{2\ell,1}, \sigma_f)$. Consequently, all edges of $P_{2\ell,1}$ belonging to $[A, A^c]$ get a negative sign in $(P_{2\ell,1}, \sigma_f)$. Clearly, $[A, A^c] = \{u_{2\ell-1}u_0, u_{\ell-1}u_\ell, v_{2\ell-1}v_0, v_{\ell-1}v_\ell\}$. Hence $\sigma^-(P_{2\ell,1}) = 4$.

Case 2. Let $n = 2\ell + 1$ for $\ell \geq 2$. By Lemma 3 and Lemma 2, we have $\sigma^-(P_{2\ell+1,1}) \geq 5$. Now we produce a parity signed graph over $P_{2\ell+1,1}$ having exactly five negative edges. Let $A = \{u_i, v_i, u_\ell \colon i < \ell\}$. Label the vertices of A and A^c with odd and even numbers, respectively. Note that the cut $[A, A^c]$ is given by $\{u_\ell u_{\ell+1}, u_{2\ell}u_0, u_\ell v_\ell, v_{\ell-1}v_\ell, v_{2\ell}v_0\}$. Each edge of $P_{2\ell+1,1}$, except these five edges of $[A, A^c]$, is positive in the parity signed graph $(P_{2\ell+1,1}, \sigma_f)$. Consequently, the number of negative edges in $(P_{2\ell+1,1}, \sigma_f)$ is five. Hence $\sigma^-(P_{2\ell+1,1}) = 5$. This completes the proof. □

4.2. The *rna* number of $P_{n,2}$

In this section, our aim is to prove the following two results.

Theorem 6. *If $\ell \geq 3$, then $\sigma^-(P_{2\ell+1,2}) = 7$.*

Theorem 7. *If $\ell \geq 4$, then $\sigma^-(P_{2\ell,2}) = 6$.*

In the light of Lemma 2, it is clear that the *rna* number of $P_{2\ell+1,2}$ cannot be 4 or 6 for $\ell \geq 3$.

Lemma 4. *If $\ell \geq 3$, then the* rna *number of $P_{2\ell+1,2}$ cannot be 5.*

Proof. We prove that no cut of size five of equal sides is possible in $P_{2\ell+1,2}$.
On the contrary, let $P_{2\ell+1,2}$ have a cut of size five of equal sides. Hence, there exists a subset A of $V(P_{2\ell+1,2})$ such that $|A| = 2\ell+1$ and $|[A, A^c]| = 5$. If A contains either all u-vertices or all v-vertices, then the set of all spokes of $P_{2\ell+1,2}$ constitutes $[A, A^c]$. This gives $|[A, A^c]| \geq 7$, a contradiction. Therefore, A contains some u-vertices as well as v-vertices.
Since $\gcd(2\ell + 1, 2) = 1$, the graph $P_{2\ell+1,2}$ has only one inner cycle induced by the v-vertices. Also, A contains both u-vertices and v-vertices. Therefore, $[A, A^c]$ must consist of two edges of C_o, two edges of C_I and one spoke. Let this spoke be u_jv_j for some $j \in \{0, \dots, 2\ell\}$. Without loss of generality, let $u_j \in A$ and $v_j \in A^c$. The conditions that $|A| = 2\ell + 1$, $u_j \in A$ and that $[A, A^c]$ contains exactly one spoke, together imply that the number of u-vertices and v-vertices in A are $\ell + 1$ and ℓ, respectively.
Consequently, there exists a path P of order $\ell + 1$ induced by the u-vertices of A. Similarly, there exists a path Q of order ℓ induced by the v-vertices of A. Let the paths P and Q be $u_r u_{r+1} \dots u_{r+\ell-1} u_{r+\ell}$ and $v_s v_{s+2} \dots v_{s+(2\ell-2)}$, respectively, for some $r, s \in \{0, 1, \dots, 2\ell\}$.
It is easy to check that at least one vertex of Q cannot have its partner among the vertices of P. This means at least one v-vertex of A has its partner in A^c, and consequently at least two spokes lie in $[A, A^c]$. This gives $|[A, A^c]| \geq 6$, a contradiction to the assumption that $|[A, A^c]| = 5$. This establishes the lemma. □

Proof of Theorem 6. By Theorem 3, Lemma 2 and Lemma 4, we have $\sigma^-(P_{2\ell+1,2}) \geq 7$. To complete the proof, we produce a parity signed graph over $P_{2\ell+1,2}$ having exactly seven negative edges. Let $A = \{u_i : i \leq \ell\} \cup \{v_i : 1 \leq i \leq \ell\}$. Clearly $|A| = |A^c| = 2\ell + 1$. Label the vertices A and A^c with odd and even integers, respectively. Observe that the set of negative edges in $(P_{2\ell+1,2}, \sigma_f)$ is $[A, A^c]$ and $[A, A^c] = \{u_0v_0, u_0u_{2\ell}, u_\ell u_{\ell+1}, v_0v_2, v_{2\ell}v_1, v_{\ell-1}v_{\ell+1}, v_\ell v_{\ell+2}\}$. Thus the number of negative edges in $(P_{2\ell+1,2}, \sigma_f)$ is seven. Hence $\sigma^-(P_{2\ell+1,2}) = 7$. $\square$

Lemma 5. *If $\ell \geq 4$, then $P_{2\ell,2}$ cannot have a cut of size four of equal sides.*

Proof. On the contrary, let $P_{2\ell,2}$ have a cut of size four with equal sides. Therefore, there exists a subset A of $V(P_{2\ell,2})$ such that $|A| = 2\ell$ and $|[A, A^c]| = 4$. Clearly, A must contain some u-vertices as well as some v-vertices. Note that the graph $P_{2\ell,2}$ has only two inner cycles given by $C_1 \coloneqq v_0v_2 \ldots v_{2\ell-2}v_0$ and $C_2 \coloneqq v_1v_3 \ldots v_{2\ell-1}v_1$. Since the u-vertices lie in both A and A^c, the cut $[A, A^c]$ must contain at least two edges of C_o. Thus the following are the only possible choices for the edges of $[A, A^c]$.

1. All four edges of $[A, A^c]$ are in $E(C_o)$.
2. Two edges of $[A, A^c]$ are in $E(C_o)$ and the remaining two edges are spokes.
3. Two edges of $[A, A^c]$ are in $E(C_o)$ and the remaining two edges are in one of the inner cycles.

Case 1. Let $[A, A^c]$ have four edges of C_o. Thus $V(C_1) \subseteq A$, $V(C_2) \subseteq A^c$ or $V(C_2) \subseteq A$, $V(C_1) \subseteq A^c$. Without loss of generality, assume that $V(C_1) \subseteq A$ and $V(C_2) \subseteq A^c$. Since $[A, A^c]$ contains no spoke, we have

$$A = \{u_0, u_2, \ldots, u_{2\ell-2}\} \cup \{v_0, v_2, \ldots, v_{2\ell-2}\} \quad \text{and}$$
$$A^c = \{u_1, u_3, \ldots, u_{2\ell-1}\} \cup \{v_1, v_3, \ldots, v_{2\ell-1}\}.$$

Thus, $[A, A^c]$ contains all the edges of C_o. That is, $|[A, A^c]| = 2\ell \geq 8$, a contradiction.
Case 2. Let $[A, A^c]$ have two edges of C_o and two spokes. Since $[A, A^c]$ contains no edge of the inner cycles, without loss of generality, assume that $\{v_0, v_2, \ldots, v_{2\ell-2}\} \subseteq A$ and $\{v_1, v_3, \ldots, v_{2\ell-1}\} \subseteq A^c$.
Recall that $[A, A^c]$ contains exactly two spokes. Consequently, the set of u-vertices in A must be $\{u_r\} \cup \{u_0, u_2, \ldots, u_{2\ell-2}\} \setminus \{u_j\}$ for some $r \in \{1, 3, \ldots, 2\ell - 1\}$ and $j \in \{0, 2, \ldots, 2\ell - 2\}$. Since $\ell \geq 4$, it is easy to check that $[A, A^c]$ will contain at least four edges of C_o, a contradiction.
Case 3. Let $[A, A^c]$ have two edges of C_o and two edges of one of the inner cycles. Without loss of generality, assume that the two edges of C_1 lie in $[A, A^c]$. Thus both A and A^c contain at least one vertex of C_1. Clearly, all the vertices of C_2 lie entirely in A or in A^c. Hence either A or A^c contains at least $(\ell + 1)$ v-vertices. If A contains at least $(\ell + 1)$ v-vertices, then the number of u-vertices in A is at most $\ell - 1$. This observation shows that at most $(\ell - 1)$ u-vertices of A can have their partners in A.

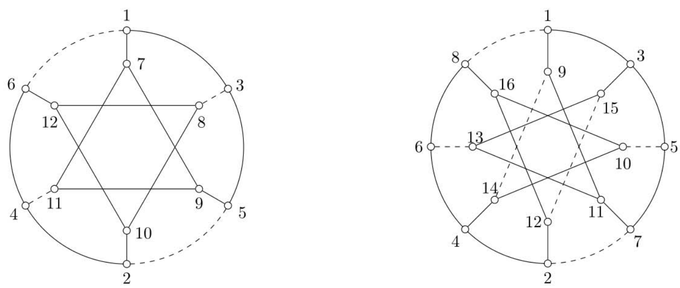
Hence at least two spokes belong to $[A, A^c]$, contradicting our assumption that $[A, A^c]$ contains no spoke. Similarly, we arrive at a contradiction if A^c contain at least $(\ell+1)$ v-vertices.
The contradictions in the preceding cases establish the lemma. □

Proof of Theorem 7. By Lemma 1 and Lemma 5, it is clear that $\sigma^-(P_{2\ell,2}) \geq 6$. Let $A = \{u_i, v_i \colon i < \ell\}$. Assign odd and even numbers to the vertices of A and A^c, respectively. Clearly, the edges of $P_{2\ell,2}$ joining the vertices of A and A^c are the only negative edges in $(P_{2\ell,2}, \sigma_f)$. The cut $[A, A^c]$ is given by $\{u_0u_{2\ell-1}, u_{\ell-1}u_\ell, v_{2\ell-2}v_0, v_{2\ell-1}v_1, v_{\ell-2}v_\ell, v_{\ell-1}v_{\ell+1}\}$. This proves that $\sigma^-(P_{2\ell,2}) = 6$. □

4.3. Some examples

In this sub-section, we compute the *rna* number of some well known generalized Petersen graphs, namely, the Petersen graph, Dürer graph, Möbius-Kantor graph, Dodecahedron, Desargues graph and the Nauru graph.

Example 1. The generalized Petersen graph $P_{5,2}$ is known as the *Petersen graph*. Since $\gcd(5, 2) = 1$, Theorem 3 gives $\sigma^-(P_{5,2}) \geq 5$. Label the vertices of $P_{5,2}$ by the map $f \colon V(P_{5,2}) \to \{1, \ldots, 10\}$ such that $f(u_i) = 2i + 1$ and $f(v_i) = 2i + 2$ for $0 \leq i \leq 4$. Thus, all the spokes of $P_{5,2}$ are negative while other edges of $P_{5,2}$ are positive in $(P_{5,2}, \sigma_f)$. Hence the *rna* number of the Petersen graph is 5. That is, $\sigma^-(P_{5,2}) = 5$.



(a) A parity signed Dürer graph with 4 negative edges

(b) A parity signed Möbius-Kantor graph with 6 negative edges

Figure 1. **Parity signed graphs over $P_{6,2}$ and $P_{8,3}$**

Example 2. The generalized Petersen graph $P_{6,2}$ is known as the *Dürer graph* and it is depicted in Figure 1-(a). By Lemma 3, we have $\sigma^-(P_{6,2}) \geq 4$. Let the map $f \colon V(P_{6,2}) \to \{1, \ldots, 12\}$ be defined by $f(v_i) = i + 7$ for $i \in \{0, \ldots, 5\}$ and

$$f(u_i) = \begin{cases} 2i + 1 & \text{for } i \in \{0, 1, 2\} \\ 2i - 4 & \text{for } i \in \{3, 4, 5\}. \end{cases}$$

This vertex labeling of $P_{6,2}$ is described in Figure 1-(a). Clearly, $(P_{6,2}, \sigma_f)$ has exactly four negative edges. Hence the *rna* number of the Dürer graph is 4.

Example 3. The generalized Petersen graph $P_{8,3}$ is known as the *Möbius-Kantor graph* and it is depicted in Figure 1-(b). By Theorem 4, we have $\sigma^-(P_{8,3}) \geq 6$. Let a mapping $f\colon V(P_{8,3}) \to \{1, \ldots, 16\}$ be defined by $f(v_0) = 9$, $f(v_1) = 15$, $f(v_2) = 10$, $f(v_3) = 11$, $f(v_4) = 12$, $f(v_5) = 14$, $f(v_6) = 13$, $f(v_7) = 16$, and

$$f(u_i) = \begin{cases} 2i + 1 & \text{for } i \in \{0, 1, 2, 3\} \\ 2i - 6 & \text{for } i \in \{4, 5, 6, 7\}. \end{cases}$$

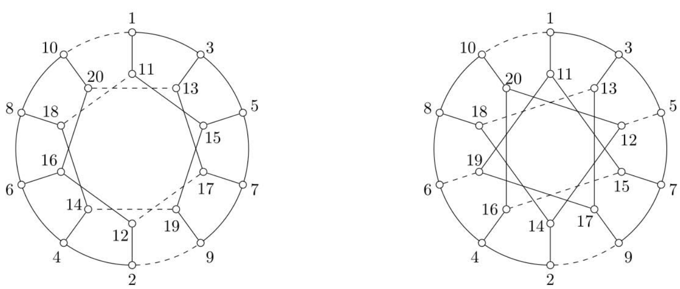
The vertex labeling f is shown in Figure 1-(b). Clearly, $(P_{8,3}, \sigma_f)$ has exactly six negative edges. Hence the *rna* number of Möbius-Kantor graph is 6.

Example 4. The generalized Petersen graph $P_{10,2}$ is known as the *Dodecahedron*. By Theorem 7, we get $\sigma^-(P_{10,2}) = 6$. The parity labeling of $P_{10,2}$ described in Theorem 7 is depicted in Figure 2-(a).

Example 5. The graph $P_{10,3}$ is known as the *Desargues* graph. Since $\gcd(10, 3) = 1$, Theorem 4 gives $\sigma^-(P_{10,3}) \geq 6$. Let the mapping $f\colon V(P_{10,3}) \to \{1, \ldots, 20\}$ be defined by $f(v_0) = 11$, $f(v_1) = 13$, $f(v_2) = 12$, $f(v_3) = 15$, $f(v_4) = 17$, $f(v_5) = 14$, $f(v_6) = 16$, $f(v_7) = 19$, $f(v_8) = 18$, $f(v_9) = 20$, and

$$f(u_i) = \begin{cases} 2i + 1 & \text{for } i \in \{0, 1, 2, 3, 4\} \\ 2i - 8 & \text{for } i \in \{5, 6, 7, 8, 9\}. \end{cases}$$

This vertex labeling is shown in Figure 2-(b). It is clear that $(P_{10,3}, \sigma_f)$ has exactly six negative edges. Hence the *rna* number of the Desargues graph is 6.



(a) A parity signed Dodecahedron graph with 6 negative edges

(b) A parity signed Desargues graph with 6 negative edges

Figure 2. **Parity signed graphs over $P_{10,2}$ and $P_{10,3}$**

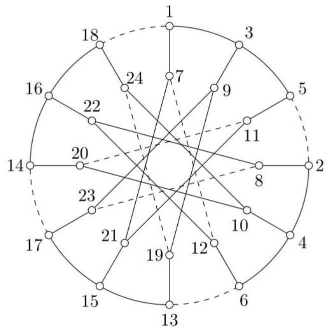


Figure 3. **A parity signed Nauru graph with 8 negative egdes**

Example 6. The generalized Petersen graph $P_{12,5}$ is known as the *Nauru* graph. The Nauru graph is depicted in Figure 3. First we show that the *rna* number of Nauru graph is at least 8.

Due to Theorem 4, we have $\sigma^-(P_{12,5}) \geq 6$. Also, Lemma 1 shows that the *rna* number of $P_{12,5}$ cannot be 7. Thus it remains to prove that $\sigma^-(P_{12,5}) \neq 6$.

Let, if possible, $P_{12,5}$ have a cut of size six of equal sides. Therefore, there exists a subset A of $V(P_{12,5})$ such that $|A| = 12$ and $|[A, A^c]| = 6$. Note that A must contain u-vertices as well as v-vertices. Since $\gcd(12, 5) = 1$, there is only one inner cycle in $P_{12,5}$. Thus the cut $[A, A^c]$ will contain even number of edges from each of C_o and C_I. We analyze three cases.

Case 1. Let $[A, A^c]$ contain four edges of C_o and two edges of C_I. Since $[A, A^c]$ contains no spoke, A must have six u-vertices and six v-vertices. Again, since exactly two edges of C_I are in $[A, A^c]$, all v-vertices of A induce a path of order six. Thus, the set of v-vertices of A is $\{v_r, v_{r+5}, v_{r+10}, v_{r+3}, v_{r+8}, v_{r+1}\}$ for some $r \in \{0, 1, \dots, 11\}$. Hence the set of u-vertices of A is $\{u_r, u_{r+5}, u_{r+10}, u_{r+3}, u_{r+8}, u_{r+1}\}$. Thus $[A, A^c]$ contains the edges $u_{r+1}u_{r+2}, u_{r+2}u_{r+3}, u_{r+3}u_{r+4}, u_{r+4}u_{r+5}$, $u_{r+5}u_{r+6}$, $u_{r+7}u_{r+8}$, $u_{r+8}u_{r+9}$, $u_{r+9}u_{r+10}, u_{r+10}u_{r+11}$ and $u_{r+11}u_r$ for some $r \in \{0, \dots, 11\}$. Hence $|[A, A^c]| \geq 12$, a contradiction.

Case 2. Let $[A, A^c]$ contain two edges of C_0 and four edges of C_I. In this case also, we proceed as in Case 1. We find that the u-vertices of A must be $u_j, u_{j+1}, u_{j+2}, u_{j+3}, u_{j+4}$ and u_{j+5} for some $j \in \{0, 1, \dots, 11\}$. Accordingly, the v-vertices of A are $v_j, v_{j+1}, v_{j+2}, v_{j+3}, v_{j+4}$ and v_{j+5}. This gives at least 12 edges in $[A, A^c]$, a contradiction.

Case 3. Let $[A, A^c]$ contain two edges of C_o, two edges of C_I and two spokes. In this case, A (or A^c) cannot have more than seven u-vertices or seven v-vertices. Otherwise, the number of spokes in $[A, A^c]$ will exceed 2 and a contradiction will occur. Note that if A has less than five u-vertices, then A^c will have more than seven u-vertices. Further, $[A^c, A] = [A, A^c]$. Therefore, if A has less than five u-vertices, then the cut $[A^c, A]$ will have more than 2 spokes, a contradiction. Thus the possible number of u-vertices of A are five, six and seven only. Also, if A has five u-vertices, then A^c has seven u-vertices. Due to $[A^c, A] = [A, A^c]$ again, the cases that A has five u-vertices and A has seven u-vertices are similar. Thus we consider two sub-cases depending on whether A has seven u-vertices or six u-vertices.

Subcase 3(i). Assume that A has seven u-vertices and five v-vertices. Since the cut $[A, A^c]$ contain only two edges from each of C_o and C_I, these seven u-vertices and five v-vertices must induce paths of order 7 and 5, respectively. Hence $A = \{u_j, u_{j+1}, u_{j+2}, u_{j+3}, u_{j+4}, u_{j+5}, u_{j+6}, v_r, v_{r+5}, v_{r+10}, v_{r+3}, v_{r+8}\}$ for some $j, r \in \{0, 1, \dots, 11\}$. Observe that these five v-vertices of A cannot be adjacent to five u-vertices of A, for any j and r. That is, at most four v-vertices of A have their partners in A. Therefore, at least one v-vertex and at least three u-vertices of A must have their partners in A^c. Consequently, $[A, A^c]$ will contain at least four spokes, a contradiction.

Subcase 3(ii). Let A have six u-vertices and six v-vertices. Since $[A, A^c]$ contains only two edges from each of C_o and C_I, the u-vertices and v-vertices of A form two paths of order six. Thus there are some $j, r \in \{0, 1, \ldots, 11\}$ such that $A = \{u_j, u_{j+1}, u_{j+2}, u_{j+3}, u_{j+4}, u_{j+5}, v_r, v_{r+5}, v_{r+10}, v_{r+3}, v_{r+8}, v_{r+1}\}$. For any j and r, observe that at most four v-vertices of A can have their partners in A. Therefore, $[A, A^c]$ will contain at least four spokes, a contradiction.

From these cases, we conclude that $\sigma^-(P_{12,5}) \geq 8$.

Now, define the mapping $f\colon V(P_{12,5}) \to \{1, \ldots, 24\}$ by

$$f(u_i) = \begin{cases} 2i + 1 & \text{for } i \in \{0, 1, 2, 6, 7, 8\} \\ 2i - 4 & \text{for } i \in \{3, 4, 5, 9, 10, 11\} \end{cases}$$

and

$$f(v_i) = \begin{cases} 2i + 7 & \text{for } i \in \{0, 1, 2, 6, 7, 8\} \\ 2i + 2 & \text{for } i \in \{3, 4, 5, 9, 10, 11\}. \end{cases}$$

This vertex labeling is shown in Figure 3. Clearly, $(P_{12,5}, \sigma_f)$ has exactly eight negative edges. Hence the *rna* number of Nauru graph is 8.

5. Regular graphs with *rna* number 1

Obvious lower and upper bounds on the *rna* number of a graph G are 1 and m, respectively, where m is the size of G. It is shown in [1, Proposition 4] that the *rna* number of a path of order n is 1, where $n \geq 2$. Acharya et al. [2] characterized the structure of those graphs whose *rna* number is 1. More precisely, we have the following theorem.

Theorem 8. *[2, Theorem 3.5] For any connected graph G, $\sigma^-(G) = 1$ if and only if G has a cut-edge joining two graphs whose orders differ by at most 1.*

Since a regular graph of even degree cannot have a cut-edge, in light of Theorem 8, the *rna* number of such a graph is at least 2. Therefore, the following problem is worth exploring.

Problem 1. If k is odd and $k \geq 3$, then what is the smallest order of a k-regular graph whose *rna* number is 1?

In this section, we find a solution to this problem. Note that an odd positive integer can be written as $4n+1$ or $4n-1$ for some n. We consider these two cases separately. For each positive integer n, we construct a $(4n+1)$-regular graph on $8n+6$ vertices with a cut-edge joining two graphs of order $4n+3$ each in Example 7.

The length of a shortest path joining the vertices x and y is called the *distance* between x and y. It is denoted by $d_G(x, y)$. The *k-th power* of a connected graph G is the graph G^k whose vertex set is $V(G)$ and two distinct vertices being adjacent in G^k if and only if their distance in G is at most k.

Example 7. Consider the cycle C_{4n+3} such that $V(C_{4n+3}) = \{v_i : i \in [4n + 3]\}$ and $E(C_{4n+3}) = \{v_iv_{i+1} : i \in [4n + 3]\}$, where the subscripts are read modulo $4n + 3$. Construct the power graph C^{2n}_{4n+3} from C_{4n+3}. Note that the degree of each vertex of C^{2n}_{4n+3} is $4n$. For each $i \in \{1, \ldots, 2n + 1\}$, insert an edge between v_i and $v_{i+(2n+1)}$ in C^{2n}_{4n+3}, and denote this new graph by G_s. Clearly, the order of G_s is $4n + 3$ and the degree of v_0 is $4n$, while the degree of all other vertices of G_s is $4n + 1$. Now take two disjoint copies of G_s and join the vertices corresponding to v_0 by an edge. This resulting graph is a $(4n + 1)$-regular graph on $8n + 6$ vertices with a cut-edge joining two graphs of order $4n + 3$.

Note that the graph G_s constructed in Example 7 can also be obtained by taking the complement of the disjoint union $M_{2n} \cup P_3$, where M_{2n} is a matching of $2n$ edges and P_3 is a path of order 3. Indeed, if G is any graph on minimum number of vertices having exactly one vertex of degree $4n$, and other vertices of degree $4n + 1$, then G must have exactly $4n + 3$ vertices, and that the complement of G must be $M_{2n} \cup P_3$. Therefore, such a graph is **unique**.

Theorem 9. *For each positive integer n, the smallest order of a $(4n + 1)$-regular graph having* rna *number 1 is $8n + 6$. Further, such a graph is unique.*

Proof. Let G be a $(4n + 1)$-regular graph having *rna* number 1. By Theorem 8, G must be obtained by joining two graphs H_1 and H_2 of equal order by a cut-edge. Therefore, exactly one vertex of H_1 (and also of H_2) must have degree $4n$ in H_1 (and in H_2), while the degree of the remaining vertices of H_1 (and also of H_2) must be $4n+1$. Thus the order of H_1 (and of H_2) must be at least $4n+3$. Hence the order of G is at least $8n + 6$. Thus the smallest order of a $(4n + 1)$-regular graph having *rna* number 1 is $8n + 6$.
From the discussion in the preceding paragraph, it is clear that the graphs H_1 and H_2 are unique. Hence a $(4n + 1)$-regular graph of smallest order having *rna* number 1 is unique, and it must be the one constructed in Example 7. □

Now we construct a $(4n - 1)$-regular graph on $12n - 2$ vertices with a cut-edge joining two graphs of order $6n - 1$ each.

Example 8. Consider the cycle C_{6n-1} such that $V(C_{6n-1}) = \{v_i : i \in [6n - 1]\}$ and $E(C_{6n-1}) = \{v_iv_{i+1} : i \in [6n - 1]\}$, where the subscripts are read modulo $6n - 1$. Construct the power graph C^{2n-1}_{6n-1} from C_{6n-1}. Note that the degree of each vertex of C^{2n-1}_{6n-1} is $4n - 2$. Now for each $i \in \{1, \ldots, 3n - 1\}$, insert an edge between v_i and $v_{i+(3n-1)}$ in C^{2n-1}_{6n-1}, and denote this new graph by G_r. Clearly the order of G_r is $6n-1$, the degree of v_0 in G_r is $4n-2$, and the degree of all other vertices in G_r is $4n - 1$. Now take two disjoint copies of G_r and join the vertices corresponding to v_0 by an edge. This resulting graph is a $(4n - 1)$-regular graph on $12n - 2$ vertices with a cut-edge joining two graphs of order $6n - 1$ each.

Figure 4. **Non-isomorphic cubic graphs of order 6**

Lemma 6. *The smallest order of a parity signed cubic graph having* rna *number* 1 *is* 10.

Proof. The only cubic graph on four vertices is K_4, which does not have a cut-edge. Further, non-isomorphic cubic graphs of order 6 and 8 are shown in Figure 4 and Figure 5, respectively. It is clear that none of these graphs contain a cut-edge. Hence the order of a cubic graph having *rna* number 1 is at least 10. Let Σ be the parity signed cubic graph as shown in Figure 6. Clearly, it is a cubic graph of order 10, and it has a cut-edge joining two graphs of the same order. Thus by Theorem 8, we have $\sigma^{-1}(\Sigma) = 1$. This completes the proof. $\square$

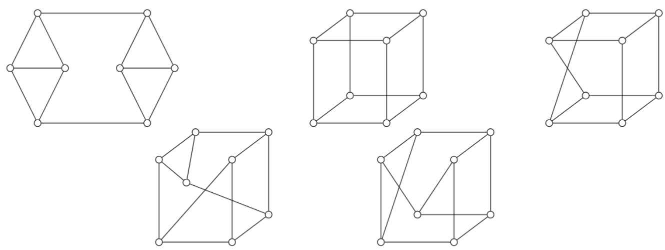


Figure 5. **Non-isomorphic cubic graphs of order 8**

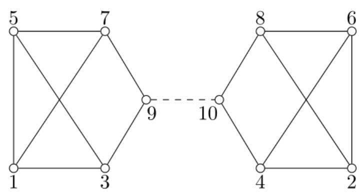


Figure 6. **A parity signed cubic graph of order 10 with exactly 1 negative edge**

Theorem 10. *For each positive integer n, the smallest order of $(4n-1)$-regular graphs having* rna *number 1 is bounded above by $12n-2$. Moreover, this bound is achieved for $n = 1$.*

Proof. The proof follows from Example 8 and Lemma 6. $\square$

6. Conclusion

In this paper, we discussed the *rna* number of generalized Petersen graphs. We determined the *rna* number of $P_{n,k}$ for $k \in \{1, 2\}$. For $k \geq 3$, the distribution of odd and even integers to the vertices of $P_{n,k}$ to obtain the value of $\sigma^-(P_{n,k})$ seems to be difficult. Thus the following problem is worth exploring.

Problem 2. What is the value of $\sigma^-(P_{n,k})$, where $n \geq 7$ and $k \geq 3$?

We also proved that the smallest order of a $(4n+1)$-regular graph having *rna* number 1 is $8n+6$. For the smallest order of $(4n-1)$-regular graphs having *rna* number 1, we obtained an upper bound of $12-2$. Moreover, we proved that for $n=1$, the smallest order of a cubic graph of *rna* number 1 is 10. This implies that best possible lower bound for the *rna* number of regular graphs is 1. To the best of our knowledge, a best possible upper bound for the *rna* number of regular graphs is not known. Hence the following problem is also interesting.

Problem 3. Determine a best possible upper bound on the *rna* number of regular graphs?

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability Statement: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

References

[1] M. Acharya and J.V. Kureethara, *Parity labeling in signed graphs*, J. Prime Research in Math. **17** (2021), no. 2, 1–7.
[2] M. Acharya, J.V. Kureethara, and T. Zaslavsky, *Characterizations of some parity signed graphs*, Australas. J. Combin. **81** (2021), no. 1, 89–100.
[3] A. Bondy and U.S.R. Murty, *Graph Theory*, Springer London, 2008.
[4] F. Harary, *On the notion of balance of a signed graph*, Michigan Math. J. **2** (1953), no. 2, 143–146.
[5] Y. Kang, X. Chen, and L. Jin, *A study on parity signed graphs: The* rna *number*, Appl. Math. Comput. **431** (2022), Article ID: 127322. https://doi.org/10.1016/j.amc.2022.127322.