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Research Article

A counterexample on the conjecture and bounds on χ_{gd}-number of Mycielskian of a graph

David A Kalarkop* **and R.Rangarajan†**

Department of Studies in Mathematics, University of Mysore
Manasagangothri, Mysuru – 570 006, India
*david.ak123@gmail.com
†rajra63@gmail.com

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Abstract: A coloring $C = (V_1, \ldots, V_k)$ of G partitions the vertex set $V(G)$ into independent sets V_i which are said to be color classes with respect to the coloring C. A vertex v is said to have a dominator (dom) color class in C if there is color class V_i such that v is adjacent to all the vertices of V_i and v is said to have an anti-dominator (anti-dom) color class in C if there is color class V_j such that v is not adjacent to any vertex of V_j. Dominator coloring of G is a coloring C of G such that every vertex has a dom color class. The minimum number of colors required for a dominator coloring of G is called the dominator chromatic number of G, denoted by $\chi_d(G)$. Global Dominator coloring of G is a coloring C of G such that every vertex has a dom color class and an anti-dom color class. The minimum number of colors required for a global dominator coloring of G is called the global dominator chromatic number of G, denoted by $\chi_{gd}(G)$. In this paper, we give a counterexample for the conjecture posed in [I. Sahul Hamid, M.Rajeswari, Global dominator coloring of graphs, Discuss. Math. Graph Theory 39 (2019), 325–339] that for a graph G, if $\chi_{gd}(G) = 2\chi_d(G)$, then G is a complete multipartite graph. We deduce upper and lower bound for the global dominator chromatic number of Mycielskian of the graph G in terms of dominator chromatic number of G.

Keywords: Global Dominator coloring, global dominator chromatic number, dominator coloring, dominator chromatic number

AMS Subject classification: 05C15, 05C69

1. Introduction

By a graph $G = (V, E)$, we mean a simple graph whose vertex set is V of order n and edge set is E of size m. For all basic graph theoretic terminologies we refer to [5]. Domination and coloring are two interesting and well known areas in graph theory.

* *Corresponding Author*

Domination has rich applications in computer science, communication networks and so on. Graph coloring has applications in scheduling, register allocations, pattern matching and so on (for more details refer to [3]).

A subset D of the vertex set V of G is said to be a dominating set if every vertex of V is in D or has a neighbor in D. The minimum cardinality of a dominating set of G is called as *domination number* of G, denoted by $\gamma(G)$. A vertex is said to dominate subset S of $V(G)$, if v is adjacent to all the vertices of S. For more information on domination, refer to [9]. A coloring of G is an assignment of colors to the vertices of the graph G such that no two adjacent vertices receive the same color. The minimum number of colors required for coloring G is said to be the *chromatic number* of G, denoted by $\chi(G)$. A vertex $v \in V(G)$ in the coloring $C = (V_1, \ldots, V_k)$ is said to have dom color class V_i (anti-dom color class V_j) if v is adjacent to all (none) of the vertices of V_i (V_j).

Dominator coloring of G is the coloring of G such that every vertex of G has a dom color class. The minimum number of colors required for dominator coloring of G is called dominator chromatic number of G, denoted by $\chi_d(G)$. The dominator coloring of G with minimum number of colors is said to be χ_d-coloring of G. Dominator coloring was studied for the first time by Gera et al. [7]. Global dominator coloring of graphs was introduced by Hamid et al. [11]. Global Dominator coloring of G is the coloring of G such that every vertex of G has a dom color class and an anti-dom color class. The minimum number of colors required for global dominator coloring of G is called global dominator chromatic number of G, denoted by $\chi_{gd}(G)$. The global dominator coloring of G with minimum number of colors is said to be χ_{gd}-coloring of G. For good number of results, conjectures and open problems on dominator coloring as well as global dominator coloring of graphs, refer to [2, 7, 8, 10, 11]. The following results are crucial to prove the main results of this paper,

Theorem 1. *[2] For any graph G, $\chi_d(G) + 1 \le \chi_d(\mu(G)) \le \chi_d(G) + 2$. Further if there exists a χ_d-coloring C of G in which every vertex v dominates a color class V_i with $v \notin V_i$, then $\chi_d(\mu(G)) = \chi_d(G) + 1$.*

$N(v) = \{u \in V(G) : uv \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$ are the open neighborhood and closed neighborhood of the vertex v of G respectively. The vertex v of G with respect to the coloring C is said to be solitary if $\{v\} \in C$ and $N(v)$ does not contain any color class. Let $C = (V_1, \ldots, V_k)$ be the coloring of G. The color class V_i, $(1 \le i \le k)$ is said to be the spare color class with respect to C if every vertex $v \in V(G)$ dominates some color class V_j, $j \neq i$, of C.

Theorem 2. *[1] Given a graph G, $\chi_d(\mu(G)) = \chi_d(G) + 1$ if and only if for some χ_d-coloring C of G:*
(i) each vertex v dominates some color class V_i with $v \notin V_i$;
(ii) a vertex v is a solitary vertex and C contains a spare color class V_i which does not contain any vertex of $N(v)$.

Theorem 3. *[11] The global dominator chromatic number of a complete m-partite graph is* $2m$*.*

Theorem 4. *[11] For any graph G, we have* $\chi_d(G) \leq \chi_{gd}(G) \leq 2\chi_d(G)$*.*

Conjecture 1. *[11]* Let G be a graph with $\Delta(G) < n-1$. Then $\chi_{gd}(G) = 2\chi_d(G)$ if and only if G is a complete multipartite graph.

Clearly if G is complete multipartite graph, then $\chi_{gd}(G) = 2\chi_d(G)$ (by Theorem 3). In this paper, we give a counterexample to conclude the conjecture is false. i.e if $\chi_{gd}(G) = 2\chi_d(G)$, then the graph G need not be a complete multipartite graph. Also motivated by the works of Arumugam et al. [2], we establish the upper and lower bound for global dominator chromatic number of Mycielskian of the graph G in terms of dominator chromatic number of G.

2. Counterexample for the Conjecture 1

In this section we give a counterexample of a graph with $\chi_{gd}(G) = 2\chi_d(G)$ which is not complete multipartite.

Lemma 1. *Let H be the graph in the Figure 1. Then* $\chi_d(H) = 3$ *and* $\chi_{gd}(H) = 6$*.*

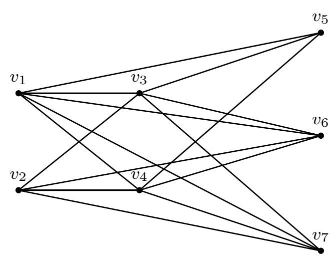

Figure 1. **Graph** H

Proof. Since the graph H has K_3 as the subgraph, minimum three colors are required for coloring of H. Hence $\chi_d(H) \geq \chi(H) \geq 3$. The coloring $C = (\{v_1, v_2\}, \{v_3, v_4\}, \{v_5, v_6, v_7\})$ is a χ_d-coloring of H. Hence, $\chi_d(H) = 3$ and $\chi(H) = 3$. Since $v_1v_3 \in E(G)$, v_1, v_3 receive different colors. Now the color of v_1 cannot be given to v_2, otherwise v_1 will not have anti-dom color class. Similarly color of v_3 cannot be given to v_4. Therefore v_1, v_2, v_3, v_4 receive different colors in a global dominator coloring of G. Therefore $\chi_{gd}(H) \geq 4$. Now v_5 cannot be given the color of v_2, otherwise v_5 will not have anti-dom color class. Therfore v_5 receive distinct color. Hence $\chi_{gd}(H) \geq 5$. If v_6, v_7 receive the color of v_5, then v_6 and v_7 will not have anti-dom color class. Therefore either v_6 or v_7 has to be given a new color. Hence $\chi_{gd}(H) \geq 6$. From the χ_{gd} coloring of H in Figure 2, we can conclude that $\chi_{gd}(H) = 6$. □

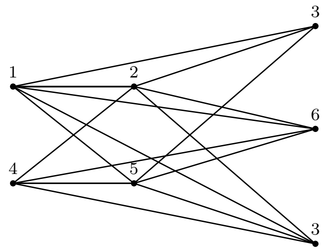

Figure 2. χ_{gd}**-coloring of** H

From the Lemma 1, it is clear that $\chi_{gd}(H) = 2\chi_d(H)$ but H is not a complete multipartite graph. Therefore if $\chi_{gd}(G) = 2\chi_d(G)$, then G need not be the complete multipartite graph. By the proof technique used in Lemma 1, we define a family of graphs $G \in \Im$ which are not complete multipartite but satisfy $\chi_{gd}(G) = 2\chi_d(G)$. The graphs $G \in \Im$ are constructed as follows,
i) V_1, V_2, V_3 are independent sets in G such that $|V_i| \geq 2$, for all $1 \leq i \leq 3$.
ii) Let $v_1 \in V_1$ and $v_3 \in V_3$. Join all the edges in G except v_1v_3.

Corollary 1. *If $G \in \Im$, then $\chi_{gd}(G) = 2\chi_d(G)$.*

Now we define a new family of graphs. $G \in \Im_1$, if for every χ_d-coloring $C = (V_1, \ldots, V_k)$ of G, the following conditions are satisfied,
i) $|V_i| \geq 2$, for all $1 \leq i \leq k$ and
ii) there exists atleast one vertex in each V_i that has at least one neighbor in each V_j, for $j = 1, 2, \ldots, k$ and $i \neq j$.
i.e in simple words, G is a k-partite graph with at least two vertices in each partite sets such that there exists at least one vertex in each partite set that has at least one neighbor in each of the other partite sets.

Theorem 5. *$\chi_{gd}(G) = 2\chi_d(G)$ if and only if $G \in \Im_1$.*

Proof. Let G be a graph with $\chi_d(G) = k$. Suppose $G \notin \Im_1$.

Case 1. If there is a χ_d-coloring $C = (V_1, \ldots, V_k)$ of G such that $|V_i| = 1$, for some $1 \leq i \leq k$. Let $v_i \in V_i$. the coloring $C' = (V_j - \{v_j\}, V_i, \{v_j\})$ for $j = 1, 2, \ldots, k$ and $i \neq j$ is a global dominator coloring of G with less than $2\chi_d(G)$ number of colors. Note that $v_i \in V_i$ cannot be adjacent to all the vertices of $V_j \in C'$, since G cannot have a vertex of degree $n - 1$ for a global dominator coloring to exist.

Case 2. There is some χ_d-coloring $C = (V_1, \dots, V_k)$ such that every vertex in some V_i has no neighbor in some V_j. Therefore every vertex in V_i has an anti-dominator color class. Let $v_i \in V_i$, then the coloring $C' = (V_j - \{v_j\}, V_i, \{v_j\})$ for $j = 1, 2, \dots, k$ and $i \neq j$ is a global dominator coloring of G with less than $2\chi_d(G)$ number of colors. Conversely, suppose $G \in \Im_1$. Then for every χ_d-coloring $C = (V_1, \dots, V_k)$, we have $|V_i| \geq 2$ and there exists atleast one vertex v_i in each V_i that has atleast one neighbor in each V_j, for $j = 1, \dots, k$ and $i \neq j$. Then the coloring $C' = (V_i - \{v_i\}, \{v_i\})$ is a global dominator coloring of G with at least $2\chi_d(G)$ number of colors. From Theorem 4, the proof follows. □

3. Bounds on Global dominator chromatic number of Mycielskian of a graph

For a graph $G = (V, E)$, the Mycielskian of G denoted by $\mu(G)$ is the graph with vertex set $V \cup V' \cup \{u\}$ where $V' = \{x' \mid x \in V\}$ and is disjoint from V, and edge set $E' = E \cup \{xy' \mid xy \in E\} \cup \{x'u \mid x' \in V'\}$. The vertices x and x' are called twins of each other and u is called the root of $\mu(G)$. For results on domination parameters in Mycielskian of a graph, refer to [1, 4, 6]. In this section, we establish the upper and lower bound for global dominator chromatic number of Mycielskian of the graph G in terms of dominator chromatic number of G.

Theorem 6. *For any graph G, we have $\chi_d(G) + 1 \leq \chi_{gd}(\mu(G)) \leq \chi_d(G) + 2$.*

Proof. We know that $\chi_{gd}(\mu(G)) \geq \chi_d(\mu(G))$ (by the Theorem 4 applied to $\mu(G)$) and $\chi_d(\mu(G)) \geq \chi_d(G) + 1$ (by the Theorem 1). So $\chi_{gd}(\mu(G)) \geq \chi_d(G) + 1$.
Let $C = (V_1, V_2, \dots, V_{\chi_d})$ be the χ_d-coloring of G. Now consider the graph $\mu(G)$ and color the vertices of $\mu(G)$ as follows,
(i) color the vertices of G by the coloring C using χ_d number of colors.
(ii) color the vertices of V' by a unique new color.
(iii) a new color to $\{u\}$
In this coloring (say C'), vertices of G have dom-color class as the coloring C is the χ_d-coloring of G and have $\{u\}$ as the anti-dom color class. The vertex u dominates the color class V' and anti-dominates all the color classes V_i. Now the vertices of V_i' dominates the color class $\{u\}$ and anti dominates the color class V_i since no vertex of V_i' will be adjacent to any vertex of V_i by the construction of Mycielskian of G. So the coloring C' is the global dominator coloring of $\mu(G)$ with at most $\chi_d(G) + 2$ number of colors. Hence $\chi_{gd}(\mu(G)) \leq \chi_d(G) + 2$. □

Lemma 2. *Let G_1 be the graph in the Figure 3. Then $\chi_{gd}(\mu(G_1)) = \chi_d(G_1) + 1$.*

Proof. Consider the graph G_1. The vertices v_1, v_2 and v_3 have to be colored using three colors. The vertex v_4 or v_5 has to be given a new color in order to achieve the

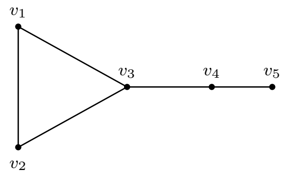

Figure 3. **Graph G_1**

χ_d as well as χ_{gd}-coloring of G_1. So $\chi_{gd}(G_1) = \chi_d(G_1) = 4$.
Consider the global dominator coloring of $\mu(G_1)$ as shown in the Figure 4.

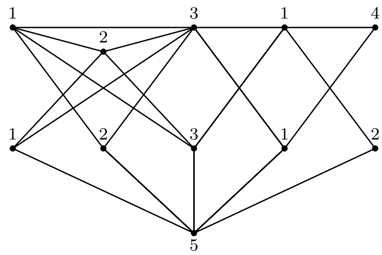

Figure 4. **Global dominator coloring of $\mu(G_1)$**

Therefore $\chi_{gd}(\mu(G_1)) \le 5$ and by the Theorem 6, $\chi_{gd}(\mu(G_1)) \ge \chi_d(G_1) + 1 = 4 + 1$. Hence $\chi_{gd}(\mu(G_1)) = \chi_d(G_1) + 1$. □

The graph G_1 in Figure 3 with χ_d-coloring $C = (\{v_1, v_4\}, \{v_2\}, \{v_3\}, \{v_5\})$ satisfies the conditions of Theorem 7 with the vertex v_5 being solitary and color class $\{v_2\}$ being the spare color class. This example motivates us to state Theorem 7.

Theorem 7. *Suppose for some χ_d-coloring C of G, the following conditions are satisfied,*
i) a vertex $v_1 \in G$ such that v_1 is solitary.
ii) C contains a spare color class V_i which does not contain any vertex of $N(v_1)$.
iii) For all $w \in N(v_1)$, w does not have neighbors in V_i.
Then $\chi_{gd}(\mu(G)) = \chi_d(G) + 1$.

Proof. Let $C = (V_1, V_2, \ldots, V_{\chi_d})$ be the χ_d-coloring of G. Let the vertex v_1 be solitary and the color class V_i (for some $1 \le i \le \chi_d$) be the spare color class with respect to the coloring C such that V_i does not contain any vertex of $N(v_1)$ and w has no neighbors in V_i, for all $w \in N(v_1)$. Consider the coloring $C' = (C - V_i) \cup \{V_i \cup \{v_1'\}, \{u\}\}$ of $\mu(G)$, where each vertex $v_j'(2 \le j \le \chi_d)$ is given a color of vertex v_j, v_1' is given the color of spare color class V_i and a new color is assigned to the vertex u . By the Theorem 2, coloring C' will be the dominator coloring of $\mu(G)$. The vertices of $V(G)$ in $\mu(G)$ anti dominates $\{u\}$, the vertex u anti dominates $\{v_1\}$ and the vertices of V' which are the twin vertices of non-neighbors of v_1 anti dominates color class $\{v_1\}$. The condition that w does not have neighbors in V_i, for all $w \in N(v_1)$ imply that the vertices of V' which are twins of neighbors of v_1 anti dominates the spare color class V_i. Thus $\chi_{gd}(\mu(G)) \leq \chi_d(G) + 1$ and the equality follows by the Theorem 6. □

Theorem 8. *Let G be the complete m-partite graph ($m \geq 2$). Then $\chi_{gd}(\mu(G)) = \chi_d(G) + 2$.*

Proof. Let G be a complete m-partite graph with vertex set V and partite sets $V_1, \ldots, V_m$. Since G is the complete m-partite graph, $\chi_d(G) = m$ where each set V_i is given a unique color. Every vertex $v_i \in V_i$ of G is adjacent to every other vertex of $V_j (i \neq j)$ and hence adjacent to every vertex of twins of V_j in $\mu(G)$.

Case 1. If the vertex u is given any one of color used in coloring G, then the vertices of V has to anti dominate a color class in V'. So at least m new colors will be required to color the vertices of V' since the vertex of V_i has to anti-dominate the color class V_i'. But $m \geq 2$ imply that $\chi_{gd}(\mu(G))$ is at least $\chi_d(G) + 2$.

Case 2. If $\{u\}$ is given a new color, then the vertices of G in $\mu(G)$ have $\{u\}$ as the anti dom-color class. Now the only possibility of giving the colors used in coloring G to the vertices of V' is by giving the color of $v \in V(G)$ to its twin vertex $v' \in V'(G)$. In that case the vertices of V' will not have anti dom-color class. Therefore a new unique color has to be given to vertices of V'. Then the vertex v' anti dominates the color class in which v lies.

From the above two cases, it is clear that at least $\chi_d(G) + 2$ number of colors are required for global dominator coloring of $\mu(G)$. So by the Theorem 6, we have $\chi_{gd}(\mu(G)) = \chi_d(G) + 2$. □

4. Open problems

Problem 1. The graph G_1 in Figure 3 is such that $\chi_d(\mu(G_1)) = \chi_{gd}(\mu(G_1)) = \chi_d(G) + 1$. This helps us to pose a question that for which graphs G, $\chi_d(\mu(G)) = \chi_{gd}(\mu(G))$?

Problem 2. Characterize graphs G such that $\chi_{gd}(\mu(G)) = \chi_d(G) + 1$.

Problem 3. Characterize graphs G such that $\chi_{gd}(\mu(G)) = \chi_d(G) + 2$.

Problem 4. Give a structural characterization for graphs G such that $\chi_{gd}(\mu(G)) = 2\chi_d(G)$.

Problem 5. The graphs $G \in \Im$ in the Corollary 1 are graphs such that $\chi_d(G) = 3$ and $\chi_{gd}(G) = 6$. One can attempt to construct graphs G such that $\chi_d(G) = k$ and $\chi_{gd}(G) = 2k$, for all $k \geq 4$.

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Data Availability. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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