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Research Article

Total chromatic number for certain classes of lexicographic product graphs

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Abstract: A total coloring of a graph G is an assignment of colors to all the elements (vertices and edges) of the graph in such a way that no two adjacent or incident elements receive the same color. The total chromatic number of G, denoted by $\chi''(G)$, is the minimum number of colors needed for a total coloring of G. The Total Coloring Conjecture (TCC) proposed independently by Behzad and Vizing claims that, $\Delta(G)+1 \leq \chi''(G) \leq \Delta(G)+2$, where $\Delta(G)$ is the maximum degree of G. The lower bound is sharp and the upper bound remains to be proved. In this paper, we prove the TCC for certain classes of lexicographic and deleted lexicographic products of graphs. Also, we obtained the lower bound for certain classes of these products.

Keywords: Total coloring, Lexicographic Product, Deleted Lexicographic Product

AMS Subject classification: 05C15

1. Introduction

All graphs considered in this paper are finite, simple and connected. Let $G = (V(G), E(G))$ be a graph and $\Delta(G)$ denote the the maximum degree of the graph G. Graph coloring is a major sub-topic of graph theory with many useful applications and unsolved problems. Vertex coloring is assigning colors to the vertices such that no two adjacent vertices are assigned the same color. The minimum number of colors required for vertex coloring is called the chromatic number, denoted by $\chi(G)$. From Brook's theorem, it is clear that $\chi(G) \leq \Delta(G)$

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except for odd cycle and complete graph for which it is $\Delta(G) + 1$. Similarly, a proper edge coloring is the assignment of colors to the edges such that no two adjacent edges receive the same color. The minimum number of colors required for edge coloring of G is called the chromatic index of the graph and it is denoted by $\chi'(G)$. Vizing proved that for any graph G, $\chi'(G)$ is either $\Delta(G)$ or $\Delta(G) + 1$. The graphs which require $\Delta(G)$ colors for its edge coloring are called class I graphs and the graphs which require $\Delta(G)+1$ colors for its edge coloring are called class II graphs.

A *total coloring* of G is a mapping $f : V(G) \cup E(G) \to C$, where C is the set of colors and f satisfies :

(a) $f(u) \neq f(v)$ for any two adjacent vertices $u, v \in V(G)$
(b) $f(e) \neq f(e')$ for any two adjacent edges $e, e' \in E(G)$ and
(c) $f(v) \neq f(e)$ for any vertex $v \in V(G)$ and any edge $e \in E(G)$ incident to v.

The *total chromatic number* of a graph G, denoted by $\chi''(G)$, is the minimum number of colors that are used in a total coloring. It is clear that $\chi''(G) \geq \Delta(G) + 1$. Behzad [1] and Vizing [12] independently conjectured (Total Coloring Conjecture (TCC)) that for every graph G, $\chi''(G) \leq \Delta(G) + 2$. The graphs that can be totally colored with $\Delta(G)+1$ colors are said to be type I graphs and the graphs with total chromatic number $\Delta(G)+2$ is said to be type II. The total coloring conjecture is a long-standing conjecture and has defined several attempts in a complete proof. It is also proved that the decidability algorithm for total coloring is NP-complete even for cubic bipartite graphs [6, 9]. But still, a lot of progress has been made in attempting TCC. It is easily seen that TCC is true for complete graphs, bipartite, complete multipartite graphs. The total coloring conjecture has also been confirmed for several other classes of graphs. Good surveys of techniques and other results on total coloring can be found in Yap [13], Borodin [2] and Geetha et al. [3].

2. Lexicographic Product

Let G and H be two graphs. The *lexicographic product* [4, 5] of graphs G and H is the graph $G \circ H$ whose vertex set is $V(G) \times V(H)$ and for which $((g, h), (g', h'))$ is an edge of $G \circ H$ precisely if $(g, g') \in E(G)$, or $g = g'$ and $(h, h') \in E(H)$. The lexicographic product is also known as graph *substitution*, a name that bears witness to the fact that $G \circ H$ can be obtained from G by substituting a copy H_g of H for every vertex g of G and then joining all vertices of H_g with all vertices of $H_{g'}$ if $(g, g') \in E(G)$. The lexicographic product is associative but not commutative. The total coloring of some classes of lexicographic product graph were discussed in [4, 8, 10]. For example it is easy to see that $K_m \circ K_n \cong K_{mn}$ is type I if m and n are odd otherwise type II.

Theorem 1. *Let G be any type I graph. If n is odd then $G \circ K_n$ is type I otherwise $G \circ K_n$ satisfies TCC.*

Proof. Let G be any type I graph with m vertices. In $G \circ K_n$, each vertex of G is replaced by a copy of K_n. The maximum degree of $G \circ K_n$ is $\Delta(G \circ K_n) = (n-1) + n\Delta(G)$. Let us consider the color classes $C_1 = \{a_1^1, a_1^2, \ldots, a_1^n\}$, $C_2 = \{a_2^1, a_2^2, \ldots, a_2^n\}$, $\ldots$, $C_{(\Delta(G)+1)} = \{a^1_{(\Delta(G)+1)}, a^2_{(\Delta(G)+1)}, \ldots, a^n_{(\Delta(G)+1)}\}$. We consider two cases.

Case 1. n is odd.

In this case, we color the elements of $G \circ K_n$ with $\Delta(G \circ K_n) + 1 = n(\Delta(G) + 1)$ colors. Let $C_1, C_2, \ldots, C_{(\Delta(G)+1)}$ be the total color classes of G. Since n is odd, we need n colors to color the elements of K_n and n colors to color the join edges between any two copies of K_n. Since G is type I we require $(\Delta(G) + 1)$ colors to color the elements of G. Assign the $\Delta(G) + 1$ set of n colors corresponding to $\Delta(G) + 1$ color classes of G. Hence $G \circ K_n$ is type I.

Case 2. n is even.

In this case, we color the elements of $G \circ K_n$ with $\Delta(G \circ K_n) + 2 = n(\Delta(G) + 1) + 1$ colors.

Here G is type I and n is even. We need $\Delta(G) + 1$ set of n or $n+1$ colors to color the elements of $G \circ K_n$. Let $C_1, C_2, \ldots, C_{(\Delta(G)+1)}$ be the total color classes of G. Assign the $\Delta(G) + 1$ set of n colors corresponding to $\Delta(G) + 1$ color classes of G. So there will be a set of n colors available at each copy of K_n. Since n is even, we need $n + 1$ colors to color all elements of K_n. Take the n colors available at each copy along with a new color to assign total coloring of K_n. In this coloring assignment, use the new color only to the edges at each copy of K_n. □

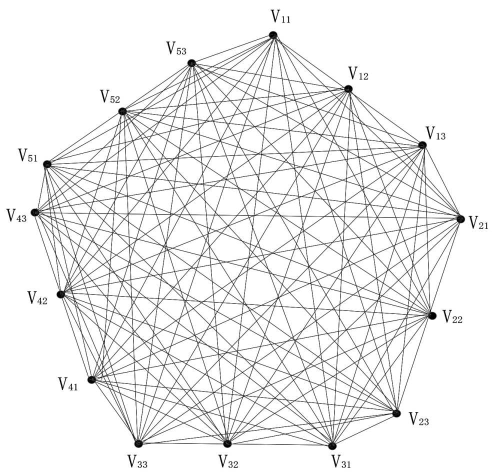

Figure 1. $K_5 \circ P_3$

The lexicographic product is not commutative and hence $G \circ K_n \ncong K_n \circ G$. In the next theorem, we prove that $K_n \circ G$ is total colorable.

Lemma 1. *[13] Let G be a graph with n vertices. If $\Delta(G) \geq \frac{3}{4}n$ then G is total colorable with $\Delta(G)+2$ colors.*

Theorem 2. *For any total colorable graph G, $K_n \circ G$ is total colorable.*

Proof. In $K_n \circ G$, each vertex of K_n is replaced by a copy of G. So each vertex in $K_n \circ G$ is adjacent to all the vertices in all the copies of G and to the vertices in the same copy as in G. So the maximum degree of $\Delta(K_n \circ G), \Delta(G) + m(n-1) \geq \frac{3}{4}mn$. From Lemma 1, $K_n \circ G$ is total colorable with $\Delta(K_n \circ G) + 2$ colors. □

Geetha and Somasundaram [4] proved that if G is a bipartite graph and H is any graph with $\chi''(K_2 \circ H) \leq \Delta(K_2 \circ H) + 2$ then $G \circ H$ satisfies TCC. Vignesh et al. [11] proved that if G is a bipartite graph and H is any total colorable graph then $G \circ H$ is total colorable. Sandhiya et al. [10] proved that if G is type I then $P_m \circ G, m \geq 3$ is type I. In the next theorem, we have generalised these results.

Lemma 2. *[13] For any integer $n \geq 3$ there exists an n edge coloring of $K_{n,n}$ such that $K_{n,n}$ has a perfect matching receiving n distinct colors.*

Theorem 3. *For any bipartite graph G and any total colorable graph H,*

$$\chi''(G \circ H) = \begin{cases} \Delta(G \circ H) + 1, & \text{if } G \text{ is unbalanced and } H \text{ is type I} \\ \leq \Delta(G \circ H) + 2, & \text{otherwise.} \end{cases}$$

Proof. Let G be a bipartite graph with partition $\{x_1, x_2, \ldots, x_m\}$ and $\{y_1, y_2, \ldots, y_n\}$ where $m \geq n$. Let H be any total colorable graph. If H is complete then by Theorem 1, it is easy to see the results. Here we assume H is not complete. In $G \circ H$, there are $m + n$ copies of H. Let us denote these copies by $H'_1, H'_2, H'_3, \ldots, H'_m, H''_1, H''_2, H''_3, \ldots, H''_n$.
The maximum degree of $G \circ H$ is $\Delta(H) + k\Delta(G)$, where k is the order of the graph H.

Case 1. Suppose G is unbalanced and H is type I.
We distinguish two situations.

Subcase 1.1. $\Delta(G) = m$.
We divide the $\Delta(H) + k\Delta(G) + 1$ colors into $\Delta(G \circ H) + 1$ color sets $C_1 = \{a_1^1, a_1^2, \ldots, a_1^{\Delta(H)+1}\}$, $C_2 = \{a_2^1, a_2^2, \ldots, a_2^k\}$, $C_3 = \{a_3^1, a_3^2, \ldots, a_3^k\}$, $\ldots$, $C_{\Delta(G)+1} = \{a_{\Delta(G)+1}^1, a_{\Delta(G)+1}^2, \ldots, a_{\Delta(G)+1}^k\}$. Since H is type I, we take $\Delta(H)+1$ colors from C_1 to color all the elements of $H''_1, H''_2, H''_3, \ldots, H''_n$. Take the $\Delta(H) + 1$ edge coloring from the total coloring of $H''_i, 1 \leq i \leq n$, and assign the colors to the edges of H'_i in the same way as in $H''_i, 1 \leq i \leq n$. G is bipartite so it is class I. Hence $\Delta(G)$ sets of k

colors are sufficient to color the join edges between $H_i', 1 \leq i \leq m$, and $H_j^{''}, 1 \leq j \leq n$. Assign the colors from $C_2, \ldots, C_{\Delta(G)+1}$ to the join edges. Since $m \geq n$, there will be at least one set of k colors missing at each copy of $H_1', H_2', H_3', \ldots, H_m'$. Assign the set of missing color corresponding to the vertices in each set of $H_1', H_2', H_3', \ldots, H_m'$. Thus we have totally used $\Delta(H) + k\Delta(G) + 1$ colors and hence $G \circ H$ is type I.

Subcase 1.2. $\Delta(G) < m$.
In this case, we further divide each color class C_i into two sets X_i and Y_i, where $X_i = \{a_i^1, a_i^2, \ldots, a_i^{\Delta(H)+1}\}$ and $Y_i = \{a_i^{\Delta(H)+2}, a_i^{\Delta(G)+3}, \ldots, a_i^k\}$, where $2 \leq i \leq \Delta(G) + 1$. Let us assume that the maximum degree attains in the copy $H_1^{'}$. First we give the total coloring for $H_1^{'}$ using the colours from C_1. Choose the colors from $X_i \cup Y_j$ to color the joins edges between $H_1^{'}$ and copies in $H^{''}$. Choose the colors from X_i to color the copies in $H^{''}$ adjacent with $H_1^{'}$. Choose the next copy with next highest degree. If the chosen copy is adjacent with $H_1^{'}$, already one set of join edges will be colored. Choose the colors accordingly to color the other join edges of the chosen copy. Repeat the same process until all the elements of $G \circ H$ are assigned colors. If there is any repetition in colors, swap the $\Delta(H) + 1$ colors between the join edges and one of its copies.

Case 2. G is unbalanced and H is type II.
We distinguish two situations.

Subcase 2.1. $\Delta(G) = m$.
Let $C_1^{'} = \{a^1, a^2, \ldots, a^{\Delta(H)+2}\}$. In this case, we consider the $\Delta(H) + km + 2$ colors from the $\Delta(G) + 1$ sets $C_1^{'}, C_2, C_3, \ldots, C_{\Delta(G)+1}$. Take the colors from $C_1^{'}$ and assign to the elements of $H_1^{''}, H_2^{''}, H_3^{''}, \ldots, H_n^{''}$. Like in the previous case, take the $\Delta(H) + 1$ edge coloring from the total coloring of $H_i^{''}$ and assign the colors to the edges in H_i' in the same way as in $H_i^{''}, 1 \leq i \leq n$. Since G is bipartite, it is class I. Hence $\Delta(G)$ sets of k colors are sufficient to color the join edges between $H_i', 1 \leq i \leq m$, and $H_j^{''}, 1 \leq j \leq n$. There will be at least one set of k colors missing at each copy of $H_i^{'}$ and assign these missing colors to its vertices in $H_i^{'}$.

Subcase 2.2. $\Delta(G) < m$.
Let $H^{''}$ be the partite set containing the maximum degree vertex. Similar to the previous case, Take the colors from $C_1^{'}$ and assign to the elements of $H_1^{''}, H_2^{''}, H_3^{''}, \ldots, H_n^{''}$. Take the $\Delta(H) + 1$ edge coloring from the total coloring of $H_i^{''}$ and assign the colors to the edges in H_i' in the same way as in $H_i^{''}, 1 \leq i \leq n$. $\Delta(G)$ sets of k colors are sufficient to color the join edges between $H_i', 1 \leq i \leq m$, and $H_j^{''}, 1 \leq j \leq n$. Using Lemma 2, Color the join edges between the copies such that the edges joining corresponding vertices are assigned n different colors. Now we need to color the vertices in all copies of $H_i^{'}$. Since H is type II, there will be a missing color at each vertex of $H_1^{''}$ (say). Recolor this missing color to the perfect matching of any one set of join edges and shift the colors in perfect matching to corresponding vertices of $H_i^{'}$ adjacent with $H_1^{''}$. Repeat this process to color the vertices in each copy of $H^{'}$.

Case 3. $G \cong K_{m,m}$
In $K_{m,m} \circ H$ there will be $2m$ copies of H. Let $H_1', H_2', H_3', \ldots, H_m'$ and $H_1^{''}, H_2^{''}, H_3^{''}, \ldots, H_m^{''}$ denote the copies of H in the partite sets. Let us take a

set of $\Delta(H)+km+2$ colors as $C_0=\{a_0^1,a_0^2,\ldots,a_0^{\Delta(H)+2}\}$, $C_1=\{a_1^1,a_1^2,\ldots,a_1^k\}$, $C_2=\{a_2^1,a_2^2,\ldots,a_2^k\}$,..., $C_m=\{a_m^1,a_m^2,\ldots,a_m^k\}$.

Color all the elements of $H_1',H_2',H_3',\ldots,H_m'$ with colors from C_0. Take the edge coloring from the total coloring of $\{H_i'\}$ and assign to the edges of $\{H_i''\}$ in the same way as in $\{H_i'\}$, $1\le i\le m$. We know that the bipartite graph is a class I graph and we assign the colors from $C_1, 1\le i\le m$ to color the join edges between the copies. Take the colors from C_1 and assign to the join edges between H_1' and H_1''. Using Lemma 2, assign the colors such that the edge joining the corresponding vertices of H_1' and H_1'' are assigned different colors.Since we used $\Delta(H)+2$ colors to color the elements of H_1', there will be one missing color at each vertex of H_1'. Now assign the color of the edges in the perfect matching to the corresponding vertices in H_1''. Recolor the edges of perfect matching by the missing color at the corresponding vertices of H_1'. Take the colors from C_i and color the join edges between H_i' and H_i'' for each i, $1\le i\le m$ and repeat the same process as the above. Assign the colors to the join edges between H_i' and H_j'', $i\ne j, 1\le i,j\le m$ with the remaining $m-1$ sets of k colors such that each of the vertex in the copies have all km colors. Hence $K_{m,m}\circ H$ is total colorable. $\square$

It is easy to prove that $P_2\circ P_3$ is type II. The above theorem can be modified further for type I graphs. For example, it is proved in [4] that if G is a bipartite graph then $G\circ P_3$ is a type I graph.

3. Deleted Lexicographic Product

The deleted lexicographic product [7] of two graphs G and H, denoted by $D_{lex}(G,H)$, is a graph with the vertex set $V(G)\times V(H)$ and the edge set $\{((g,h),(g',h')) : (g,g')\in E(G)$ and $h\ne h'$, or $(h,h')\in E(H)$ and $g=g'$. Similar to lexicographic product, $D_{lex}(G,H)$ and $D_{lex}(H,G)$ are not necessarily isomorphic. Note that $D_{lex}(G,H) = G\circ H\setminus kG$, where kG denotes the graph consisting of k vertex disjoint copies of G and $G\circ H\setminus kG$ denotes the deletion of kG from $G\circ H$. Vignesh et. al [11] discussed the TCC for certain classes of deleted lexicographic product graphs. They proved that if G and H are class I graphs then $D_{lex}(G,H)$ is type I. Also for any graph H, $D_{lex}(P_m,H), m\ge 3$, is type I.
Sandhiya et al. [10] generalized the above result to class I graph G with any graph H. The following theorems are due to Sandhiya et al. [10].

Theorem 4. *[10] For any class-I graph G and a graph H, $D_{lex}(G,H)$ is type I.*

Theorem 5. *[10] $D_{lex}(C_m,K_n)$ is type I, where n is odd.*

We extended the Theorem 5 to any graph G with complete graph K_n.

Theorem 6. *For any graph G,*

$$D_{lex}(G, K_n) = \begin{cases} \Delta(D_{lex}(G, K_n) + 1, & \text{when } n \text{ is odd} \\ \leq \Delta(D_{lex}(G, K_n) + 2, & \text{when } n \text{ is even.} \end{cases}$$

Proof. Let G be a graph with m vertices. In $D_{lex}(G, K_n)$, there will be m copies of K_n. The maximum degree $\Delta(D_{lex}(G, K_n)) = (n-1)(\Delta(G) + 1)$. We consider two cases.

Case 1. n is odd.
In this case, we choose a set of $(n-1)(\Delta(G) + 1) + 1$ colors. The edges of G are partitioned into $\Delta(G) + 1$ independent sets. Correspondingly the join edges between the copies of K_n are partitioned into $\Delta(G) + 1$ independent sets namely $S_1, S_2, \ldots, S_{\Delta(G)+1}$. We take $(n-1)(\Delta(G) + 1)$ colors to color the join edges in each partition $S_1, S_2, \ldots, S_{\Delta(G)+1}$. Since we partition the join edges into $\Delta(G) + 1$ sets, there will be a set of $n-1$ missing colors at each copy of K_n. Also we have one unused color c (say). Here n is odd, we need n colors to color the elements of K_n. Using the set of $n-1$ missing colors at each copy and the color c, we color the elements in each copy of K_n. In this coloring assignment, we make sure that the corresponding vertex in all the copies of K_n receive the color c.

Case 2. n is even.
Similar to the previous case, the edges of G are partitioned into $\Delta(G)+1$ independent sets. Correspondingly the join edges between the copies of K_n are partitioned into $\Delta(G) + 1$ independent sets namely $S_1, S_2, \ldots, S_{\Delta(G)+1}$. We take $(n-1)(\Delta(G) + 1)$ colors to color the join edges in each partition $S_1, S_2, \ldots, S_{\Delta(G)+1}$. Since we partition the join edges into $\Delta(G)+1$ sets, there will be a set of $n-1$ missing colors at each copy of K_n. Take two new colors c_1 and c_2. Here n is even, we need $n + 1$ colors to color the elements of K_n. Using the set of $n-1$ missing colors at each copy and the colors c_1 and c_2, we color the elements in each copy of K_n. In this coloring assignment, we make sure that the corresponding vertex in all the copies of K_n receive the either of the colors c_1 or c_2. □

Conflict of interest. The authors declare that they have no conflict of interest.

Data Availability. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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