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Short Note

Triangular tile latching system

K. Kanchan Devi[†] **and S. Arumugam**[*]

National Centre for Advanced Research in Discrete Mathematics (n-CARDMATH),
Kalasalingam Academy of Research and Education, Krishnankoil-626126, India
[†]kanchandevi.klu@gmail.com
[*]s.arumugam.klu@gmail.com

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Abstract: A triangular tile latching system consists of a set Σ of equilateral triangular tiles with at least one latchable side and an attachment rule which permits two tiles to get latched along a latchable side. In this paper we determine the language generated by a triangular tile latching system in terms of planar graphs.

Keywords: Attachment rule, latching system, planar graph

AMS Subject classification: 05C10, 68Q45

1. Introduction

Tiling or tessellation is a branch of combinatorial geometry which is concerned with the ways shapes fit together to fill the Euclidean plane without gaps or overlaps. Various methods have been developed to generate intricate tiling patterns. Puzzle grammar has been proposed by Nivat et al. [7] and has been subsequently investigated in [12].
Pasting system, a two-dimensional model that generates tiling patterns by pasting square tiles at the edges, was introduced and investigated by Robinson in [8] and [10]. Iso-triangular picture languages have been studied in [1, 3, 6] and [2]. Motivated by this concept, triangular pasting system, that pastes edge-to-edge isosceles right triangles was introduced in [6]. The extended pasting scheme introduced in [9] generated certain kolam patterns. Generation of tiling patterns and tessellation using tile pasting system has also been discussed in [11].

* *Corresponding Author*

In this paper we present a variant of triangular tile pasting system [6], namely triangular tile latching system. In this system equilateral triangular tiles are latched edge to edge to generate polygonal patterns. The grammar rule used in this system is motivated by [4], where the graph grammar rules were used to model and direct self-organization of programmable parts (robots). The study in this paper pertains to the formation of two-dimensional patterns only. Though the concepts of pasting and latching are mathematically equivalent, in [4] latching is used to indicate that two programmable parts are physically attached to each other.
In Section 2 we define latching system and illustrate the same with examples. In section 3 we present theorems that give a complete description of the language generated by a latching system, which depends on the number of latchable tiles. For basic terminology in graph theory we refer to Chartrand, Lesniak and Zhang [5].
By a graph $G = (V, E)$ we mean, a finite undirected graph with neither loops nor multiple edges. The order $|V|$ and the size $|E|$ are denoted by n and m respectively.

Definition 1. A graph $G = (V, E)$ is called a planar graph if it can be drawn in a plane in such a way that no two edges in G intersect in a point other than a vertex of G, where each edge of G is a simple arc or a Jordan arc. Such a drawing of G is called a plane graph. A plane graph divides the rest of the plane into regions, which are called the faces of G. Exactly one face of G is unbounded and is called the exterior face of G. The remaining faces are called interior faces. The number of faces in a plane graph is denoted by r.

The following theorem is known as Euler's polyhedron formula.

Theorem 1. [5] *Let G be a connected plane graph with n vertices, m edges and r faces. Then $n - m + r = 2$.*

Definition 2. [5] Let G be a plane graph. Its geometric dual G^* is constructed by placing one vertex in each region of G and if two regions have a common edge x, then the corresponding vertices are joined by an edge x^* such that x^* crosses only x.

2. Latching system

A tile is a topological disk whose boundary is a simple closed polygon. In this paper we consider equilateral triangular tiles and each side length is taken as one unit.

Let $A = \{0, 1\}$ be a binary set of symbols, where 1 represents the possibility of latching and 0 otherwise. Let T be a triangular tile whose sides are labelled from A.

Definition 3. A labelled tile T is said to be i-side latchable if i sides are labelled with 1, where $1 \leq i \leq 3$.

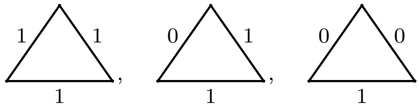


Figure 1. **3-sides latchable, 2-sides latchable and 1-side latchable tiles**

3-side latchable, 2-side latchable and 1-side latchable tiles are given in Figure 1. Let Σ be a finite set of latchable tiles. In what follows we omit the label 0 so that a side with no label represents an unlatchable side.

Definition 4. The graph $G = K_4 - e$ is called a dimer. An attachment rule P is a rule which attaches the edges with label 1 of tiles T_1 and T_2 to form a dimer in which the latched edge is unlabeled. This implies that no further latching is possible with that side.

The attachment rule P is given in Figure 2.

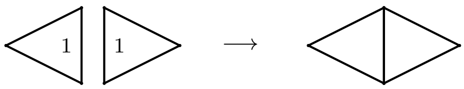
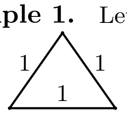


Figure 2. **Attachment rule giving a dimer**

Definition 5. A latching system is a triple S= (Σ, P, t_0), where Σ is a set of i-side latchable triangular tiles where $1 \leq i \leq 3$, the tile t_0 in Σ is the starting symbol and P is the attachment rule given in Definition 4.

Given a set of n triangular latchable tiles, the latching system S terminates when no further latching is possible. For a latching system S, starting from t_0 and applying the rules P parallely, several tiling patterns are generated. The set of all tiling patterns derived by S is denoted by $L(S)$ and is called the language set of the latching system S. The attachment process terminates if there is no possibility of any further latching.

Example 1. Let S_1=(Σ_1, P, t_0) where Σ_1 is a set of 3-side latchable tiles,

$t_0 = \Big\{$ 

$\Big\}$ and P is the attachment rule given in Definition 4.

The first few members of $L(S_1)$ are given in Figure 3.

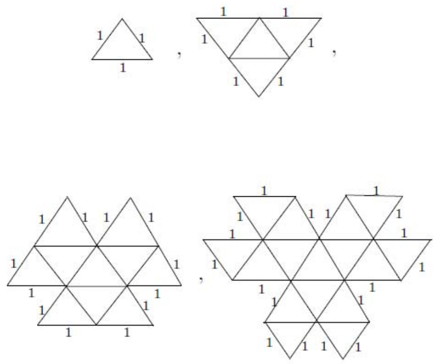


Figure 3. **Members of** $L(S_1)$

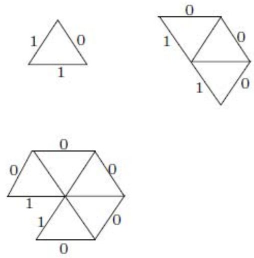
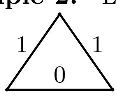


Figure 4. **Members of** $L(S_2)$

Example 2. Let S_2=(Σ_2, P, t_0) where Σ_2 is a set of 2-side lachable tiles, t_0= $\left\{\right.$ 

$\left.\right\}$ and P is the attachment rule in Definition 4. A few members of $L(S_2)$ are given in Figure 4.

3. Theorems on Latchability

A tiling pattern generated by a latching system S is a plane graph G where each tile is a face of G. In this section we give a complete description of $L(S)$ in terms of plane graphs.

Theorem 2. *Let $S = (\Sigma, P, t_0)$ be a latching system where Σ is a set of 1-side latchable tiles and $|\Sigma|$=k. Then $G \in L(S)$ if and only if G is a plane graph with $\lceil k/2 \rceil$ components, at most one component is a triangle and each of the remaining components is a dimer.*

Proof. Let T_1, $T_2 \in \Sigma$. Since T_1 and T_2 are one-side latchable tiles, exactly one side of T_1 and exactly one side of T_2 are labelled edges. When these labelled sides are latched, we obtain a dimer D such that no side of D is labelled. Hence no further latching with D is possible. Hence D is a component of G. If k is even, by applying $k/2$ times the rule P, we get a plane graph G with $k/2$ components, each isomorphic to a dimer. Similarly, if k is odd, we obtain a plane graph G with $(k-1)/2$ components each isomorphic to a dimer and one component a triangle. □

We now proceed to determine $L(S)$ where Σ is a set of 2-side latchable tiles. For this purpose we introduce a notation and a definition.
Let G be a planar graph and let G^* be the geometric dual of G. Let $H^* = G^* - v_0$, where v_0 is the vertex in the exterior face of G.

Definition 6. A polygonal tiling pattern G is called a linear polygonal tiling pattern if H^* is a path.

Example 3. The polygonal tiling patterns given in Figure 5 along with the path H^* are linear polygonal tiling patterns.

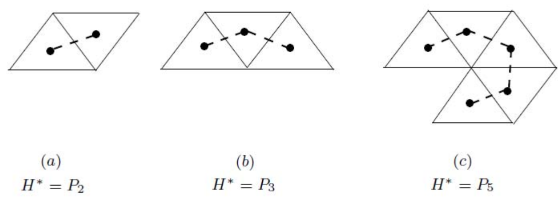


Figure 5. **Linear Polygonal Tiling Pattern**

Example 4. The polygonal tiling patterns given in Figure 6 is a nonlinear polygonal tiling patterns. The graph H^* has a vertex of degree 3 and it is not a path.

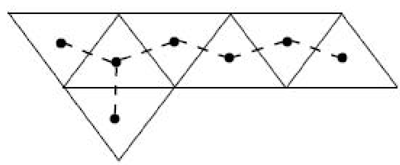


Figure 6. **Nonlinear Polygonal Tiling Pattern**

Theorem 3. *Let $S = (\Sigma, P, t_0)$ be a latching system where Σ is a set of 2-side latchable tiles and $|\Sigma|{=}k$. Then $G \in L(S)$ if and only if G is a linear polygonal tiling pattern.*

Proof. The proof is by induction on k. When $k = 2$, G is a dimer with $H^* = P_2$ and hence the theorem is true. Suppose the theorem is true if $|\Sigma|=r-1$ where $r \geq 3$. Let $G_{r-1} \in L(S)$. Then G_{r-1} is a linear polygonal tiling pattern with

$$H^* = P_{r-1} = (v_1, v_2, \ldots, v_{r-1}).$$

Let T_i be the tile corresponding to $v_i, 1 \leq i \leq r-1$. Let $2 \leq i \leq r-2$. Since degree of v_i in H^* is 2, the tile T_i has been latched along two of its sides with tiles T_{i-1} and T_{i+1}. Since T_i is a 2-side latchable tile, no further latching is possible with T_i. Also the tiles T_1 and T_{r-1} has one latchable side. Hence the r^{th} tile T_r can be latched only with T_1 or T_{r-1}. Hence the tiling pattern G_r obtained from G_{r-1} by latching T_r is a linear polygonal tiling pattern where H^* is a path on r vertices. Thus the theorem is true for r and the proof is complete by induction. □

Theorem 4. *Let $S = (\Sigma, P, t_0)$ be a latching system where Σ is a set of 3-side latchable tiles and $|\Sigma|=k$. Then $G \in L(S)$ if and only if G is a connected plane graph with $k+2$ vertices, $2k+1$ edges and $k+1$ faces and all faces except the exterior face are triangles.*

Proof. The proof is by induction on k. If $k = 1$, then $n = 3$, $m = 3$, $r = 2$ and hence the result is true. Assume that $k \geq 2$ and the theorem is true for $k-1$. Let G be any plane graph generated by $k-1$ triangular tiles. Hence by induction $n = k+1$, $m = 2k-1$ and $r = k$. Now when the k^{th} tile latches with one triangle in G, then the number of vertices increases by 1 and the number of edges increases by 2. Hence it follows from Theorem 1 that the number of faces increases by 1. Hence it follows that

$$n = k+2, m = 2k+1 \text{ and } r = k+1$$

Therefore the result follows by induction. □

A tiling pattern with exactly one hole is called an annular tiling pattern.

Corollary 1. *If $S = (\Sigma, P, t_0)$ is a latching system where Σ is a set of 3-side latchable tiles, then an annular tiling pattern cannot be generated by S.*

4. Conclusion

In this paper we have introduced a tile latching system for generating tiling patterns using equilateral triangular tiles. This work can be further extended by using tiles in the shape of a regular pentagon or a hexagon. Another approach is to consider a system which allows tiles of different shapes so as to generate semi-regular tessellations. It follows from Theorem 2, Theorem 3 and Theorem 4 that $G \notin L(S)$ where G is a connected plane graph with an annular region. All the tiling patterns considered in this paper are two dimensional patterns. Formulating a latching system for the construction of three dimensional tiling patterns is another significant problem. Results in this direction will be reported in a subsequent paper.

The following problems arise naturally.

Problem 1. If $S = (\Sigma, P, t_0)$ where Σ is a union of three families of equilateral triangular tiles consisting of i-side latchable tiles where $1 \leq i \leq 3$, then determine $L(S)$.

Problem 2. Determine a latching system S such that $L(S)$ is precisely a set of connected annular tiling patterns with exactly one annular region.

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Conflict of interest. The author declares that they have no conflict of interest.

Data Availability. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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