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Research Article

Balance theory: An extension to conjugate skew gain graphs

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Abstract: We extend the notion of balance from the realm of signed and gain graphs to conjugate skew gain graphs which are skew gain graphs where the labels on the oriented edges get conjugated when we reverse the orientation. We characterize the balance in a conjugate skew gain graph in several ways especially by dealing with its adjacency matrix and the g-Laplacian matrix. We also deal with the concept of anti-balance in conjugate skew gain graphs.

Keywords: Skew gain graphs, Adjacency matrix, Laplacian matrix, Eigenvalues

AMS Subject classification: 05C22, 05C50

1. Introduction

In this paper, we extend the theory of balance in signed graphs and complex unit gain graphs to conjugate skew gain graphs, a particular class of skew gain graphs and deal with the properties of the adjacency and the g-Laplacian matrices (the terminology as in [8] which differs a bit from the usual Laplacian matrix) of the conjugate skew gain graphs. We denote by $\mathbb{C}^{\times}$, the set of non-zero complex numbers. The real part of a complex number z is denoted by $\Re(z)$ and imaginary part by $\Im(z)$. Regarding the basic definitions and other details of graphs, signed graphs, gain graphs and skew gain graphs, the reader may refer, respectively, to [1, 3, 9, 10]. Note that all the underlying graphs in this article are simple and finite. From now onwards, the notation $\overrightarrow{E}$ stands for the collection of oriented edges such that for an edge $uv \in E$ of a graph $G = (V, E)$, we have two oriented counterparts $\overrightarrow{uv}$ and $\overrightarrow{vu}$ in $\overrightarrow{E}$. Let us begin with the definition of a conjugate skew gain graph.

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Definition 1. Let $G = (V, E)$ be a graph with some prescribed orientation for the edges. A *conjugate skew-gain graph* $\Phi = (G, \mathbb{C}^\times, \varphi)$ (or for brevity, a csg-graph, which we also denote by G^φ) is such that the *conjugate skew gain function* $\varphi : \overrightarrow{E} \to \mathbb{C}^\times$ satisfies $\varphi(\overrightarrow{vu}) = \overline{\varphi(\overrightarrow{uv})}$.

The *conjugate skew gain*, $\varphi(C)$, of a cycle $C : v_0v_1 \dots v_nv_0$ in a csg-graph, is the product $\varphi(v_0v_1)\varphi(v_1v_2)\dots\varphi(v_nv_0)$ of the conjugate skew gains of its edges. Also, when the underlying graph is a path P_n or a cycle C_n, we call the corresponding structures to be *conjugate skew gain path* or *conjugate skew gain cycle*, respectively. When a vertex v is adjacent with a vertex u, we write $v \sim u$ and $v \sim e$ denotes that the edge e is incident with the vertex v. Also if not explicitly mentioned, the order of the underlying graph G of a csg-graph G^φ will be taken as n.

1.1. Adjacency matrix of conjugate skew gain graphs

The adjacency matrix $A(\Phi) = (a_{ij})_n$ of a csg-graph $\Phi = G^\varphi$ is defined as the square matrix of order $n = |V(G)|$ where

$$a_{ij} = \begin{cases} \varphi(\overrightarrow{v_iv_j}) & \text{if } v_i \sim v_j \\ 0 & \text{otherwise} \end{cases}$$

such that whenever $a_{ij} \neq 0,\ a_{ji} = \varphi(\overrightarrow{v_jv_i}) = \overline{a}_{ij}$. Note that by definition, the adjacency matrix of a csg-graph is hermitian and hence its eigenvalues are all real numbers. Hence by ordering these eigenvalues of $A(\Phi)$ as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The spectral radius $\rho(A(\Phi))$ of $A(\Phi)$ is the maximum of the absolute value of the eigenvalues. We denote the charactersitic polynomial of a csg-graph Φ by $\Psi(\Phi,\ x) = \det(xI -\ A(\Phi))$. We define, as usual, a subgraph of a graph as an *elementary subgraph* [2], if its components consist only of K_2 or cycles. In the following formulae, we take the sum over all elementary subgraphs $L \in \mathfrak{L}_i$ where $\mathfrak{L}_i$ denotes the collection of all elementary subgraphs L of order i. For $i = 0, 1$, we take $a_i(\Phi) = 1, 0$ respectively in order to avoid confusion. Also the notation $K(L)$ is used to denote the number of components in L and $C(L)$ denotes the number of cycles in L. Adapting similar techniques as in [5], a Sach's type formula for a csg-graph $\Phi = G^\varphi$ can be derived as follows.

Theorem 1. *If $\Phi = G^\varphi$ is a conjugate skew gain graph where $G = (V,\ E)$ is a graph of order n, and if $\Psi(\Phi, x) = \sum_{i=0}^{n} a_i(\Phi)x^{n-i}$ then*

$$a_i(\Phi) = \sum_{L \in \mathfrak{L}_i} (-1)^{K(L)} 2^{C(L)} \Big(\prod_{K_2 \in L} \prod_{e \in K_2} |\varphi(e)|^2 \Big) \Big(\prod_{C \in L} \Re(\varphi(C)) \Big). \tag{1}$$

For a vector $\boldsymbol{x} \in \mathbb{C}^n$, as usual, $\boldsymbol{x}^*$ denotes the conjugate transpose of the column matrix $\boldsymbol{x}$. Then an easy calculation proves the expression in the following lemma.

Lemma 1. *If G^φ is a csg-graph, then*

$$x^* A(G^\varphi) x = 2 \sum_{e=uv\in E} \Re\Big(\varphi(\overrightarrow{uv})\overline{x_u} x_v\Big) \tag{2}$$

for all $x \in \mathbb{C}^n$.

1.2. The g-Laplacian matrix of conjugate skew gain graphs

For an oriented edge $\vec{e}_j = \overrightarrow{v_i v_k}$ we take v_i as the tail of that edge and v_k as its head and we write $t(\vec{e}_j) = v_i$ and $h(\vec{e}_j) = v_k$. We define the *absolute-degree* $d_a(v)$ of a vertex v in a csg-graph $\Phi = G^\varphi$ as $d_a(v) = \sum_{\vec{e}: v\sim\vec{e}} |\varphi(\vec{e})|$. The definition of the g-Laplacian matrix (adapated by making suitable changes in [8]) of a csg-graph is as follows:

Definition 2. Given a csg-graph $\Phi = G^\varphi$ its g-Laplacian matrix is defined as $L_g(\Phi) = D_a(\Phi) - A(\Phi)$ where the diagonal matrix $D_a(\Phi)$, called the abolute-degree matrix of Φ, is $\operatorname{diag}\big(d_a(v)\big)$.

Theorem 2. *$L_g(\Phi)$ is positive semi-definite.*

Proof. For any $x \in \mathbb{C}^n$, a simple computation shows that $L_g(\Phi)$ satisfies

$$x^* L_g(\Phi) x = \sum_{uv\in E} \Big[|\varphi(\overrightarrow{uv})|\Big(|x_u| - |x_v|\Big)^2 + 2\Big(|\varphi(\overrightarrow{uv})\overline{x}_u x_v| - \Re\big(\varphi(\overrightarrow{uv})\overline{x}_u x_v\big)\Big)\Big]. \tag{3}$$

Since for any complex number z, $|z| - \Re(z) \geq 0$, we conclude from Equation (3) that $L_g(\Phi)$ is positive semi-definite. □

The incidence matrix for a csg-graph Φ (by adapting the one in [8]) can be defined as follows:

Definition 3 ([8]). Given a csg-graph $\Phi = G^\varphi$ its (oriented) incidence matrix is defined as $\mathrm{H}(\Phi) = (b_{ij})$ where

$$b_{ij} = \begin{cases} |\varphi(\vec{e}_j)|^2 & \text{if } t(\vec{e}_j) = v_i, \\ -\overline{\varphi(\vec{e}_j)}|\varphi(\vec{e}_j)| & \text{if } h(\vec{e}_j) = v_i, \\ 0 & \text{otherwise.} \end{cases}$$

Now we give the definition of a matrix operation for the incidence matrix $\mathrm{H}(\Phi)$ as follows:
$\mathrm{H}^\#$ is the transpose of the matrix obtained by replacing each column element of H as under:
(i) $|\varphi(\vec{e}_j)|^2$ replaced by $(|\varphi(\vec{e}_j)|)^{-1}$ and
(ii) $-\overline{\varphi(\vec{e}_j)}|\varphi(\vec{e}_j)|$ replaced by $-(\overline{\varphi(\vec{e}_j)})^{-1} = -\dfrac{\varphi(\vec{e}_j)}{|\varphi(\vec{e}_j)|^2}$.

The following theorem suitably adapted for a csg-graph has been proved generally for a skew gain graph in [8] and it will be used in our discussions later.

Theorem 3 ([8]). *For a csg-graph $\Phi = G^\varphi$, $L_g(\Phi) = \mathrm{H}(\Phi)\mathrm{H}^\#(\Phi)$.*

A 1-tree is a connected unicyclic graph and a 1-forest is a disjoint union of 1-trees. A spanning subgraph of G which is a 1-forest is called as an essential spanning subgraph of G. We denote the collection of all essential spanning subgraphs of Φ by $\mathfrak{E}(\Phi)$.The following is the matrix tree theorem for csg-graphs which is a special case of a general theorem in [8].

Theorem 4 ([8]). *If $\Phi = G^\varphi$ is a csg-graph on n vertices, then*

$$\det(L_g(\Phi)) = \sum_{\Psi\in\mathfrak{E}(\Phi)} 2^{N(C_\Psi)} \prod_{\psi\in\Psi} \Big[\Big(|\varphi(C_\psi)| - \Re(\varphi(C_\psi))\Big)\Big(\prod_{\vec{e}\in E(\psi)\setminus E(C_\psi)} |\varphi(\vec{e})|\Big)\Big]$$

where the summation runs over all essential spanning subgraphs Ψ of Φ and $\psi\in\Psi$ denotes the component 1-trees ψ in the spanning 1-forest Ψ.

As a passing reference, note that as corollaries to the above theorem, in the case of a csg-graph $\Phi = G^\varphi$ where G is a 1-forest, then $\det L_g(\Phi) = 2^{N(C_\Phi)} \prod_{\Psi\in\mathfrak{E}(\Phi)} \Big(|\varphi(C_\Psi)| - \Re(\varphi(C_\Psi))\Big)\Big(\prod_{\vec{e}\in E(\Psi)\setminus E(C_\Psi)} |\varphi(\vec{e})|\Big)$ where the product runs over all component 1-trees Ψ having unique cycle C_Ψ and $N(C_\Phi)$ is the total number of unicycles in the 1-forest and in the case of a csg-graph with the underlying graph as a cycle C, it simply means $\det L_g(\Phi) = 2\Big(|\varphi(C)| - \Re(\varphi(C))\Big)$.

2. Notion of balance in conjugate skew gain graphs

The concept of balance in social networks, which are till now modelled only in terms of signed or gain graphs, is discussed at length in the literature and the details about various ways of characterizing balance in such discrete structures can be found in various articles; for instance see [4, 9]. Many people worked on the balancing aspects of complex unit graphs (See for example [6, 7]). Recall that a complex unit gain graph is only a particular case of csg-graphs where they deal with the gains from the group $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Infact, it is so as $|z| = 1$ implies $\overline{z} = \frac{1}{z}$. So our attempt is to extend balance theory by having the conjugate skew gains to the entire complex plane sans zero, though generally, the balance theory as such cannot be easily extended to the realm of skew gain graphs with a general involutive automorphism.

Definition 4. A csg-graph $\Phi = G^\varphi$ is said to be balanced, if the conjugate skew gain $\varphi(C) = \prod_{e\in E(C)} \varphi(e)$ of every cycle C in Φ satisfies $\varphi(C) = |\varphi(C)|$.

The usual balance theory in signed and complex unit gain graphs coincides with the above one when we take $\varphi(C) = 1$ for every cycle C. Spectral characterization of balance in gain graphs can be found in [4] and spectral properties of complex unit gain graphs can be found in [6, 7].

Remark 1. Note that $\varphi(C) = |\varphi(C)| \Leftrightarrow |\varphi(C)| = \Re(\varphi(C)) > 0 \Leftrightarrow \overline{\varphi(C)} = |\varphi(C)|$.

In the following lines, we provide basics of balance theory in csg-graphs and the highlight is that a balanced csg-graph behaves like a positively weighted graph. We require the definition of an important operation, called a switching in conjugate skew gain graphs to move ahead. A function $\zeta : V \to \mathbb{T} \subset \mathbb{C}^{\times}$ is called a *switching function* for a csg-graph $\Phi = G^{\varphi}$, if it results in a csg-graph $\Phi^{\zeta} = G^{\varphi^{\zeta}}$ where φ^{ζ} satisfies

$$\varphi^{\zeta}(\overrightarrow{uv}) = \overline{\zeta(u)}\varphi(\overrightarrow{uv})\zeta(v).$$

Whether the switched graph is indeed a csg-graph or not is established in the following result.

Lemma 2. *If $\zeta : V \to \mathbb{T}$ is a switching function which switches a csg-graph Φ to Φ^{ζ}, then $\varphi^{\zeta}(\overrightarrow{vu}) = \overline{\varphi^{\zeta}(\overrightarrow{uv})}$.*

Proof. We have $\varphi^{\zeta}(\overrightarrow{uv}) = \overline{\zeta(u)}\varphi(\overrightarrow{uv})\zeta(v)$. Therefore,

$$\overline{\varphi^{\zeta}(\overrightarrow{uv})} = \zeta(u)\overline{\varphi(\overrightarrow{uv})}\overline{\zeta(v)} = \overline{\zeta(v)}\varphi(\overrightarrow{vu})\zeta(u) = \overline{\zeta(v)}\varphi(\overrightarrow{vu})\zeta(u) = \varphi^{\zeta}(\overrightarrow{vu}).$$

□

We call two csg-graphs $\Phi = G^{\varphi}$ and $\Psi = G^{\psi}$ to be *switching equivalent*, if there exists a switching function $\zeta : V \to \mathbb{T}$ such that $\Phi = \Psi^{\zeta}$. Switching preserves many features of the two csg-graphs including their eigenvalues. Corresponding to a csg-graph $\Phi = G^{\varphi}$, we have a weighted graph denoted by $G^{|\varphi|}$ with each edge conjugate skew gains replaced by its modulus value (Note that $G^{|\varphi|}$ is also a conjugate skew gain graph). Indeed, the following is a very important result.

Lemma 3. *The csg-graph $\Phi = G^{\varphi}$ switches to $G^{|\varphi|}$ if and only if there exists a switching function $\zeta : V \to \mathbb{T}$ satisfying $\arg(\varphi(\overrightarrow{uv})) = \arg(\zeta(u)) - \arg(\zeta(v)) \pm 2\,k\pi$ for every oriented edge $\overrightarrow{uv}$.*

Proof. Let $\zeta : V \to \mathbb{T}$ switch the csg-graph G^{φ} to $G^{|\varphi|}$. By the definition this means

$$\begin{aligned}\varphi^{\zeta}(\overrightarrow{uv}) &= \overline{\zeta(u)}\varphi(\overrightarrow{uv})\zeta(v)\\ &= e^{-i\arg(\zeta(u))}|\varphi(\overrightarrow{uv})|e^{i\arg(\varphi(\overrightarrow{uv}))}e^{i\arg(\zeta(v))}\\ &= |\varphi(\overrightarrow{uv})|e^{i[\arg(\varphi(\overrightarrow{uv}))-\arg(\zeta(u))+\arg(\zeta(v))]}.\end{aligned}$$

Therefore, $\varphi^{\zeta}(\overrightarrow{uv}) = |\varphi(\overrightarrow{uv})|$ if and only if, $\arg(\varphi(\overrightarrow{uv})) = \arg(\zeta(u)) - \arg(\zeta(v)) \pm 2\ k\pi$. □

We define such a switching function ζ in the above lemma to be a *potential function* for the conjugate skew gain function φ.

Theorem 5. *A csg-graph G^{φ} is balanced if, and only if, it can be switched to $G^{|\varphi|}$.*

Proof. Suppose that G^{φ} gets switched to $G^{|\varphi|}$. By Lemma 3, there is a potential function ζ for φ satisfying $\arg(\varphi(\overrightarrow{uv})) = \arg(\zeta(u)) - \arg(\zeta(v)) \pm 2\ n\pi$. Note that for each edge $\overline{uv}$ in G^{φ}, $\varphi(\overrightarrow{uv}) = |\varphi(\overrightarrow{uv})|e^{-i(\arg(\zeta(u))-\arg(\zeta(v)))}$. Thus, if $C_k^{\varphi} : v_1\overrightarrow{e_1}v_2\overrightarrow{e_2}v_3 \cdots v_k\overrightarrow{e_k}v_1$ is a cycle in G^{φ}, then

$$\begin{aligned}
\varphi(C_k^{\varphi}) &= \prod_{\overrightarrow{e}\in\overrightarrow{E}(C_k)} \varphi(\overrightarrow{e}) \\
&= \prod_{\overrightarrow{e}\in\overrightarrow{E}(C_k)} |\varphi(\overrightarrow{e})|e^{i\arg(\varphi(\overrightarrow{e}))} \\
&= \Big[\prod_{\overrightarrow{uv}\in\overrightarrow{E}(C_k)} |\varphi(\overrightarrow{e})|\Big] e^{-i\sum(\arg(\zeta(u))-\arg(\zeta(v)))} \\
&= |\varphi(C_k^{\varphi})|
\end{aligned}$$

establishing the balance of G^{φ}.
Conversely assume that G^{φ} is balanced. Without loss of generality, we assume that G is connected. Let T be a spanning tree and u be considered as its root vertex. If $v, w \in V(G)$, denoting T_{vw} to be the unique path in T from v to w, define $\zeta : V(G) \to \mathbb{T}$ by $\zeta(v) = \varphi(T_{vu})/|\varphi(T_{vu})|$ and $\zeta(u) = 1$, it can be verified that $\varphi^{\zeta}(\overrightarrow{e}) = |\varphi(\overrightarrow{e})|$ for $e \notin E(T)$ and for $e \notin E(T)$ lying on a cycle C, $\varphi^{\zeta}(\overrightarrow{e}) = |\varphi(\overrightarrow{e})|$ if C is balanced. Thus this is the required switching function ζ which switches G^{φ} to $G^{|\varphi|}$. □

In the following theorem and other results that follow, the matrix $D(\zeta)$ corresponding to a switching function ζ is the diagonal matrix $\text{diag}(\zeta(v_i))$ where $v_i : i = 1, 2 \ldots, n$ are the vertices in the graph. We omit the proof of the following theorem as the two equations therein can be verified easily with the help of the definitoin of the switching function.

Theorem 6. *If two conjugate skew gain graphs Φ_1 and Φ_2, having the same underlying graph G, are switching equivalent and the corresponding switching function is ζ, then*

$$\textit{(i)}\ A(\Phi_2) = D^*(\zeta)A(\Phi_1)D(\zeta) \tag{4}$$

$$\textit{(ii)}\ L_g(\Phi_2) = D^*(\zeta)L_g(\Phi_1)D(\zeta). \tag{5}$$

Now we provide a characterization of balance in a csg-graph using the eigenvalues of its adjacency matrix.

Theorem 7. *A csg-graph G^{φ} is balanced if and only if, the eigenvalues of G^{φ} and $G^{|\varphi|}$ conicide.*

Proof. Let G^{φ} be balanced. Then for every cycle C in G^{φ}, $\varphi(C) = |\varphi(C)|$ or equivalently, $\Re(\varphi(C)) = |\varphi(C)|$, then from the Equation (1) in Theorem 1, the corresponding coefficients of the characteristic polynomials of G^{φ} and $G^{|\varphi|}$ will be identical, since

$$\begin{aligned} a_i(G^{\varphi}) &= \sum_{L\in\mathfrak{L}_i} (-1)^{K(L)} 2^{C(L)} \Big(\prod_{K_2\in L} \prod_{e\in K_2} |\varphi(e)|^2 \Big) \Big(\prod_{C\in L} \Re(\varphi(C)) \Big) \\ &= \sum_{L\in\mathfrak{L}_i} (-1)^{K(L)} 2^{C(L)} \Big(\prod_{K_2\in L} \prod_{e\in K_2} |\varphi(e)|^2 \Big) \Big(\prod_{C\in L} |\varphi(C)| \Big) \\ &= a_i(G^{|\varphi|}). \end{aligned}$$

Thus the eigenvalues of G^{φ} and $G^{|\varphi|}$ conicide. Conversely, assume that G^{φ} is unbalanced. Therefore, there are unbalanced cycles in it. Let k be the smallest order of such an unbalanced cycle C with $|\varphi(C)| \neq \Re(\varphi(C))$ or equivalently $|\varphi(C)| - \Re(\varphi(C)) > 0$ and all such elementary subgraphs of order k are cycles. Hence, comparing the coefficients, we see that

$$a_k(G^{\varphi}) - a_k(G^{|\varphi|}) = \sum_{L\in\mathfrak{L}_k} (-1)^{K(L)} 2^{C(L)} \Big(\prod_{C\in L} \big(|\varphi(C)| - \Re(\varphi(C))\big) \Big) \neq 0.$$

This in turn shows that the characteristic polynomials differ and hence they have different eigenvalues. This completes the proof. □

Lemma 4. *If G is a connected graph,* $\operatorname{rank}(H(G^{\varphi})) = n-1$ *or n according as G^{φ} is balanced or not.*

Proof. Suppose G^{φ} is not balanced. This means there is a cycle C for which $\varphi(C) \neq |\varphi(C)|$. Then concidering the square submatrix, say $H_1(G^{\varphi})$ of $H(G^{\varphi})$, corresponding the 1-tree containing this cycle C, it is a simple calculation to see that (by relabelling the vertices, if required) $\det(H_1(G^{\varphi})) = |\varphi(C)| \Big[|\varphi(C)| - \overline{\varphi(C)} \Big] \Big[\prod_{e\notin E(C)} |\varphi(e)| \overline{\varphi(e)} \Big] \neq 0$. Hence $\operatorname{rank}\big(H(G^{\varphi})\big) = n$. A similar computation will provide the result in the case of balanced G^{φ}. □

Theorem 8. *If G is a connected graph,* $\operatorname{rank}\big(L_g(G^{\varphi})\big) = n-1$ *or n according as G^{φ} is balanced or not.*

Proof. Proof follows from Theorem 3 and Lemma 4. □

Now we characterize the balance using g-Laplacian matrix.

Theorem 9. *A connected csg-graph G^{φ} is balanced if and only if* $\det\big(L_g(G^{\varphi})\big) = 0$.

Proof. Let G^{φ} be balanced. Then, as $\varphi(C) = |\varphi(C)|$ for every cycle C in G^{φ}, the matrix tree theorem for conjugate skew graphs in Theorem 4 proves the one way implication. Conversely assume that $\det(L_g(G^{\varphi}) = 0$, but G^{φ} is not balanced. This gives the existence of atleast one cycle C in G^{φ} that is not balanced. i.e., $\varphi(C) \neq |\varphi(C)|$ for this cycle. The $\mathrm{rank}(L_g(G^{\varphi}))$ is indeed the order of the largest square submatrix of $L_g(G^{\varphi})$ which is non-singular. Taking the incidence matrix corresponding to the cycle C above, using Lemma 4 and Theorem 8, we see that $\mathrm{rank}(L_g(G^{\varphi})) = n$ and hence a contradiction. This proves the result. □

3. On some spectral aspects of conjugate skew gain graphs

Though we wish to work on various spectral bounds with respect to the adjacency and g-Laplacian matrices of a csg-graph somewhere else in great detail, we discuss a few aspects in this regard in the following. Recall that abolute-degree of a vertex v is $d_a(v) = \sum\limits_{\vec{e}:v\sim\vec{e}} |\varphi(\vec{e})|$ and we define Δ_a as the maximum of all the absolute-degrees of the vertices. Similarly δ_a is the minimum of all absolute-degrees. Also a csg-graph is said to be absolute-degree k-regular if $d_a(v)$ of each vertex v is a constant k.

Theorem 10. $\rho(A(\Phi)) \leq \Delta_a$.

Proof. Let $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)^T \in \mathbb{C}^n \setminus \{0\}$ be the eigenvector corresponding to $\lambda_{\max}$ where $|\lambda_{\max}| = \rho(A(\Phi))$ and let $|x_k| = \max\limits_{i\in\{1,2,\cdots,n\}} |x_i|$. Then $A\boldsymbol{x} = \lambda_{\max}\boldsymbol{x}$ implies $|\lambda_{\max}||x_k| \leq \Big(\sum\limits_{e:k\sim e} |\varphi(e)|\Big)|x_k| = d_a(v_k)|x_k| \leq \Delta_a|x_k|$. Therefore, $|\lambda_{\max}| \leq \Delta_a$ as $|x_k| \neq 0$. □

Theorem 11. *If G^{φ} is an absolute-degree k-regular csg-graph, then $\lambda_i^{L_g} = k - \lambda_i$, where $\lambda_i^{L_g}$ are the g-Laplacian eigenvalues and λ_i are the adjacency eigenvalues of G^{φ}.*

Proof. As G^{φ} is an absolute-degree k-regular, $L_g(\Phi) = kI - A(\Phi)$. This gives the required relation $\lambda_i^{L_g} = k - \lambda_i$ between the eigenvalues of $L_g(\Phi)$ and $A(\Phi)$. □

The above Theorem 11 has the following Corollary 1 which extends the result about a cycle in a complex unit gain graph given in [7] to a conjugate skew gain cycle.

Lemma 5 ([7]). *If $\Psi = (C_n, \mathbb{T}, \varphi)$ is a complex unit gain cycle with $\varphi(C_n) = e^{i\theta}$, then the adjacency eigenvalues and Laplacian eigenvalues of Ψ, respectively, are given by*

$$\lambda_j(A(\Psi)) = 2\cos\left(\frac{\theta + 2\pi j}{n}\right),\ j = 1, 2, \ldots, n \tag{6}$$

$$\lambda_j(L(\Psi)) = 2 - 2\cos\left(\frac{\theta + 2\pi j}{n}\right),\ j = 1, 2, \ldots, n. \tag{7}$$

Corollary 1. *If C_n^{φ} is a absolute-degree k-regular csg-gain graph with the underlying graph as the cycle C_n and $\arg\big(\varphi(C_n)\big) = \theta$, then its adjacency and g-Laplacian spectrum, respectively, are given by*

$$\lambda_j(A(C_n^{\varphi})) = k\cos\left(\frac{\theta + 2\pi j}{n}\right),\ j = 1, 2, \cdots, n \tag{8}$$

$$\lambda_j^{L_g}(C_n^{\varphi}) = k - k\cos\left(\frac{\theta + 2\pi j}{n}\right),\ j = 1, 2, \cdots, n. \tag{9}$$

Proof. If e_i for $i = 1,\ 2, \cdots\ n$ are the edges of the cycle C_n, then C_n^{φ} is absolute-degree k-regular only when $|\varphi(e_i)| = k$ for all the edgese_i. Then, $A(C_n^{\varphi})$ will be k times the adjacency matrix of the corresponding complex unit gain graphs with unit gains $e^{\arg(\varphi(e_i))}$ as the elements in the corresponding positions. This proves the corollary. □

4. Anti-balance and its impacts

We define the negative $-\Phi = G^{-\varphi}$ of a csg-graph $\Phi = G^{\varphi}$ as that csg-graph, where each conjugate skew gain in $-\Phi$ is the negative of those in Φ. Indeed, it can be easily checked that $-\Phi$ is also a conjugate skew gain graph. A csg-graph Φ is said to be anti-balanced if its negative $-\Phi$ is balanced.

Theorem 12. *A conjugate skew gain graph G^{φ} is both balanced and anti-balanced if and only if G is bipartite.*

Proof. Let $\Phi = G^{\varphi}$ be the given csg-graph and letting $\varphi_1 = -\varphi$, its negation is $-\Phi = G^{\varphi_1}$. Let C be a cycle in G. Then $\varphi_1(C) = (-1)^{l(C)}\varphi(C)$ where $l(C)$ is the length of the cylce. Now assume first that G is bipartite. As there are no odd cycles, this implies that $\varphi_1(C) = \varphi(C)$ for every cycle C in Φ and $-\Phi$, proving the first part. Conversely assume that Φ is both balanced and anitblanced or in other words Φ and $-\Phi$ are balanced. Suppose on the contrary that G is not bipartite. then there is an odd cycle C in G. For any cylces or in particular for this cycle, $\varphi_1(C) = (-1)^{l(C)}\varphi(C)$. Using the condition of balance this means $|\varphi_1(C)| = \varphi_1(C) = (-1)\varphi(C)$ implying that $\varphi(C) = -\varphi(C)$. So, $\varphi(C) = 0$, a contradiction. □

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