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Short Note

Multiplicative Zagreb indices of trees with given domination number

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Abstract: In [On extremal multiplicative Zagreb indices of trees with given domination number, Applied Mathematics and Computation 332 (2018), 338–350] Wang et al. presented bounds on the multiplicative Zagreb indices of trees with given domination number. We fill in the gaps in their proofs of Theorems 3.1 and 3.3 and we correct Theorem 3.3.

Keywords: Extremal tree, bound, correction

AMS Subject classification: 05C09, 05C05, 05C07

1. Introduction

Multiplicative Zagreb indices have many applications and they have been studied especially in the past decade. Bounds on the multiplicative Zagreb indices of trees with prescribed order were presented in [8] and [10], chemical graphs were studied in [1] and [4], and graphs of given order and size in [3], graph operations in [7]. Bounds on the classical Zagreb indices for trees with prescribed domination number were studied in [2]. Zagreb indices were studied also in [5] and [6].
We denote the vertex set of a graph G by $V(G)$. The degree $d_G(v)$ of $v \in V(G)$ is the number of edges incident with v. A tree is a connected graph containing no cycles.

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A dominating set D is a subset of $V(G)$ where each vertex not in D is a neighbour of a vertex in D. The domination number $\gamma(G)$ of G is the cardinality of a minimum dominating set.
The first multiplicative Zagreb index of G is defined as

$$\prod_1(G) = \prod_{v\in V(G)} (d_G(v))^2$$

and the second multiplicative Zagreb index is

$$\prod_2(G) = \prod_{v\in V(G)} (d_G(v))^{d_G(v)}.$$

Wang, Wang and Liu [9] presented bounds on $\Pi_1(G)$ and $\Pi_2(G)$ for trees G having domination number γ and n vertices in their Section 3. In Case 1 (proofs of Theorems 3.1 and 3.3),

$$\frac{f\left(\gamma\left\lfloor\frac{n-1}{\gamma}\right\rfloor+\gamma-n+1\right)}{f(0)} \quad \text{and} \quad \frac{g\left(\gamma\left\lfloor\frac{n-1}{\gamma}\right\rfloor+\gamma-n+1\right)}{g(0)}$$

for the functions f and g (defined in the next sections) were studied. In Case 2,

$$\frac{f\left(\gamma\left\lfloor\frac{n-1}{\gamma}\right\rfloor-n+1\right)}{f(-\gamma+1)} \quad \text{and} \quad \frac{g\left(\gamma\left\lfloor\frac{n-1}{\gamma}\right\rfloor-n+1\right)}{g(-\gamma+1)}$$

were investigated. We correct these values. Those terms play very important roles in the proofs. Then it is necessary to compare $f(-\gamma+1)$ and $f(0)$. Similarly, we compare $g(-\gamma+1)$ and $g(0)$. Note that we cannot assume that f and g are monotonous on the interval $[-\gamma+1, \gamma-1]$.
We also correct Theorem 3.3 of [9] and show that

$$\Pi_2(G) \;\geq\; 2^{4(\gamma-1)}\left(\left\lfloor\frac{n}{\gamma}\right\rfloor-1\right)^{\left(\left\lfloor\frac{n}{\gamma}\right\rfloor-1\right)\left(\gamma-n+\gamma\left\lfloor\frac{n}{\gamma}\right\rfloor\right)}\left\lfloor\frac{n}{\gamma}\right\rfloor^{\left\lfloor\frac{n}{\gamma}\right\rfloor\left(n-\gamma\left\lfloor\frac{n}{\gamma}\right\rfloor\right)}.$$

2. First multiplicative Zagreb index

Theorem 3.1 presented in [9] says that for $1 \leq \gamma \leq \frac{n}{3}$, if G is a tree having domination number γ and n vertices, then

$$\Pi_1(G) \leq 4^{2\gamma-2}\left\lfloor\frac{n-\gamma}{\gamma}\right\rfloor^{2\left(2\gamma-n+\gamma\left\lfloor\frac{n-\gamma}{\gamma}\right\rfloor\right)}\left(\left\lfloor\frac{n-\gamma}{\gamma}\right\rfloor+1\right)^{2\left(n-\gamma-\gamma\left\lfloor\frac{n-\gamma}{\gamma}\right\rfloor\right)}$$

with the equality if and only if G is $D_{n,\gamma}$ (which is a set of trees given in their paper).

Let us study the proof of Theorem 3.1. The smallest dominating set in a tree G having domination number γ and n vertices is denoted by D. We define $\overline{D} = V(G) \setminus D$. We use l for the number of edges uv where $u \in D$ and $v \in \overline{D}$, k is the number of edges between vertices in D and p is the number of edges between vertices in $\overline{D}$. Every tree has $n-1$ edges, so we obtain $k+l+p = n-1$. Since $|\overline{D}| = n-\gamma$ and every vertex of $\overline{D}$ is joined to at least one vertex of D, we get $l \geq n-\gamma$. Since $k+l+p = n-1$, we have $k+p \leq \gamma - 1$ and consequently,

$$-\gamma + 1 \leq k - p \leq \gamma - 1.$$

In [9] it was shown that

$$\begin{aligned}\Pi_1(G) \leq f(k-p) &= (q+1)^{2(n-1+(k-p)-\gamma q)} q^{2(1-n+\gamma-(k-p)+\gamma q)} 2^{2(\gamma-1+p-k)} \\ &= (q+1)^{2(k-p)} q^{-2(k-p)} 2^{-2(k-p)} \\ &\quad (q+1)^{2(n-1-\gamma q)} q^{2(1-n+\gamma+\gamma q)} 2^{2(\gamma-1)},\end{aligned}$$

where $q = \left\lfloor \frac{n-1+k-p}{\gamma} \right\rfloor \geq 2$.
We need to obtain the largest value of $f(k-p)$. One cannot assume that

$$f(k-p) = \left(\frac{1/2}{q/(q+1)}\right)^{2(k-p)} (q+1)^{2(n-1-\gamma q)} q^{2(1-n+\gamma+\gamma q)} 2^{2(\gamma-1)}$$

is a decreasing function for $-\gamma+1 \leq k-p \leq \gamma-1$, because $f(k-p)$ contains $(q+1)^{2(n-1-\gamma q)} q^{2(1-n+\gamma+\gamma q)}$ and we have $q = \left\lfloor \frac{n-1+k-p}{\gamma} \right\rfloor$. So q contains $k-p$. Since $\gamma \leq \frac{n}{3}$ and $n \geq 3$,

$$c = \left\lfloor \frac{n-1}{\gamma} \right\rfloor \geq \left\lfloor \frac{n-1}{\frac{n}{3}} \right\rfloor = \left\lfloor 3 - \frac{3}{n} \right\rfloor = 2.$$

We often use c instead of $\left\lfloor \frac{n-1}{\gamma} \right\rfloor$ in computations below. Let us consider two cases.

Case 1. $0 \leq k-p \leq \gamma - 1$.
If $k-p = \gamma \left\lfloor \frac{n-1}{\gamma} \right\rfloor + \gamma - n + 1$, then $\frac{n-1+k-p}{\gamma} = \left\lfloor \frac{n-1}{\gamma} \right\rfloor + 1$ which gives

$$q = \left\lfloor \frac{n-1+k-p}{\gamma} \right\rfloor = \left\lfloor \frac{n-1}{\gamma} \right\rfloor \quad \text{for} \quad 0 \leq k-p < \gamma \left\lfloor \frac{n-1}{\gamma} \right\rfloor + \gamma - n + 1 \tag{1}$$

and

$$q = \left\lfloor \frac{n-1+k-p}{\gamma} \right\rfloor = \left\lfloor \frac{n-1}{\gamma} \right\rfloor + 1$$

for

$$\gamma \left\lfloor \frac{n-1}{\gamma} \right\rfloor + \gamma - n + 1 \leq k-p \leq \gamma - 1. \tag{2}$$

Since $0 < \frac{q+1}{2q} = \frac{1/2}{q/(q+1)} < 1$, we have $\left(\frac{1/2}{q/(q+1)}\right)^{2t_1} > \left(\frac{1/2}{q/(q+1)}\right)^{2t_2}$, where $t_1 < t_2$. So, we have decreasing $f(k-p)$ on the interval presented in (1), so $f(k-p)$ is largest for $k-p=0$. Similarly, if $k-p$ belongs to the interval presented in (2), then $f(k-p)$ is largest for $k-p = \gamma\left\lfloor\frac{n-1}{\gamma}\right\rfloor + \gamma - n + 1 = \gamma c + \gamma - n + 1 = z$.
Let us compare $f(z)$ and $f(0)$ to get the largest value of $f(k-p)$ for $0 \le k-p \le \gamma - 1$. We have

$$\begin{aligned} f(z) &= (c+2)^{2z}(c+1)^{-2z}2^{-2z} \\ &\quad (c+2)^{2(n-1-\gamma(c+1))}(c+1)^{2(1-n+\gamma+\gamma(c+1))}2^{2(\gamma-1)} \\ &= (c+2)^0(c+1)^{2\gamma}2^{-2z}2^{2(\gamma-1)} \end{aligned}$$

and

$$f(0) = (c+1)^{2(n-1-\gamma c)}c^{2(1-n+\gamma+\gamma c)}2^{2(\gamma-1)}.$$

Then

$$\frac{f(z)}{f(0)} = \frac{(c+1)^{2\gamma}2^{-2z}2^{2(\gamma-1)}}{(c+1)^{2(n-1-\gamma c)}c^{2(1-n+\gamma+\gamma c)}2^{2(\gamma-1)}} = \frac{(c+1)^{2z}}{c^{2z}2^{2z}} = \left(\frac{c+1}{2c}\right)^{2z} < 1$$

for every $c \ge 2$ since $0 < \frac{c+1}{2c} < 1$ and $2z > 0$. Thus $f(z) < f(0)$.

Case 2. $-\gamma + 1 \le k - p \le 0$.
If $k-p = \gamma\left\lfloor\frac{n-1}{\gamma}\right\rfloor - n + 1$, then $\frac{n-1+k-p}{\gamma} = \left\lfloor\frac{n-1}{\gamma}\right\rfloor$ which gives

$$q = \left\lfloor\frac{n-1+k-p}{\gamma}\right\rfloor = \left\lfloor\frac{n-1}{\gamma}\right\rfloor \quad \text{for} \quad \gamma\left\lfloor\frac{n-1}{\gamma}\right\rfloor - n + 1 \le k-p \le 0 \tag{3}$$

and

$$q = \left\lfloor\frac{n-1+k-p}{\gamma}\right\rfloor = \left\lfloor\frac{n-1}{\gamma}\right\rfloor - 1$$

for

$$-\gamma + 1 \le k-p < \gamma\left\lfloor\frac{n-1}{\gamma}\right\rfloor - n + 1. \tag{4}$$

Again, $f(k-p)$ is decreasing on the interval presented in (3), so $f(k-p)$ is largest for $k-p = \gamma\left\lfloor\frac{n-1}{\gamma}\right\rfloor - n + 1 = \gamma c - n + 1$. Similarly, if $k-p$ belongs to the interval presented in (4), then $f(k-p)$ is largest for $k-p = -\gamma+1$.
Let us compare $f(\gamma c - n + 1)$ and $f(-\gamma+1)$ to obtain the maximum of $f(k-p)$ for $-\gamma + 1 \le k-p \le 0$. Note that if $n = t\gamma$ (n is a multiple of γ), then the interval presented in (4) is an empty set; since $\gamma\left\lfloor\frac{n-1}{\gamma}\right\rfloor - n + 1 = \gamma(t-1) - t\gamma + 1 = -\gamma + 1$. Thus in that case $f(\gamma c - n + 1) = f(-\gamma + 1)$ is the unique maximum. So we can assume that n is not a multiple of γ. We have

$$\begin{aligned} f(\gamma c - n + 1) &= (c+1)^{2(n-1+(\gamma c-n+1)-\gamma c)}c^{2(1-n+\gamma-(\gamma c-n+1)+\gamma c)} \\ &\quad 2^{2(\gamma-1-(\gamma c-n+1))} \\ &= (c+1)^0 c^{2\gamma}2^{2(n-\gamma(c-1)-2)} \end{aligned}$$

and

$$f(-\gamma+1) = c^{2(n-1-\gamma+1-\gamma(c-1))}(c-1)^{2(1-n+\gamma+(\gamma-1)+\gamma(c-1))}2^{2(\gamma-1+\gamma-1)}$$
$$= c^{2(n-\gamma c)}(c-1)^{2(\gamma(c+1)-n)}2^{2(2\gamma-2)}. \quad (5)$$

Then

$$\frac{f(\gamma c-n+1)}{f(-\gamma+1)} = \frac{c^{2\gamma}2^{2(n-\gamma(c-1)-2)}}{c^{2(n-\gamma c)}(c-1)^{2(\gamma(c+1)-n)}2^{2(2\gamma-2)}}$$
$$= \frac{c^{2(\gamma c+\gamma-n)}}{(c-1)^{2(\gamma c+\gamma-n)}2^{2(\gamma c+\gamma-n)}} = \left(\frac{c}{2(c-1)}\right)^{2(\gamma c+\gamma-n)} < 1$$

for each $c \geq 3$ since $0 < \frac{c}{2(c-1)} < 1$ and $\gamma c+\gamma-n = \gamma\left\lfloor\frac{n-1}{\gamma}\right\rfloor+\gamma-n > \gamma\left(\frac{n-\gamma}{\gamma}\right)+\gamma-n = 0$ (note that $\left\lfloor\frac{n-1}{\gamma}\right\rfloor > \frac{n-\gamma}{\gamma}$ if n is not a multiple of γ). Thus $f(\gamma c-n+1) < f(-\gamma+1)$.
If $c = 2$, then the interval presented in (4) is an empty set, therefore the solution is trivial.

So $f(-\gamma+1)$ is the largest value of $f(k-p)$ if $-\gamma+1 \leq k-p \leq 0$, and $f(0)$ is the maximum value if $0 \leq k-p \leq \gamma-1$. It remains to compare $f(-\gamma+1)$ and $f(0)$. We have

$$\frac{f(0)}{f(-\gamma+1)} = \frac{(c+1)^{2(n-1-\gamma c)}c^{2(1-n+\gamma+\gamma c)}2^{2(\gamma-1)}}{c^{2(n-\gamma c)}(c-1)^{2(\gamma(c+1)-n)}2^{2(2\gamma-2)}}$$
$$= \frac{(c+1)^{2(n-1-\gamma c)}c^{2(\gamma c+\gamma-n)}}{c^{2(n-1-\gamma c)}(c-1)^{2(\gamma c+\gamma-n)}2^{2(\gamma-1)}}$$
$$= \left(\frac{c+1}{c}\right)^{2(n-1-\gamma c)}\left(\frac{c}{c-1}\right)^{2(\gamma c+\gamma-n)}\frac{1}{2^{2(n-1-\gamma c)}2^{2(\gamma c+\gamma-n)}}$$
$$= \left(\frac{c+1}{2c}\right)^{2(n-1-\gamma c)}\left(\frac{c}{2(c-1)}\right)^{2(\gamma c+\gamma-n)}.$$

If $n-1$ is not divisible by γ, then $c = \left\lfloor\frac{n-1}{\gamma}\right\rfloor < \frac{n-1}{\gamma}$, which implies that $2(n-1-\gamma c) > 2\left(n-1-\gamma\left(\frac{n-1}{\gamma}\right)\right) = 0$. Since $0 < \frac{c+1}{2c} < 1$, we obtain

$$0 < \left(\frac{c+1}{2c}\right)^{2(n-1-\gamma c)} < 1.$$

We have $0 < \frac{c}{2(c-1)} \leq 1$ and $\gamma c+\gamma-n = \gamma\left\lfloor\frac{n-1}{\gamma}\right\rfloor+\gamma-n \geq \gamma\left(\frac{n-\gamma}{\gamma}\right)+\gamma-n = 0$, therefore

$$0 < \left(\frac{c}{2(c-1)}\right)^{2(\gamma c+\gamma-n)} \leq 1.$$

Thus

$$\frac{f(0)}{f(-\gamma+1)} = \left(\frac{c+1}{2c}\right)^{2(n-1-\gamma c)}\left(\frac{c}{2(c-1)}\right)^{2(\gamma c+\gamma-n)} < 1$$

and $f(0) < f(-\gamma+1)$.
If $n-1$ is divisible by γ, then $c = \left\lfloor \frac{n-1}{\gamma} \right\rfloor = \frac{n-1}{\gamma}$, which implies that $2(n-1-\gamma c) = 0$ and

$$\left(\frac{c+1}{2c}\right)^{2(n-1-\gamma c)} = 1.$$

We obtain $\gamma c + \gamma - n = \gamma\left(\frac{n-1}{\gamma}\right) + \gamma - n = \gamma - 1 > 0$ for $\gamma > 1$ (note that if $\gamma = 1$, then obviously $f(-\gamma+1) = f(0)$ is the unique maximum). We have $c = \frac{n-1}{\gamma} \geq 3$ (otherwise if $\frac{n-1}{\gamma} = 2$, then $n = 2\gamma+1$ which in combination with $\gamma \leq \frac{n}{3}$ implies that $n = 3$ and $\gamma = 1$, a trivial case). Therefore $0 < \frac{c}{2(c-1)} < 1$ and

$$0 < \left(\frac{c}{2(c-1)}\right)^{2(\gamma c+\gamma-n)} < 1.$$

Thus

$$\frac{f(0)}{f(-\gamma+1)} = \left(\frac{c+1}{2c}\right)^{2(n-1-\gamma c)} \left(\frac{c}{2(c-1)}\right)^{2(\gamma c+\gamma-n)} < 1$$

and $f(0) < f(-\gamma+1)$ which implies that $f(-\gamma+1)$ is the maximum value of $f(k-p)$ for $-\gamma+1 \leq k-p \leq \gamma-1$. So, from (5) we have

$$\begin{aligned} \Pi_1(G) &\leq f(-\gamma+1) \\ &= \left\lfloor \frac{n-1}{\gamma} \right\rfloor^{2\left(n-\gamma\left\lfloor \frac{n-1}{\gamma}\right\rfloor\right)} \left(\left\lfloor \frac{n-1}{\gamma} \right\rfloor - 1\right)^{2\left(\gamma\left\lfloor \frac{n-1}{\gamma}\right\rfloor+\gamma-n\right)} 2^{4(\gamma-1)}. \end{aligned} \tag{6}$$

In [9] the set of extremal trees was presented and it was shown that

$$f(-\gamma+1) = 4^{2\gamma-2} \left\lfloor \frac{n-\gamma}{\gamma} \right\rfloor^{2\left(2\gamma-n+\gamma\left\lfloor \frac{n-\gamma}{\gamma}\right\rfloor\right)} \left(\left\lfloor \frac{n-\gamma}{\gamma} \right\rfloor + 1\right)^{2\left(n-\gamma-\gamma\left\lfloor \frac{n-\gamma}{\gamma}\right\rfloor\right)}$$

which can be simplified as

$$2^{4(\gamma-1)} \left(\left\lfloor \frac{n}{\gamma} \right\rfloor - 1\right)^{2\left(\gamma-n+\gamma\left\lfloor \frac{n}{\gamma}\right\rfloor\right)} \left\lfloor \frac{n}{\gamma} \right\rfloor^{2\left(n-\gamma\left\lfloor \frac{n}{\gamma}\right\rfloor\right)} \tag{7}$$

since $\left\lfloor \frac{n-\gamma}{\gamma} \right\rfloor = \left\lfloor \frac{n}{\gamma} \right\rfloor - 1$. It is easy to check that (6) is equal to (7), hence the proof is complete now.

3. Second multiplicative Zagreb index

Theorem 3.3 stated in [9] says that for $1 \leq \gamma \leq \frac{n}{3}$, if G is a tree having domination number γ and n vertices, then

$$\begin{aligned} \Pi_2(G) \geq\ & 4^{2\gamma-2} \left\lfloor \frac{n-\gamma}{\gamma} \right\rfloor^{\left\lfloor \frac{n-\gamma}{\gamma}\right\rfloor\left(2\gamma-n+\gamma\left\lfloor \frac{n-\gamma}{\gamma}\right\rfloor\right)} \\ & \left(\left\lfloor \frac{n-\gamma}{\gamma} \right\rfloor + 1\right)^{\left\lfloor \frac{n-\gamma}{\gamma}\right\rfloor\left(n-\gamma-\gamma\left\lfloor \frac{n-\gamma}{\gamma}\right\rfloor\right)}, \end{aligned}$$

with the equality if and only if G is $D_{n,\gamma}$. Let us note that the correct sharp bound is

$$\Pi_2(G) \geq 4^{2\gamma-2} \left\lfloor \frac{n-\gamma}{\gamma} \right\rfloor^{\left\lfloor \frac{n-\gamma}{\gamma} \right\rfloor \left(2\gamma-n+\gamma\left\lfloor \frac{n-\gamma}{\gamma} \right\rfloor\right)} \left(\left\lfloor \frac{n-\gamma}{\gamma} \right\rfloor + 1\right)^{\left(\left\lfloor \frac{n-\gamma}{\gamma} \right\rfloor+1\right)\left(n-\gamma-\gamma\left\lfloor \frac{n-\gamma}{\gamma} \right\rfloor\right)}$$

which can be simplified as

$$2^{4(\gamma-1)} \left(\left\lfloor \frac{n}{\gamma} \right\rfloor - 1\right)^{\left(\left\lfloor \frac{n}{\gamma} \right\rfloor-1\right)\left(\gamma-n+\gamma\left\lfloor \frac{n}{\gamma} \right\rfloor\right)} \left\lfloor \frac{n}{\gamma} \right\rfloor^{\left\lfloor \frac{n}{\gamma} \right\rfloor\left(n-\gamma\left\lfloor \frac{n}{\gamma} \right\rfloor\right)} \tag{8}$$

since $\left\lfloor \frac{n-\gamma}{\gamma} \right\rfloor = \left\lfloor \frac{n}{\gamma} \right\rfloor - 1$.

We study the proof of Theorem 3.3. Again, D is a minimum dominating set in a tree G having domination number γ and n vertices. $\overline{D} = V(G) \setminus D$ and the number of edges uv with $u \in D$ and $v \in \overline{D}$ is l. Let k and p be the number of edges between vertices in D and between vertices in $\overline{D}$, respectively. We have

$$-\gamma + 1 \leq k - p \leq \gamma - 1.$$

In [9] it was shown that

$$\begin{aligned} \Pi_2(G) \geq g(k-p) &= (q+1)^{(q+1)(n-1+(k-p)-\gamma q)} \\ &\quad q^{q(1-n+\gamma-(k-p)+\gamma q)} 2^{2(\gamma-1+p-k)} \\ &= (q+1)^{(q+1)(k-p)} q^{-q(k-p)} 2^{-2(k-p)} \\ &\quad (q+1)^{(q+1)(n-1-\gamma q)} q^{q(1-n+\gamma+\gamma q)} 2^{2(\gamma-1)}, \end{aligned}$$

where $q = \left\lfloor \frac{n-1+k-p}{\gamma} \right\rfloor \geq 2$.
We need to obtain the minimum value of $g(k-p)$. One cannot assume that

$$g(k-p) = \left(\frac{1^1/2^2}{q^q/(q+1)^{q+1}}\right)^{k-p} (q+1)^{(q+1)(n-1-\gamma q)} q^{q(1-n+\gamma+\gamma q)} 2^{2(\gamma-1)}$$

is increasing for $-\gamma + 1 \leq k - p \leq \gamma - 1$, because $g(k-p)$ contains $(q+1)^{(q+1)(n-1-\gamma q)} q^{q(1-n+\gamma+\gamma q)}$ and we have $q = \left\lfloor \frac{n-1+k-p}{\gamma} \right\rfloor$. So q contains $k-p$. Let $c = \left\lfloor \frac{n-1}{\gamma} \right\rfloor$.

Case 1. $0 \leq k - p \leq \gamma - 1$.
Since $\frac{1/2^2}{q^q/(q+1)^{q+1}} = \frac{(q+1)^{q+1}}{2^2 q^q} > 1$, we get $\left(\frac{1/2^2}{q^q/(q+1)^{q+1}}\right)^{t_1} < \left(\frac{1/2^2}{q^q/(q+1)^{q+1}}\right)^{t_2}$, where $t_1 < t_2$. So, we have increasing $g(k-p)$ on the interval presented in (1), so $g(k-p)$

is minimum if $k-p=0$. Similarly, if $k-p$ belongs to the interval presented in (2), then $g(k-p)$ is minimum for $k-p=\gamma\left\lfloor\frac{n-1}{\gamma}\right\rfloor+\gamma-n+1=\gamma c+\gamma-n+1=z$. Let us compare $g(z)$ and $g(0)$ to get the minimum for $g(k-p)$ for $0\le k-p\le\gamma-1$. We have

$$\begin{aligned}
g(z) &= (c+2)^{(c+2)z}(c+1)^{-(c+1)z}2^{-2z}\\
&\quad (c+2)^{(c+2)(n-1-\gamma(c+1))}(c+1)^{(c+1)(1-n+\gamma+\gamma(c+1))}2^{2(\gamma-1)}\\
&= (c+2)^0(c+1)^{(c+1)\gamma}2^{-2z}2^{2(\gamma-1)}
\end{aligned}$$

and

$$g(0)=(c+1)^{(c+1)(n-1-\gamma c)}c^{c(1-n+\gamma+\gamma c)}2^{2(\gamma-1)}.$$

Then

$$\begin{aligned}
\frac{g(z)}{g(0)} &= \frac{(c+1)^{(c+1)\gamma}2^{-2z}2^{2(\gamma-1)}}{(c+1)^{(c+1)(n-1-\gamma c)}c^{c(1-n+\gamma+\gamma c)}2^{2(\gamma-1)}}\\
&= \frac{(c+1)^{(c+1)z}}{c^{cz}2^{2z}}=\left(\frac{(c+1)^{c+1}}{2^2c^c}\right)^z>1
\end{aligned}$$

for every $c\ge 2$ since $\frac{(c+1)^{c+1}}{2^2c^c}>1$. Thus $g(z)>g(0)$.

Case 2. $-\gamma+1\le k-p\le 0$.
We have increasing $g(k-p)$ on the interval presented in (3), so $g(k-p)$ is the smallest for $k-p=\gamma\left\lfloor\frac{n-1}{\gamma}\right\rfloor-n+1=\gamma c-n+1$. Similarly, if $k-p$ belongs to the interval presented in (4), then $g(k-p)$ is the smallest for $k-p=-\gamma+1$.
Let us compare $g(\gamma c-n+1)$ and $g(-\gamma+1)$ to get the minimum for $g(k-p)$ for $-\gamma+1\le k-p\le 0$. Note that if $n=t\gamma$ (n is a multiple of γ), then the interval presented in (4) is an empty set. In that case $g(\gamma c-n+1)=g(-\gamma+1)$ is the unique minimum. So we can assume that n is not a multiple of γ. We have

$$\begin{aligned}
g(\gamma c-n+1) &= (c+1)^{(c+1)(n-1+(\gamma c-n+1)-\gamma c)}c^{c(1-n+\gamma-(\gamma c-n+1)+\gamma c)}\\
&\quad 2^{2(\gamma-1-(\gamma c-n+1))}\\
&= (c+1)^0c^{c\gamma}2^{2(n-\gamma(c-1)-2)}
\end{aligned}$$

and

$$\begin{aligned}
g(-\gamma+1) &= c^{c(n-1-\gamma+1-\gamma(c-1))}(c-1)^{(c-1)(1-n+\gamma+(\gamma-1)+\gamma(c-1))}\\
&\quad 2^{2(\gamma-1+\gamma-1)}\\
&= c^{c(n-\gamma c)}(c-1)^{(c-1)(\gamma(c+1)-n)}2^{2(2\gamma-2)}. \qquad (9)
\end{aligned}$$

Then

$$\begin{aligned}
\frac{g(\gamma c-n+1)}{g(-\gamma+1)} &= \frac{c^{c\gamma}2^{2(n-\gamma(c-1)-2)}}{c^{c(n-\gamma c)}(c-1)^{(c-1)(\gamma(c+1)-n)}2^{2(2\gamma-2)}}\\
&= \frac{c^{c(\gamma c+\gamma-n)}}{(c-1)^{(c-1)(\gamma c+\gamma-n)}2^{2(\gamma c+\gamma-n)}}\\
&= \left(\frac{c^c}{2^2(c-1)^{c-1}}\right)^{\gamma c+\gamma-n}>1
\end{aligned}$$

for every $c \geq 3$ since $\frac{c^c}{2^2(c-1)^{c-1}} > 1$ and $\gamma c + \gamma - n = \gamma \left\lfloor \frac{n-1}{\gamma} \right\rfloor + \gamma - n > \gamma \left(\frac{n-\gamma}{\gamma} \right) + \gamma - n = 0$. Thus $g(\gamma c - n + 1) > g(-\gamma + 1)$.
If $c = 2$, then the interval presented in (4) is an empty set, therefore the solution is trivial.

So $g(-\gamma + 1)$ is the minimum for $g(k - p)$ if $-\gamma + 1 \leq k - p \leq 0$, and $g(0)$ is the minimum value if $0 \leq k - p \leq \gamma - 1$. It remains to compare $g(-\gamma + 1)$ and $g(0)$. We have

$$\begin{aligned}
\frac{g(0)}{g(-\gamma+1)} &= \frac{(c+1)^{(c+1)(n-1-\gamma c)} c^{c(1-n+\gamma+\gamma c)} 2^{2(\gamma-1)}}{c^{c(n-\gamma c)} (c-1)^{(c-1)(\gamma(c+1)-n)} 2^{2(2\gamma-2)}} \\
&= \frac{(c+1)^{(c+1)(n-1-\gamma c)} c^{c(\gamma c+\gamma-n)}}{c^{c(n-1-\gamma c)} (c-1)^{(c-1)(\gamma c+\gamma-n)} 2^{2(\gamma-1)}} \\
&= \left(\frac{(c+1)^{c+1}}{c^c} \right)^{n-1-\gamma c} \left(\frac{c^c}{(c-1)^{c-1}} \right)^{\gamma c+\gamma-n} \\
&\quad \frac{1}{2^{2(n-1-\gamma c)} 2^{2(\gamma c+\gamma-n)}} \\
&= \left(\frac{(c+1)^{c+1}}{2^2 c^c} \right)^{n-1-\gamma c} \left(\frac{c^c}{2^2(c-1)^{c-1}} \right)^{\gamma c+\gamma-n}.
\end{aligned}$$

If $n - 1$ is divisible by γ, then $c = \left\lfloor \frac{n-1}{\gamma} \right\rfloor < \frac{n-1}{\gamma}$, which gives $n - 1 - \gamma c > n - 1 - \gamma \left(\frac{n-1}{\gamma} \right) = 0$. Since $\frac{(c+1)^{c+1}}{2^2 c^c} > 1$, we obtain

$$\left(\frac{(c+1)^{c+1}}{2^2 c^c} \right)^{n-1-\gamma c} > 1.$$

We get $\frac{c^c}{2^2(c-1)^{c-1}} \geq 1$ and $\gamma c + \gamma - n = \gamma \left\lfloor \frac{n-1}{\gamma} \right\rfloor + \gamma - n \geq \gamma \left(\frac{n-\gamma}{\gamma} \right) + \gamma - n = 0$, therefore

$$\left(\frac{c^c}{2^2(c-1)^{c-1}} \right)^{\gamma c+\gamma-n} \geq 1.$$

Then

$$\frac{g(0)}{g(-\gamma+1)} = \left(\frac{(c+1)^{c+1}}{2^2 c^c} \right)^{n-1-\gamma c} \left(\frac{c^c}{2^2(c-1)^{c-1}} \right)^{\gamma c+\gamma-n} > 1$$

and $g(0) > g(-\gamma + 1)$.
If $n - 1$ is divisible by γ, then $c = \left\lfloor \frac{n-1}{\gamma} \right\rfloor = \frac{n-1}{\gamma}$, which implies that $n - 1 - \gamma c = 0$ and

$$\left(\frac{(c+1)^{c+1}}{2^2 c^c} \right)^{n-1-\gamma c} = 1.$$

We obtain $\gamma c + \gamma - n = \gamma \left(\frac{n-1}{\gamma} \right) + \gamma - n = \gamma - 1 > 0$ for $\gamma > 1$ (note that if $\gamma = 1$, then obviously $g(-\gamma + 1) = g(0)$ is the unique minimum). We have $c = \frac{n-1}{\gamma} \geq 3$

(otherwise if $\frac{n-1}{\gamma}=2$, then $n=2\gamma+1$ which in combination with $\gamma\leq\frac{n}{3}$ implies that $n=3$ and $\gamma=1$, a trivial case). Therefore $\frac{c^c}{2^2(c-1)^{c-1}}>1$ and

$$\left(\frac{c^c}{2^2(c-1)^{c-1}}\right)^{\gamma c+\gamma-n}>1.$$

Thus

$$\frac{g(0)}{g(-\gamma+1)}=\left(\frac{(c+1)^{c+1}}{2^2c^c}\right)^{2(n-1-\gamma c)}\left(\frac{c^c}{2^2(c-1)^{c-1}}\right)^{\gamma c+\gamma-n}>1$$

and $g(0)>g(-\gamma+1)$ which implies that $g(-\gamma+1)$ is the minimum of $g(k-p)$ for $-\gamma+1\leq k-p\leq\gamma-1$. So, from (9) we have

$$\begin{aligned}\Pi_2(G) &\geq g(-\gamma+1)\\ &= \left\lfloor\frac{n-1}{\gamma}\right\rfloor^{\left\lfloor\frac{n-1}{\gamma}\right\rfloor\left(n-\gamma\left\lfloor\frac{n-1}{\gamma}\right\rfloor\right)}\\ &\quad\left(\left\lfloor\frac{n-1}{\gamma}\right\rfloor-1\right)^{\left(\left\lfloor\frac{n-1}{\gamma}\right\rfloor-1\right)\left(\gamma\left\lfloor\frac{n-1}{\gamma}\right\rfloor+\gamma-n\right)}2^{4(\gamma-1)},\end{aligned}$$

which is equal to (8), hence the proof is complete now.

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References

[1] Z.H. Ahsan and S.H. Arshad, *Computing multiplicative topological indices of some chemical nenotubes and networks*, Open J. Discrete Appl. Math. **2** (2019), no. 3, 7–18.

[2] B. Borovićanin and B. Furtula, *On extremal Zagreb indices of trees with given domination number*, Appl. Math. Comput. **279** (2016), 208–218 https://doi.org/10.1016/j.amc.2016.01.017.

[3] A. Javed, M.K. Jamil, J.B. Liu, and A. Ali, *Extremal (n,m)-graphs w.r.t. general multiplicative Zagreb indices*, Comb. Chem. High Throughput Screen. **25** (2022), no. 3, 476–482 http://doi.org/10.2174/1386207323999201103222640.

[4] R. Kazemi, *Note on the multiplicative Zagreb indices*, Discrete Appl. Math. **198** (2016), 147–154 https://doi.org/10.1016/j.dam.2015.06.028.

[5] J.V. Kureethara, A. Asok, and I.N. Cangul, *Inverse problem for the forgotten and the hyper Zagreb indices of trees*, Commun. Comb. Optim. **7** (2022), no. 2, 203–209
https://doi.org/10.22049/cco.2021.27034.1182.

[6] I. Milovanović, M. Matejić, E. Milovanović, and R. Khoeilar, *A note on the first Zagreb index and coindex of graphs*, Commun. Comb. Optim. **6** (2021), no. 1, 41–51
https://doi.org/10.22049/cco.2020.26809.1144.

[7] F.F. Nezhad, A. Iranmanesh, A. Tehranian, and M. Azari, *Strict lower bounds on the multiplicative Zagreb indices of graph operations*, Ars Combin. **117** (2014), 399–409.

[8] T. Vetrík and S. Balachandran, *General multiplicative Zagreb indices of trees*, Discrete Appl. Math. **247** (2018), 341–351
https://doi.org/10.1016/j.dam.2018.03.084.

[9] S. Wang, C. Wang, and J.B. Liu, *On extremal multiplicative Zagreb indices of trees with given domination number*, Applied Math. Comput. **332** (2018), 338–350
https://doi.org/10.1016/j.amc.2018.03.058.

[10] K. Xu and H. Hua, *A unified approach to extremal multiplicative Zagreb indices for trees, unicyclic and bicyclic graphs*, MATCH Commun. Math. Comput. Chem. **68** (2012), no. 1, 241–256.