# Algorithmic complexity of triple Roman dominating functions on graphs 

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#### Abstract

Given a graph $G=(V, E)$, a function $f: V \rightarrow\{0,1,2,3,4\}$ is a triple Roman dominating function (TRDF) of $G$, for each vertex $v \in V$, (i) if $f(v)=0$, then $v$ must have either one neighbour in $V_{4}$, or either two neighbours in $V_{2} \cup V_{3}$ (one neighbour in $V_{3}$ ) or either three neighbours in $V_{2}$, (ii) if $f(v)=1$, then $v$ must have either one neighbour in $V_{3} \cup V_{4}$ or either two neighbours in $V_{2}$, and if $f(v)=2$, then $v$ must have one neighbour in $V_{2} \cup V_{3} \cup V_{4}$. The triple Roman domination number of $G$ is the minimum weight of an TRDF $f$ of $G$, where the weight of $f$ is $\sum_{v \in V} f(v)$. The triple Roman domination problem is to compute the triple Roman domination number of a given graph. In this paper, we study the triple Roman domination problem. We show that the problem is NP-complete for the star convex bipartite and the comb convex bipartite graphs and is APX-complete for graphs of degree at most 4. We propose a linear-time algorithm for computing the triple Roman domination number of proper interval graphs. We also give an $(2 H(\Delta(G)+1)-1)$-approximation algorithm for solving the problem for any graph $G$, where $\Delta(G)$ is the maximum degree of $G$ and $H(d)$ denotes the first $d$ terms of the harmonic series. In addition, we prove that for any $\varepsilon>0$ there is no $(1 / 4-\varepsilon) \ln |V|$-approximation polynomial-time algorithm for solving the problem on bipartite and split graphs, unless NP $\subseteq$ DTIME $\left(|V|^{O(\log \log |V|)}\right)$.


Keywords: Triple Roman domination, Approximation algorithm, NP-complete, Proper interval graph, APX-complete

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## 1. Introduction

Let $G=(V, E)$ be a graph such that $V$ denotes the vertex set of $G$ and $E$ denotes the edge set of $G$. Let $N_{G}(v)=\{u \in V: u v \in E\}, N_{G}[v]=N_{G}(v) \cup\{v\}$ and $\operatorname{deg}_{G}(v)=\left|N_{G}(v)\right|$. A pendant vertex is a vertex with degree one. The maximum degree of a graph $G$, denoted by $\Delta(G)$, is $\Delta(G)=\max \left\{\operatorname{deg}_{G}(v): v \in V\right\}$. A graph

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is complete if there is an edge between any pair of its vertices. The induced subgraph $G[S]$ for any $S \subseteq V$ is the graph whose vertex set is $S$ and edge set consists of all edges in $E$ that have both endpoints in $S$. A clique of $G$ is a subset $S \subseteq V$ such that $G[S]$ is a complete graph. A graph $G=(V, E)$ is an intersection graph for a family of nonempty sets $\mathcal{F}$ if each vertex in $V$ is corresponding to a set in $\mathcal{F}$ and two vertices are adjacent in $G$ if and only if the intersection of their corresponding sets in $\mathcal{F}$ is nonempty. An interval graph $G=(V, E)$ is an intersection graph for a family of intervals on the real line. A proper interval graph is an interval graph in which no interval properly contains another. A tree is a connected graph with no cycles. A tree $T=(V, E)$ is called a star if $|V|=2$ or $|V| \geq 3$ and $T$ contains exactly one vertex that is not pendant and is called the central vertex of the star. A path is a tree with exactly two pendant vertex and a comb graph is a tree that is obtained by attaching a pendant vertex to each vertex of a path. A graph $G=(V, E)$ is called a bipartite graph if $V$ can be partitioned into two subsets $X$ and $Y$ such that each edge in $E$ has one end in $X$ and one end in $Y$, denoted by $G=(X, Y, E)$. Let $G=(X, Y, E)$ be a bipartite graph. The graph $G$ is a tree convex bipartite graph [12] if there is a tree $T=(X, F)$ such that the induced subgraph $T\left[N_{G}(v)\right]$ is connected for each vertex $v \in Y$. When $T$ is a star (resp., comb), then $G$ is a star (resp., comb) convex bipartite graph. Let $H(d)$ denote the first $d$ terms of the harmonic series, that is, $H(d)=\sum_{i=1}^{d} 1 / i$. Note that $H(d) \leq \ln (d)+1$.
Given a graph $G=(V, E)$ and a function $f$ from $V$ to $\{0,1, \ldots, t\}$, where $t>0$ is an integer, the weight of $f$, denoted by $w(f)$, is equal to $\sum_{v \in V} f(v)$. We denote $f$ by $\left(V_{0}, V_{1}, \ldots, V_{k}\right)$, where $V_{i}=\{v \in V: f(v)=i\}$ for all $0 \leq i \leq t$. A function $f: V \rightarrow\{0,1,2\}$ is a Roman dominating function (RDF) of $G$, if each vertex $v \in V$ with $f(v)=0$ is adjacent to a vertex $u \in D$ with $f(u)=2$. The Roman domination number of $G$ is the minimum weight of an RDF $f$ of $G$. Beeler et al. [6] initiated the study of double Roman dominating functions, a stronger version of Roman domination functions. A double Roman dominating function (DRDF) of $G$ is a function $f: V \rightarrow$ $\{0,1,2,3\}$ such that for each $v \in V$ :

1. with $f(v)=0$, there is a vertex $u \in N_{G}(v)$ with $f(u)=3$ or there are vertices $x, y \in N_{G}(v)$ with $f(x)=f(y)=2$, and
2. with $f(v)=1$, there is a vertex $u \in N_{G}(v)$ with $f(u)>1$.

The double Roman domination number of $G$ is the minimum weight of an DRDF $f$ of $G$.
The double Roman domination of graphs has been studied in the literature, for example [16, 17]. In 2019, Abdollahzadeh Ahangar et al. [1] introduced a generalization of the DRDFs in which any undefended place could be defended from a sudden attack with, at least, $k$ legions without leaving any neighboring strong-city without military forces.
We use the notation used in [1]. Let $G=(V, E)$ be a graph and let $f: V \rightarrow$ $\{0,1, \ldots, k+1\}$ for a given positive integer $k$. Given a vertex $v \in V$, the active neighbourhood of $v$, denoted by $A N(v)$, is the set of vertices $w \in N_{G}(v)$ such that
$f(w) \geq 1$ and let $A N[v]=A N(v) \cup\{v\}$. The function $f$ is a $[k]-\mathrm{RDF}$, if for each vertex $v \in V$ with $f(v)<k$,

$$
f(A N[v]) \geq|A N(v)|+k
$$

Denote the minimum weight of an $[k]$-RDF of $G$ by $\gamma_{[k R]}(G)$. Note that for $k \in\{1,2\}$ the $[k]$-RDF definition matches that of the RDF and DRDF. Authors [1] focused their attention to the triple Roman domination number $(k=3)$ case, so that for any vertex $v \in V$ with $f(v)<3$, it must happen that $f(A N[v]) \geq|A N(v)|+3$. More precisely, for each vertex $v \in V$, the following conditions hold.

1. If $h(v)=0$, then $v$ must have either one neighbour in $V_{4}$, or either two neighbours in $V_{2} \cup V_{3}$ (one neighbour in $V_{3}$ ) or either three neighbours in $V_{2}$.
2. If $h(v)=1$, then $v$ must have either one neighbour in $V_{3} \cup V_{4}$ or either two neighbours in $V_{2}$.
3. If $h(v)=2$, then $v$ must have one neighbour in $V_{2} \cup V_{3} \cup V_{4}$.

The triple Roman domination problem is to compute the triple Roman domination number, the minimum weight of a triple Roman dominating function (TRDF), of a given graph. Authors [1] proved that the triple Roman domination problem is NPcomplete for chordal graphs and bipartite graphs. Moreover, they showed that it is possible to compute the triple Roman domination number of bounded clique-width graphs in linear-time. Triple Roman domination has been studied by several authors $[2,4,9,10,14]$.
The organization of the paper as follows. In Section 2, we prove that the triple Roman domination problem is NP-complete even for the star convex bipartite graphs and the comb convex bipartite graphs. In Section 3, we propose a linear-time algorithm for computing the triple Roman domination number of proper interval graphs. In Section 4, we prove that for any $\varepsilon>0$ there is no $(1 / 4-\varepsilon) \ln |V|$-approximation polynomial-time algorithm for solving the triple Roman domination problem on bipartite and split graphs, unless NP $\subseteq$ DTIME $\left(|V|^{O(\log \log |V|)}\right)$. In Section 5 , we first give an $(2 H(\Delta(G)+1)-1)$-approximation algorithm for computing the triple Roman domination number of graphs. Finally, APX-completeness of the triple Roman domination problem for graphs of degree at most 4 is proven.

## 2. NP-complete results

In this section, we show that the decision version of triple Roman domination problem is NP-complete even when restricted to the star convex bipartite graphs and the comb convex bipartite graphs. For this purpose, we present polynomial-time reductions from a well-known NP-complete problem, 3-SAT [11], to the triple Roman domination problem. The 3-SAT problem and the decision version of triple Roman
domination problem are defined as follows:
3-SAT
Instance: A boolean formula $\Phi$ in 3 -conjunctive normal form.
Question: Is $\Phi$ satisfiable?
Let $C=\left\{c_{1}, \cdots, c_{l}\right\}$ be a set of $l \geq 1$ clauses, let $X=\left\{x_{1}, \cdots, x_{k}\right\}$ be a set of $k \geq 3$ variables and let $[1, t]=\{1,2, \ldots, t\}$, where $t$ is a positive integer. $\Phi=\{C, X\}$ is called an instance of 3 -SAT if the clause $c_{j}, j \in[1, l]$, is of the form $c_{j}=\left\{y_{1 j}, y_{2 j}, y_{3 j}\right\}$ such that each of $y_{1 j}, y_{2 j}$ and $y_{3 j}$ is either a variable or the negation of a variable in $X$.

Triple Roman Domination (TRD)
Instance: A graph $G$ and a positive integer $t$.
Question: Is there an TRDF $f$ of $G$ with $w(f) \leq t$ ?

Theorem 1. The TRD problem is NP-complete even for the star convex bipartite graphs.

Proof. Clearly, the TRD problem is in NP because for a given graph $G$, a positive integer $t$ and a function $f$ on $G$ we can check in polynomial-time whether $f$ is an TRDF of $G$ with $w(f) \leq t$. In the rest of the proof, we transfer an instance $\Phi=\{C, X\}$ of the 3 -SAT problem to an instance $\left(G_{\Phi}, 8 k\right)$ of the TRD problem. Let $i \in[1, k]$ and $j \in[1, l]$.

- Add a path $u_{i}^{f} u_{i} u_{i}^{t}$ such that $u_{i}$ is adjacent to new pendants $b_{i}^{1}, b_{i}^{2}, b_{i}^{3}, b_{i}^{4}$ and both $u_{i}^{f}$ and $u_{i}^{t}$ are adjacent to new vertices $a_{i}^{1}, a_{i}^{2}, a_{i}^{3}, a_{i}^{4}$ for each $x_{i} \in X$.
- Add a vertex $z_{j}$ for each $c_{j} \in C$.
- Add an edge $u_{i}^{t} z_{j}$ if $x_{i} \in c_{j}$ for each $c_{j} \in C$.
- Add an edge $u_{i}^{f} z_{j}$ if $\neg x_{i} \in c_{j}$, where $\neg x_{i}$ is the negation of $x_{i}$, for each $c_{j} \in C$.
- Add a new vertex $o$ such that is adjacent to both $u_{i}^{t}$ and $u_{i}^{f}$ for each $i \in[1, k]$.

Let $G_{\Phi}$ be the resulting graph. See Figure 1(a). The graph $G_{\Phi}=(A, B, E)$ is a star convex bipartite graph with an associated star tree $T=(A, F)$, see Figure 1(b), where $A=\left\{o, z_{j}, u_{i}, a_{i}^{1}, a_{i}^{2}, a_{i}^{3}, a_{i}^{4}: i \in[1, k], j \in[1, l]\right\}, B=\left\{u_{i}^{f}, u_{i}^{t}, b_{i}^{1}, b_{i}^{2}, b_{i}^{3}, b_{i}^{4}: i \in[1, k]\right\}$ and $F=\{o y: y \in A \backslash\{o\}\}$.

Claim 1. The boolean formula $\Phi$ is satisfiable if and only if there is an TRDF $f$ on $G_{\Phi}$ with $w(f) \leq 8 k$.

Proof of Claim. Assume that $\Phi$ is satisfiable. Let $T$ be a truth assignment for variables in $X$ for which $\Phi$ evaluates to true. We construct sets $V_{0}$ and $V_{4}$ on the vertex set of $G_{\Phi}$ as follows. Initialize $V_{0}$ to be $\left\{o, z_{j}, a_{i}^{1}, \ldots, a_{i}^{4}, b_{i}^{1}, \ldots, b_{i}^{4}: i \in[1, k], j \in[1, l]\right\}$


Figure 1. (a) Constructing a star convex bipartite graph $G_{\Phi}$ from a given instance $\Phi=\{C, X\}$ of the 3-SAT problem, where $X=\left\{x_{1}, x_{2}, x_{3}, x_{4}\right\}, C=\left\{c_{1}, c_{2}, c_{3}\right\}, c_{1}=\left\{x_{1}, \neg x_{2}, \neg x_{3}\right\}, c_{2}=$ $\left\{\neg x_{1}, x_{2}, \neg x_{3}\right\}$ and $c_{3}=\left\{x_{2}, x_{3}, \neg x_{4}\right\}$ and (b) an associated star tree with $G_{\Phi}$.
and $V_{4}$ to be $\left\{u_{i}: i \in[1, k]\right\}$. If $T$ assigns the value false (resp., true) to $x_{i}$, then we add the vertex $u_{i}^{f}$ (resp., $u_{i}^{t}$ ) to $V_{4}$ and $u_{i}^{t}$ (resp., $u_{i}^{f}$ ) to $V_{0}$. Function $f=\left(V_{0}, \emptyset, \emptyset, \emptyset, V_{4}\right)$ is an TRDF on $G_{\Phi}$ with $w(f)=8 k$.
Conversely, let $f=\left(V_{0}, V_{1}, V_{2}, V_{3}, V_{4}\right)$ be an TRDF on $G_{\Phi}$ with $w(f) \leq 8 k$. We fix indices $i$ and $j$, where $1 \leq i \leq k$ and $1 \leq j \leq l$. It gets that $f\left(u_{i}\right)+\sum_{s=1}^{4} f\left(b_{i}^{s}\right) \geq 4$ and $f\left(u_{i}^{f}\right)+f\left(u_{i}^{t}\right)+\sum_{s=1}^{4} f\left(a_{i}^{s}\right) \geq 4$ and so $S_{i}=f\left(u_{i}^{f}\right)+f\left(u_{i}\right)+f\left(u_{i}^{t}\right)+\sum_{s=1}^{4}\left(f\left(a_{i}^{s}\right)+f\left(b_{i}^{s}\right)\right) \geq 8$. Since $f$ is an TRDF on $G_{\Phi}$ with $w(f) \leq 8 k$, it obtains that $S_{i}=8$ and so $f(o)=f\left(z_{j}\right)=0$. This holds only when $\sum_{s=1}^{4}\left(f\left(a_{i}^{s}\right)+f\left(b_{i}^{s}\right)\right)=0, f\left(u_{i}\right)=4$ and either $f\left(u_{i}^{f}\right)=4$ and $f\left(u_{i}^{t}\right)=0$ or $f\left(u_{i}^{f}\right)=0$ and $f\left(u_{i}^{t}\right)=4$. If $f\left(u_{i}^{f}\right)=4$ and $f\left(u_{i}^{t}\right)=0$ (resp., $f\left(u_{i}^{f}\right)=0$ and $f\left(u_{i}^{t}\right)=4$ ), then we assign the value false (resp., true) to the variable $x_{i}$. We claim that this assignment satisfies $\Phi$. Assume $c_{j}=\left\{y_{1 j}, y_{2 j}, y_{3 j}\right\}$. By constructing $G_{\Phi}$, for each $s \in\{1,2,3\}$, if $y_{s j}=x_{i}$, for some $1 \leq i \leq k$, then $z_{j}$ is adjacent to $u_{i}^{t}$ and otherwise, adjacent to $u_{i}^{f}$. Since $f\left(z_{j}\right)=f(o)=0$ and $\left|N_{G_{\Phi}}\left(z_{j}\right)\right|=3, f(y)=4$ for some $y \in N_{G_{\Phi}}\left(z_{j}\right)$. Assume without loss of generality that the vertex $y$ is corresponding to $y_{1 j}$ and $y_{1 j} \in\left\{x_{i}, \neg x_{i}\right\}$, for some $i \in[1, k]$, where $\neg x_{i}$ is the negation of $x_{i}$. If $y_{1 j}=\neg x_{i}$ (resp., $y_{1 j}=x_{i}$ ), then $f\left(u_{i}^{f}\right)=4$ and $f\left(u_{i}^{t}\right)=0$ (resp., $f\left(u_{i}^{f}\right)=0$ and $f\left(u_{i}^{t}\right)=4$ ) and so $x_{i}$ has the value false (resp., true). It causes to satisfy the clause $c_{j}$ and so the boolean formula $\Phi$ is satisfiable. This completes the proof of the claim. $\triangleleft$

We can compute $G_{\Phi}$ in polynomial time with respect to the size of $|X|$ and $|C|$. This completes the proof of the theorem.

Theorem 2. The TRD problem is NP-complete even for the comb convex bipartite graphs.

Proof. We transfer an instance $\Phi=\{C, X\}$ of the 3-SAT problem to an instance $\left(H_{\Phi}, 8 k\right)$ of the TRD problem as follows. Let $i \in[1, k]$ and $j \in[1, l]$.


Figure 2. (a) Constructing a comb convex bipartite graph $H_{\Phi}$ from a given instance $\Phi=\{C, X\}$ of the 3 -SAT problem, where $X=\left\{x_{1}, x_{2}, x_{3}, x_{4}\right\}, C=\left\{c_{1}, c_{2}, c_{3}\right\}, c_{1}=\left\{x_{1}, \neg x_{2}, \neg x_{3}\right\}$, $c_{2}=\left\{\neg x_{1}, x_{2}, \neg x_{3}\right\}$ and $c_{3}=\left\{x_{2}, x_{3}, \neg x_{4}\right\}$, (b) an associated comb tree with $H_{\Phi}$. Note that some edges of $H_{\Phi}$ are not drawn.

- Add a path $u_{i}^{f} u_{i} u_{i}^{t}$ such that $u_{i}$ is adjacent to new pendants $b_{i}^{1}, b_{i}^{2}, b_{i}^{3}, b_{i}^{4}$ and both $u_{i}^{f}$ and $u_{i}^{t}$ are adjacent to new vertices $a_{i}^{1}, a_{i}^{2}, a_{i}^{3}, a_{i}^{4}$ for each $x_{i} \in X$.
- Add a vertex $z_{j}$ for each $c_{j} \in C$.
- Add an edge $u_{i}^{t} z_{j}$ if $x_{i} \in c_{j}$ for each $c_{j} \in C$.
- Add an edge $u_{i}^{f} z_{j}$ if $\neg x_{i} \in c_{j}$, where $\neg x_{i}$ is the negation of $x_{i}$, for each $c_{j} \in C$.
- Add new vertices $u_{i}^{\prime}, z_{j}^{\prime}, d_{i}^{1}, d_{i}^{2}, d_{i}^{3}, d_{i}^{4}$ for each $i \in[1, k]$ and $j \in[1, l]$ such that each of these vertices is adjacent to $u_{i}^{f}$ and $u_{i}^{t}$ for all $i \in[1, k]$.

Let $H_{\Phi}$ be the resulting graph, see Figure 2(a), and let $A=$ $\left\{z_{j}, z_{j}^{\prime}, u_{i}, u_{i}^{\prime}, a_{i}^{1}, a_{i}^{2}, a_{i}^{3}, a_{i}^{4}, d_{i}^{1}, d_{i}^{2}, d_{i}^{3}, d_{i}^{4} \quad: \quad i \quad[1, k], j \in[1, l]\right\}$ and $B=$ $\left\{u_{i}^{f}, u_{i}^{t}, b_{i}^{1}, b_{i}^{2}, b_{i}^{3}, b_{i}^{4}: i \in[1, k]\right\}$. The graph $H_{\Phi}=(A, B, E)$ is a comb convex bipartite graph with an associated comb tree $T=(A, F)$, see Figure 2(b), where $T$ includes the path $u_{1}^{\prime} \ldots u_{k}^{\prime} z_{1}^{\prime} \ldots z_{l}^{\prime} d_{1}^{1} d_{1}^{2} d_{1}^{3} d_{1}^{4} \ldots d_{k}^{1} d_{k}^{2} d_{k}^{3} d_{k}^{4}$ such that $u_{i}^{\prime}$ is adjacent to $u_{i}, z_{j}^{\prime}$ is adjacent to $z_{j}$, and $d_{i}^{s}$ is adjacent to $a_{i}^{s}$ for all $i \in[1, k], j \in[1, l]$ and $s \in[1,4]$. Similar to Claim 1, we can obtain the following result.

Claim 2. The boolean formula $\Phi$ is satisfiable if and only if there is an TRDF $f$ on $H_{\Phi}$ such that $w(f) \leq 8 k$.

Recall that the TRD problem is in NP. We can compute $H_{\Phi}$ in polynomial time with respect to the size of $|X|$ and $|C|$. This completes the proof of the theorem.

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Algorithm 3.1: \(\operatorname{TRDNPIG}(G, 1, \ldots, n)\)
Input: A proper interval graph \(G=(V, E)\) with \(|V|=n\) such that \((1, \ldots, n)\) is a consecutive
        ordering of vertices in \(V\).
    Output: The triple Roman domination number of \(G\).
Compute \(\operatorname{Min}(1), \ldots, \operatorname{MIN}(n)\);
\(\gamma_{[3 R]}^{0}(1) \leftarrow \infty ; \gamma_{[3 R]}^{3}(1) \leftarrow 3 ; \gamma_{[3 R]}^{4}(1) \leftarrow 4 ; i \leftarrow 1\);
3 while \(i<n\) do
        \(i \leftarrow i+1 ;\)
        \(v \leftarrow \operatorname{MIN}(i) ;\)
        \(\gamma_{[3 R]}^{0}(i) \leftarrow \gamma_{[3 R]}^{4}(v) ;\)
        \(\gamma_{[3 R]}^{3}(i) \leftarrow \gamma_{[3 R]}(i+1)+3 ;\)
        \(\gamma_{[3 R]}^{4}(i) \leftarrow \gamma_{[3 R]}(v-1)+4 ;\)
        \(\gamma_{[3 R]}(i) \leftarrow \min \left\{\gamma_{[3 R]}^{0}(i), \gamma_{[3 R]}^{3}(i), \gamma_{[3 R]}^{4}(i)\right\} ;\)
    end while
    return \(\gamma_{[3 R]}(n)\);
```


## 3. Proper interval graphs

In this section we propose a linear algorithm (Algorithm 3.1) for computing the triple Roman domination number of a given proper interval graph. Let $G$ be a graph of order $n$. An ordering $\left(v_{1}, v_{2}, \ldots, v_{n}\right)$ of vertices of $G$ is a consecutive ordering if $v_{i} v_{k} \in E$ for some $1 \leq i<k \leq n$ implies both $v_{i} v_{j} \in E$ and $v_{j} v_{k} \in E$ for every $i<j<k$.

Theorem 3 ([13]). A graph $G$ is a proper interval graph if and only if $G$ has a consecutive ordering of its vertices.

Booth and Lueker [7] proposed a linear-time algorithm for testing whether a graph is a proper interval graph, and give a consecutive ordering if the answer is positive. For a given proper interval $G=(V, E)$ of order $n$, let $V=\{1, \ldots, n\}$ and let $1 \leq i \leq j \leq n$ and $a \in\{0,1,2,3,4\}$.

- $[i, j)=\{i \leq k<j\}$,
- $(i, j]=\{i<k \leq j\}$,
- $(i, j)=\{i<k<j\}$,
- $G[i, j]=G[\{i \leq k \leq j\}]$,
- $\operatorname{MIN}(i)=\min N_{G}[i]$,
- $\gamma_{[3 R]}^{a}(i)=\min \{w(f): f$ is an TRDF on $G[1, i]$ with $f(i)=a\}$.

To prove that Algorithm 3.1 works correctly we need the following results.

Proposition 1. Given a proper interval graph $G=(V, E)$ with $|V|=n$ and a consecutive ordering $(1, \ldots, n)$ of vertices of $G$, let $1 \leq i \leq j \leq n$.


Figure 3. Illustrating an TRDF $g$ on $H$ such that $g(n+1)=4$ and an TRDF $f$ on $H[1, n]$ such that $f(n)=4$; note that some edges of $H$ are not drawn.
(i) For all $S \subseteq V$, the induced subgraph $G[S]$ is also a proper interval graph.
(ii) If $i j \in E$, then $[i, j]$ is a clique of $G$.
(iii) $\operatorname{MIN}(i) \leq \operatorname{MIN}(j)$.

Lemma 1. Given a proper interval graph $G=(V, E)$ with $|V|=n$ and a consecutive ordering $(1, \ldots, n)$ of vertices of $G$, $\gamma_{[3 R]}^{4}(1) \leq \gamma_{[3 R]}^{4}(2) \leq \cdots \leq \gamma_{[3 R]}^{4}(n)$.

Proof. The proof is by induction on $n$. Clearly, $\gamma_{[3 R]}^{4}(1)=\gamma_{[3 R]}^{4}(2)=4$ and so the claim holds for $n=2$. Assume the claim holds for all proper interval graphs of order $n \geq 2$. Let $H$ be a proper interval graph of order $n+1$ with a consecutive ordering $(1, \ldots, n+1)$ of vertices of $H$. By Proposition 1, the induced subgraph $H[1, n]$ is a proper interval graph of order $n$ and so

$$
\begin{equation*}
\gamma_{[3 R]}^{4}(1) \leq \cdots \leq \gamma_{[3 R]}^{4}(n) . \tag{1}
\end{equation*}
$$

Let $g$ be an TRDF on $H$ such that its weight is minimum and $g(n+1)=4$. So, $w(g)=\gamma_{[3 R]}^{4}(n+1)$. See Figure 3. By Proposition 1 and since $H$ is connected, $\operatorname{MIN}(n) \leq \operatorname{MIN}(n+1) \leq n$ and $[\operatorname{MIN}(n), n]$ is a clique of $H$. So, each vertex adjacent to $n+1$ is also adjacent to $n$. Let $f$ be a new function from $[1, n]$ to $[0,4]$ as follows: $f(i)=g(i)$ for all $i \in[1, n-1]$ and $f(n)=4$. Clearly, $w(f)=w(g)-g(n)$, where $g(n) \in[0,4]$, and so $w(f) \leq w(g)$. Since each vertex adjacent to $n+1$ is also adjacent to $n, f$ is an TRDF on $H[1, n]$ such that $f(n)=4$. Hence, $\gamma_{[3 R]}^{4}(n) \leq w(f) \leq \gamma_{[3 R]}^{4}(n+1)$. This, together with Inequality (1), completes the proof of the lemma.

Lemma 2. Given a proper interval graph $G=(V, E)$ with $|V|=n$ and a consecutive ordering $(1, \ldots, n)$ of vertices of $G$, if $i \in[2, n]$; then $\gamma_{[3 R]}^{0}(i)=\gamma_{[3 R]}^{4}(\operatorname{MIN}(i))$.

Proof. Let $f=\left(V_{0}, V_{1}, V_{2}, V_{3}, V_{4}\right)$ be an TRDF on $G[1, i]$ such that its weight is minimum and $f(i)=0$. So, $w(f)=\gamma_{[3 R]}^{0}(i)$. Since $f(i)=0$, the vertex $i$ must have either (i) one neighbour in $V_{4}$, or either (ii) two neighbours in $V_{2} \cup V_{3}$ (one neighbour in $V_{3}$ ) or either (iii) three neighbours in $V_{2}$.


Figure 4. Illustrating two TRDFs $f$ and $g$ on $G[1, i]$; note that some edges of $G$ are not drawn.

We claim that Cases (ii) and (iii) do not occur. Assume that $i$ has two neighbours in $V_{2} \cup V_{3}$ (one neighbour in $V_{3}$ ), that is, $f(a)=3$ and $f(b)=2$ for some $a, b \in[\operatorname{MIN}(i), i)$. We first consider the case that $a<b$. By Proposition 1, $\operatorname{MIN}(a) \leq \operatorname{MIN}(b) \leq \operatorname{MIN}(i)$ and $[\operatorname{MIN}(x), x]$ is a clique of $G$ for each vertex $x$. So, each vertex adjacent to both vertices $a, b$ is also adjacent to $a$. Let $g$ be a new function on $G[1, i]$ as $g(k)=f(k)$ for all $k \in[1, i] \backslash\{a, b\}, g(a)=4$ and $g(b)=0$. We get that $g$ is an TRDF on $G[1, i]$ such that $g(i)=0$ and $w(g)<w(f)$, contradicting that $f$ is an TRDF on $G[1, i]$ such that its weight is minimum and $f(i)=0$. See Figure 4(a). Similarly, if $b<a$ or $i$ has three neighbours in $V_{2}$, then we can obtain a new TRDF $g$ on $G[1, i]$ such that $g(i)=0$ and $w(g)<w(f)$, see Figure $4(\mathrm{~b})$, a contradiction. This proves the claim. Now, assume that $i$ has one neighbour in $V_{4}$. So, $f(a)=4$ for some $a \in[\operatorname{MIN}(i), i)$. See Figure $4(\mathrm{c})$. By Proposition 1, $\operatorname{MIN}(a) \leq \operatorname{MIN}(i)$ and $[\operatorname{MIN}(a), a]$ is a clique of $G$. So, each vertex adjacent to $i$ is also adjacent to $a$ in the induced subgraph $G[1, i]$. Let $f^{\prime}$ be the restriction of $f$ to $G[1, a]$. We get that $w\left(f^{\prime}\right) \leq w(f)$ and $f^{\prime}$ is an TRDF on $G[1, a]$ such that $f^{\prime}(a)=4$. Thus, $\gamma_{[3 R]}^{4}(a) \leq w\left(f^{\prime}\right) \leq w(f)=\gamma_{[3 R]}^{0}(i)$. Since $\operatorname{MIN}(i) \leq a$, by Lemma 1, $\gamma_{[3 R]}^{4}(\operatorname{MIN}(i)) \leq \gamma_{[3 R]}^{4}(a)$ and so $\gamma_{[3 R]}^{4}(\operatorname{MIN}(i)) \leq \gamma_{[3 R]}^{0}(i)$.
Conversely, let $g$ be an TRDF on $G[1, \operatorname{MIN}(i)]$ such that its weight is minimum and $g(\operatorname{MIN}(i))=4$. So, $w(g)=\gamma_{[3 R]}^{4}(\operatorname{MIN}(i))$. Let $h$ be a function on $G[1, i]$ as $h(k)=g(k)$ for all $k \in[1, \operatorname{MIN}(i)]$ and $h(j)=0$ for all $j \in(\operatorname{MIN}(i), i]$. We have $w(h)=w(g)=$ $\gamma_{[3 R]}^{4}(\operatorname{MIN}(i))$. Because $[\operatorname{MIN}(i), i]$ is a clique of $G, h$ is an TRDF on $G[1, i]$ such that $h(i)=0$. Hence, $\gamma_{[3 R]}^{0}(i) \leq w(h)=\gamma_{[3 R]}^{4}(\operatorname{MIN}(i))$. This implies that $\gamma_{[3 R]}^{0}(i)=$ $\gamma_{[3 R]}^{4}(\operatorname{MIN}(i))$ and completes the proof of the lemma.

Theorem 4 ([1]). For a given graph $G$, there exists an TRDF on $G$ with minimum weight that does not assign an 1 to any vertex in $G$.

Lemma 3. Given a proper interval graph $G=(V, E)$ with $|V|=n$ and a consecutive ordering $(1, \ldots, n)$ of vertices of $G$, let $i \in[2, n]$; then
(i) $\gamma_{[3 R]}^{0}(i) \leq \gamma_{[3 R]}^{2}(i)$ and
(ii) $\gamma_{[3 R]}^{3}(i)=\gamma_{[3 R]}(i-1)+3$ or $\gamma_{[3 R]}^{0}(i) \leq \gamma_{[3 R]}^{3}(i)$.

Proof. We first prove (i). Let $f=\left(V_{0}, V_{1}, V_{2}, V_{3}, V_{4}\right)$ be an TRDF on $G[1, i]$ such that its weight is minimum and $f(i)=2$, that is, $w(f)=\gamma_{[3 R]}^{2}(i)$. Since $f(i)=2$,


Figure 5. Illustrating two TRDFs $f$ and $g$ on $G[1, i]$; note that some edges of $G$ are not drawn.
the vertex $i$ has one neighbour $j$ in $V_{2} \cup V_{3} \cup V_{4}$, that is, $j \in[\operatorname{MIN}(i), i)$ such that $f(j) \geq 2$. See Figure 5(a). Let $g$ be a new function on $G[1, i]$ as $g(x)=f(x)$ for all $x \in[1, i) \backslash\{j\}, g(j)=4$ and $g(i)=0$. We obtain that $g$ is an TRDF on $G[1, i]$ such that $g(i)=0$ and $w(g) \leq w(f)$. Thus, $\gamma_{[3 R]}^{0}(i) \leq w(g) \leq \gamma_{[3 R]}^{2}(i)$. This completes the proof of (i).
Now, we prove (ii). Let $f=\left(V_{0}, V_{1}, V_{2}, V_{3}, V_{4}\right)$ be an TRDF on $G[1, i]$ such that its weight is minimum and $f(i)=3$, that is, $w(f)=\gamma_{[3 R]}^{3}(i)$. We distinguish two cases depending on whether $f(x) \geq 1$ for some $x \in[\operatorname{MIN}(i), i)$.

Case 1. Assume $f(j) \geq 1$ for some $j \in[\operatorname{MIN}(i), i)$. Let $g$ be a new function on $G[1, i]$ as $g(x)=f(x)$ for all $x \in[1, i) \backslash\{j\}, g(j)=4$ and $g(i)=0$. See Figure 5(b). We see that $g$ is an TRDF on $G[1, i]$ such that $g(i)=0$ and $w(g) \leq w(f)$. Hence, $\gamma_{[3 R]}^{0}(i) \leq w(g) \leq \gamma_{[3 R]}^{3}(i)$.

Case 2. Assume that $f(x)=0$ for all $x \in[\operatorname{MIN}(i), i)$. We have $\operatorname{MIN}(i)<i$ and so $f(i-1)=0$. Since $f$ is an TRDF on $G[1, i], f(j) \geq 2$ for some $j \in[\operatorname{MIN}(i-1), \operatorname{MIN}(i))$ and so $\operatorname{MIN}(i-1)<\operatorname{MIN}(i)$.

- Assume that $f(j)=2$. Since $f$ is an TRDF on $G[1, i]$, the vertex $j$ has one neighbour $k$ in $V_{2} \cup V_{3} \cup V_{4}$.
If $f(k) \geq 3$, then let $g$ be a new function on $G[1, i]$ as $g(x)=f(x)$ for all $x \in[1, i) \backslash\{k, j, \operatorname{MIN}(i)\}, g(k)=g(\operatorname{MIN}(i))=4$ and $g(j)=g(i)=0$. See Figure $5(\mathrm{c})$. We get that $g$ is an TRDF on $G[1, i]$ such that $g(i)=0$ and $w(g) \leq w(f)$. Thus, $\gamma_{[3 R]}^{0}(i) \leq \gamma_{[3 R]}^{3}(i)$.
Now, assume that $f(k)=2$. Let $l$ be a vertex such that $f(l) \geq 2, l<k$, and $f(x) \leq 1$ for all $x \in(l, k)$, that is, $l=\max \{x<k: f(x) \geq 2\}$. (If such vertex does not exist, then we obtain that $k=1$ and $j=2=\operatorname{MIN}(i-1)$ and $\gamma_{[3 R]}^{0}(i) \leq \gamma_{[3 R]}^{3}(i)$.) We construct a new function $g$ on $G[1, i]$ as $g(x)=f(x)$ for all $x \in[1, i) \backslash\{l, j, \operatorname{MIN}(i)\}, g(l)=3, g(j)=g(i)=0$, and $g(\operatorname{MIN}(i))=4$. See Figure $5(\mathrm{~d})$. We get that $g$ is an TRDF on $G[1, i]$ such that $g(i)=0$ and $w(g) \leq w(f)$. Hence, $\gamma_{[3 R]}^{0}(i) \leq \gamma_{[3 R]}^{3}(i)$.
- Assume that $f(j)=3$. Let $k$ be a vertex such that $f(k) \geq 2, k<j$, and $f(x) \leq 1$ for all $x \in(k, j)$, that is, $k=\max \{x<j: f(x) \geq 2\}$. (If such


Figure 6. Illustrating an TRDF $f$ on $G[1, i]$ with $f(i)=f(j)=2$ such that $j$ is adjacent to $i$; note that some edges of $G$ are not drawn.
vertex does not exist, then we obtain that $j=1=\operatorname{MIN}(i-1)$ and $\gamma_{[3 R]}^{0}(i) \leq$ $\gamma_{[3 R]}^{3}(i)$.) We construct a new function $g$ on $G[1, i]$ as $g(x)=f(x)$ for all $x \in[1, i) \backslash\{k, j, \operatorname{MIN}(i)\}, g(k)=3, g(j)=2, g(\operatorname{MIN}(i))=4$, and $g(i)=0$. See Figure $6(\mathrm{a})$. We get that $g$ is an TRDF on $G[1, i]$ such that $g(i)=0$ and $w(g) \leq w(f)$. Thus, $\gamma_{[3 R]}^{0}(i) \leq \gamma_{[3 R]}^{3}(i)$.

- Assume that $f(j)=4$. See Figure 6(b). Let $f^{\prime}$ be the restriction of $f$ to $G[1, i-1]$. Since $f^{\prime}(j)=4$, we see that $f^{\prime}$ is an TRDF on $G[1, i-1]$. So, $\gamma_{[3 R]}(i-1) \leq w\left(f^{\prime}\right)=w(f)-3=\gamma_{[3 R]}^{3}(i)-3$. Conversely, assume that $g$ is an TRDF on $G[1, i-1]$ such that its weight is minimum. So, $w(g)=\gamma_{[3 R]}(i-1)$. Let $h=g \cup\{(i, 3)\}$. We get that $h$ is an TRDF on $G[1, i]$ such that $h(i)=3$. Hence, $\gamma_{[3 R]}^{3}(i) \leq w(h)=w(g)+3=\gamma_{[3 R]}(i-1)+3$. This, together with $\gamma_{[3 R]}(i-1) \leq \gamma_{[3 R]}^{3}(i)-3$, implies $\gamma_{[3 R]}^{3}(i)=\gamma_{[3 R]}(i-1)+3$.

This completes the proof of (ii) and so the proof of the lemma.

Lemma 4. Given a proper interval graph $G=(V, E)$ with $|V|=n$ and a consecutive ordering $(1, \ldots, n)$ of vertices of $G$, let $i \in[2, n]$. If $\operatorname{MIN}(i) \geq 2$, then $\gamma_{[3 R]}^{4}(i)=\gamma_{[3 R]}(\operatorname{MIN}(i)-$ $1)+4$, otherwise, $\gamma_{[3 R]}^{4}(i)=4$.

Proof. If $\operatorname{MIN}(i)=1$, that is, $[1, i]$ is a clique of $G$, then clearly $\gamma_{[3 R]}^{4}(i)=4$. Note that $i \geq 2$. In the rest of the proof, we assume that $\operatorname{MIN}(i) \geq 2$. Let $f=\left(V_{0}, V_{1}, V_{2}, V_{3}, V_{4}\right)$ be an TRDF on $G[1, i]$ such that its weight is minimum and $f(i)=4$, that is, $w(f)=$ $\gamma_{[3 R]}^{4}(i)$. Clearly, $f(x) \neq 1$ for all $x \in[\min (i), i)$. If $f(u), f(v) \geq 2$ for some $u, v \in$ $[\min (i), i)$ with $u<v$, then let $g$ be a new function on $G[1, i]$ as $g(x)=f(x)$ for all $x \in[1, i] \backslash\{u, v\}, g(u)=4$ and $g(v)=0$. We get that $g$ is an TRDF on $G[1, i]$ such that $g(i)=4$ and $w(g) \leq w(f)$. See Figure 7 (a). If $f(u) \geq 2$ for exactly one vertex $u \in[\min (i), i)$, then let $g$ be a new function on $G[1, i]$ as $g(x)=f(x)$ for all $x \in[1, i] \backslash\{\min (i)-1, u\}, g(\min (i)-1)=f(u)$ and $g(u)=0$. We get that $g$ is an TRDF on $G[1, i]$ such that $g(i)=4$ and $w(g) \leq w(f)$. See Figure 7(b). So, in the rest of the proof, we assume that $f(u)=0$ for all $u \in[\min (i), i)$.


Figure 7. Illustrating two TRDFs $f$ and $g$ on $G[1, i]$; note that some edges of $G$ are not drawn.

Let $f^{\prime}$ be the restriction of $f$ to $G[1, \min (i))$. We see that $f^{\prime}$ is an TRDF on $G[1, \min (i))$. So, $\gamma_{[3 R]}(\min (i)-1) \leq w\left(f^{\prime}\right)=w(f)-4=\gamma_{[3 R]}^{4}(i)-4$. Conversely, assume that $g$ is an TRDF on $G[1, \min (i))$ such that its weight is minimum, that is, $w(g)=\gamma_{[3 R]}(\min (i)-1)$. Let $h$ be a function on $G[1, i]$ as $h(x)=g(x)$ for all $x \in[1, \min (i)), h(y)=0$ for all $y \in[\min (i), i)$ and $h(i)=4$. We see that $h$ is an TRDF on $G[1, i]$ such that $h(i)=4$. Hence, $\gamma_{[3 R]}^{4}(i) \leq w(h)=w(g)+4=$ $\gamma_{[3 R]}(\min (i)-1)+4$.

Theorem 5. Given a proper interval graph $G=(V, E)$ with $|V|=n$ and a consecutive ordering $(1, \ldots, n)$ of vertices of $G$, Algorithm 3.1 computes $\gamma_{[3 R]}(G)$ in $O(n)$ time.

Proof. Let $i \in[2, n]$. By Theorem 4, let $f=\left(V_{0}, \emptyset, V_{2}, V_{3}, V_{4}\right)$ be an TRDF on $G[1, i]$ such that its weight is minimum. So, $w(f)=\gamma_{[3 R]}(G[1, i])$. Clearly, $f(i) \in$ $\{0,2,3,4\}$ and so $\gamma_{[3 R]}(G[1, i])=\gamma_{[3 R]}(i)=\min \left\{\gamma_{[3 R]}^{0}(i), \gamma_{[3 R]}^{2}(i), \gamma_{[3 R]}^{3}(i), \gamma_{[3 R]}^{4}(i)\right\}$. By Lemma 3, $\gamma_{[3 R]}(i)=\min \left\{\gamma_{[3 R]}^{0}(i), \gamma_{[3 R]}^{3}(i), \gamma_{[3 R]}^{4}(i)\right\}$. It obtains that $\gamma_{[3 R]}^{0}(1)$ is not defined, $\gamma_{[3 R]}^{3}(1)=3$ and $\gamma_{[3 R]}^{4}(1)=4$. By Lemmas 2, 3 and 4, the output of Algorithm 3.1 on input $(G, 1, \ldots, n)$ is $\gamma_{[3 R]}^{3}(G)$. It follows from (Algorithm 2 of) [5] that we can compute all values $\operatorname{MIN}(1), \ldots, \operatorname{MIN}(n)$ in $O(n)$ time. So, it deduces that the running time of Algorithm 3.1 is $O(n)$.

## 4. Lower bound on the approximation ratio

In this section, a lower bound on the approximation factor of the triple Roman domination problem is established. Before we give our lower bound on the approximation factor of the triple Roman domination problem, we have to introduce the Min Dom Set and the Min Triple Roman Dom Function problems, formalized as follows.

## Min Dom Set

Instance: A graph $G=(V, E)$.
Solution: A DS $S$ of $G$, where a subset $S \subseteq V$ is called a dominating set (DS) of $G$

| Algorithm 4.1: $B$ |
| :--- |
| Input: A graph $G=(V, E)$. |
| Output: An DS of $G$. |
| 1 Compute an TRDF $f=\left(V_{0}, V_{1}, V_{2}, V_{3}, V_{4}\right)$ of $G$ using algorithm $A ;$ |
| $2 D=V_{1} \cup V_{2} \cup V_{3} \cup V_{4} ;$ |
| 3 return $D ;$ |

if each vertex not in $S$ is adjacent to one vertex of $S$.
Measure: $|S|$.
Min Triple Roman Dom Function
Instance: A graph $G=(V, E)$.
Solution: An TRDF $f$ of $G$.
Measure: $w(f)$.
Theorem 6 ([8]). For a given bipartite or split graph $G=(V, E)$, there is no ( $1-$ $\varepsilon) \ln |V|$-approximation polynomial-time algorithm for any $\varepsilon>0$ to solve the Min Dom Set problem, unless $N P \subseteq$ DTIME $\left(|V|^{O(\log \log |V|)}\right)$.

Theorem 7 ([1]). For a given graph $G, 2 \gamma(G) \leq \gamma_{[3 R]}(G) \leq 4 \gamma(G)$.
Theorem 8. For a given bipartite or split graph $G=(V, E)$, there is no $(1 / 4-\varepsilon) \ln |V|-$ approximation polynomial-time algorithm for any $\varepsilon>0$ to solve the Min Triple Roman Dom Function problem, unless NP $\subseteq$ DTIME $\left(|V|^{O(\log \log |V|)}\right)$.

Proof. Assume there is an algorithm $A$ that can approximate within ratio $\alpha>0$ the Min Triple Roman Dom Function problem. Let $f^{*}$ be an TRDF of $G$ such that $w\left(f^{*}\right)=\gamma_{[3 R]}(G)$ and let $D^{*}$ be an DS of $G$ such that $\left|D^{*}\right|=\gamma(G)$. By Theorem 7, $2 \gamma(G) \leq \gamma_{[3 R]}(G) \leq 4 \gamma(G)$ and so $w\left(f^{*}\right) \leq 4\left|D^{*}\right|$. Algorithm 4.1 on input $G$ returns an DS $D$ of $G$ such that $|D|=\left|V_{1}\right|+\left|V_{2}\right|+\left|V_{3}\right|+\left|V_{4}\right| \leq\left|V_{1}\right|+2\left|V_{2}\right|+3\left|V_{3}\right|+4\left|V_{4}\right|=$ $w(f) \leq \alpha \times w\left(f^{*}\right) \leq 4 \alpha\left|D^{*}\right|$. Thus, Algorithm 4.1 can approximate the Min Dom Set problem within the ratio $4 \alpha$. Assume that there is some fixed $\varepsilon>0$ such that can approximate the Min Triple Roman Dom Function problem within ratio $\alpha=(1 / 4-\varepsilon) \ln |V|$ using algorithm $A$. Then, the Min Dom Set problem can be approximated within ratio $\left(1-\varepsilon^{\prime}\right) \ln |V|$ by Algorithm 4.1, where $\varepsilon^{\prime}=4 \varepsilon$, which is a contradiction to Theorem 6. This completes the proof of the theorem.

## 5. An approximation algorithm and APX-Completeness

In this section, we first give an approximation algorithm for computing the triple Roman domination number of graphs. Next, we prove that the triple Roman domination
problem is APX-complete for graphs of degree at most 4. To present an approximation algorithm for computing the triple Roman domination number of graphs, we need the following results.

Theorem 9 ([8]). There is an $(H(\Delta(G)+1)-1 / 2)$-approximation algorithm for computing a minimum DS of any given graph $G$, where $H(d)=\sum_{i=1}^{d} 1 / i$.

Theorem 10. For a given graph $G=(V, E)$, there is an $(2 H(\Delta(G)+1)-1)$ approximation algorithm for computing an TRDF of $G$ with minimum weight.

Proof. By Theorem 9, let $A$ be an approximation algorithm that computes an DS of $G$ and let $D$ be the output of Algorithm $A$ on input $A$. So, $|D| \leq(H(\Delta(G)+$ $1)-1 / 2)\left|D^{*}\right|$, where $D^{*}$ is a minimum DS of $G$. Assume that $f=(V \backslash D, \emptyset, \emptyset, \emptyset, D)$. We get that $f$ is an TRDF of $G$ such that $w(f)=4|D|$. Thus, $w(f) \leq 4(H(\Delta(G)+$ 1) $-1 / 2)\left|D^{*}\right|$. Let $f^{*}$ be an TRDF of $G$ such that $w\left(f^{*}\right)=\gamma_{[3 R]}(G)$. Hence, by Theorem $7, w(f) \leq 4(H(\Delta(G)+1)-1 / 2)\left|D^{*}\right| \leq 2(H(\Delta(G)+1)-1 / 2) w\left(f^{*}\right)$. This completes the proof of the theorem.

To show that the triple Roman domination problem is APX-complete, we use the L-reduction notation, see [3, 15]. Let $F$ and $G$ be two NP optimization problems. An $L$-reduction is a polynomial time transformation $h$ from instances of $F$ to instances of $G$, if for some positive constants $\alpha$ and $\beta$ and each instance $x$ of $F$

1. $\operatorname{OPT}_{G}(h(x)) \leq \alpha \cdot \operatorname{OPT}_{F}(x)$, and
2. we can find a solution $y^{\prime}$ of $x$ with $\mathrm{m}_{F}\left(x, y^{\prime}\right)=c_{1}$ in polynomial time such that $\left|\mathrm{OPT}_{F}(x)-c_{1}\right| \leq \beta\left|\mathrm{OPT}_{G}(h(x))-c_{2}\right|$ for every feasible solution $y$ of $h(x)$ with objective value $\mathrm{m}_{G}(h(x), y)=c_{2}$.

To prove that a problem $P \in \mathrm{APX}$ is APX-complete, we need to give an L-reduction from some APX-complete problem to $P$. We formalize the considered problems as follows.

Min Dom Set-B
Instance: A graph $G=(V, E)$ with degree at most B.
Solution: A DS $D$ of $G$.
Measure: $|D|$.

Min Triple Roman Dom Function-B
Instance: A graph $G=(V, E)$ with degree at most B.
Solution: A TRDF $f$ of $G$.
Measure: $w(f)$.

Theorem 11 ([3]). Min Dom Set-3 is APX-complete.

Theorem 12. Min Triple Roman Dom Function-4 is APX-complete.

Proof. By Theorem 10, Min Triple Roman Dom Function-4 is in APX and by Theorem 11, Min Dom Set-3 is APX-complete. It is enough to construct an Lreduction $h$ from Min Dom Set-3 to Min Triple Roman Dom Function-4. Let $G^{\prime}=\left(V^{\prime}, E^{\prime}\right)$ be a graph constructed from a given graph $G=(V, E)$ with degree at most 3 as $V^{\prime}=V \cup\left\{a_{v}: v \in V\right\}$ and $E^{\prime}=E \cup\left\{v a_{v}: v \in V\right\}$. We get that $G^{\prime}$ is a graph with degree at most 4. For a given DS $D$ of $G$, let $f=\left(V_{0}, V_{1}=\emptyset, V_{2}=\emptyset, V_{3}, V_{4}\right)$, where $V_{4}=D, V_{3}=\left\{a_{v}: v \in V \backslash D\right\}$, and $V_{0}=V^{\prime} \backslash\left(V_{3} \cup V_{4}\right)$. We get that $w(f)=3\left|V_{3}\right|+4\left|V_{4}\right|=3(|V|-|D|)+4|D|=|D|+3|V|$. Since each vertex in $V_{0}$ is adjacent to a vertex in $V_{4}$ and $V_{1}=V_{2}=\emptyset$, the function $f$ is an TRDF of $G^{\prime}$ such that $w(f) \leq|D|+3|V|$. In particular, $w\left(f^{*}\right) \leq\left|D^{*}\right|+3|V|$, where $D^{*}$ is an DS of $G$ with $\left|D^{*}\right|=\gamma(G)$ and $f^{*}$ is an TRDF of $G^{\prime}$ with $w\left(f^{*}\right)=\gamma_{[3 R]}\left(G^{\prime}\right)$. Since $G$ is a graph with degree at most 3 and $D^{*}$ is an DS of $G,|V| \leq \sum_{v \in D^{*}}\left(d_{G}(v)+1\right) \leq 4\left|D^{*}\right|$. Hence, $w\left(f^{*}\right) \leq\left|D^{*}\right|+3|V| \leq\left|D^{*}\right|+12\left|D^{*}\right|=13\left|D^{*}\right|$.
Conversely, let $g$ be an TRDF of $G^{\prime}$. Assume that for some $v \in V$, we have either $g\left(a_{v}\right)=4$, or either $g\left(a_{v}\right)=3$ and $g(v) \geq 1$, or either $g\left(a_{v}\right) \in\{1,2\}$. We obtain that $g\left(a_{v}\right)+g(v) \geq 4$. Let $g^{\prime}=\left(V_{0}^{\prime}, V_{1}^{\prime}, V_{2}^{\prime}, V_{3}^{\prime}, V_{4}^{\prime}\right)$ be a new function of $G^{\prime}$ such that for all $v \in V$ if either $g\left(a_{v}\right)=4$, or either $g\left(a_{v}\right)=3$ and $g(v) \geq 1$, or either $g\left(a_{v}\right) \in\{1,2\}$, then $g^{\prime}\left(a_{v}\right)=0$ and $g^{\prime}(v)=4$, otherwise, $g^{\prime}(v)=g(v)$ and $g^{\prime}\left(a_{v}\right)=g\left(a_{v}\right)$. We obtain that $g^{\prime}$ is an TRDF of $G^{\prime}$ such that $w\left(g^{\prime}\right) \leq w(g), g^{\prime}\left(a_{v}\right) \in\{0,3\}$ for all $v \in V$, and if $g^{\prime}\left(a_{v}\right)=3$ for some $v \in V$, then $g^{\prime}(v)=0$. Hence, $\left|V_{1}^{\prime}\right|=\left|V_{2}^{\prime}\right|=0$, $\left|V_{3}^{\prime}\right|+\left|V_{4}^{\prime}\right|=|V|$, and $V_{4}^{\prime} \subseteq V$ and so $w\left(g^{\prime}\right)=3\left|V_{3}^{\prime}\right|+4\left|V_{4}^{\prime}\right|=3|V|+\left|V_{4}^{\prime}\right|$. Since each vertex $v \in V_{0}^{\prime} \cap V$ is adjacent to a vertex in $S=V_{4}^{\prime}$, the set $S$ is an DS of $G$ such that $|S|=\left|V_{4}^{\prime}\right|=w\left(g^{\prime}\right)-3|V| \leq w(g)-3|V|$. In particular, $\left|D^{*}\right| \leq w\left(f^{*}\right)-3|V|$ and so $w\left(f^{*}\right)=\left|D^{*}\right|+3|V|$. We obtain that $|S|-\left|D^{*}\right| \leq w(g)-3|V|-\left(w\left(f^{*}\right)-3|V|\right)=$ $w(g)-w\left(f^{*}\right)$. As a result, $h$ is an L-reduction such that $\alpha=13$ and $\beta=1$.

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