Communications in Combinatorics and Optimization
Vol. 9, No. 1 (2024), pp. 131-143
https://doi.org/10.22049/CCO.2022.27240.1218

CCO
Commun. Comb. Optim.

Research Article

On equitable near proper coloring of graphs

Sabitha Jose†, **Libin Chacko Samuel**‡, **Sudev Naduvath***

Department of Mathematics, CHRIST (Deemed to be University), Bangalore-560029, Karnataka, India
†sabitha.jose@res.christuniversity.in
‡libin.samuel@res.christuniversity.in
*sudev.nk@christuniversity.in

Received: 2 May 2021; Accepted: 3 September 2022
Published Online: 6 November 2022

Abstract: A defective vertex coloring of a graph is a coloring in which some adjacent vertices may have the same color. An edge whose adjacent vertices have the same color is called a bad edge. A defective coloring of a graph G with minimum possible number of bad edges in G is known as a near proper coloring of G. In this paper, we introduce the notion of equitable near proper coloring of graphs and determine the minimum number of bad edges obtained from an equitable near proper coloring of some graph classes.

Keywords: Improper coloring, equitable coloring, near proper coloring, equitable near proper coloring

AMS Subject classification: 05C15, 05C38

1. Introduction

For terms and definitions in graph theory, we refer to [1, 4]. Unless mentioned otherwise, all graphs discussed in this paper are finite, simple, connected and undirected.

A *proper coloring* of a graph G is an assignment of colors to the vertices of a graph so that no two adjacent vertices receive the same color. An *equitable coloring* of a graph G is an assignment of colors to the vertices of a graph such that the number of vertices in any two color classes differ by at most one. The smallest integer k for which G is equitably k-colorable is called the *equitable chromatic number* of G and is denoted by $\chi_e(G)$.

* *Corresponding Author*

An *improper coloring* or a *defective coloring* of a graph G is a vertex coloring in which adjacent vertices are allowed to have the same color. The edges whose end vertices receive the same color are called as *bad edges*. A *near proper coloring* of G is a coloring which minimises the number of bad edges by restricting the number of color classes that can have adjacency among their own elements (see [2, 3]). The number of bad edges which result from a near proper coloring of G is denoted by $b_k(G)$.

Motivated by the studies mentioned above, in this paper, we discuss about equitable near proper coloring of certain basic graph classes.

2. Equitable Near Proper Coloring of Graphs

An *equitable near proper coloring* of a graph G is an improper coloring in which the vertex set can be partitioned into k color classes $V_1, V_2, \dots, V_k$ such that $||V_i|-|V_j|| \leq 1$ for any $1 \leq i \neq j \leq k$ and the number of bad edges is minimised by restricting the number of color classes that can have adjacency among their own elements. The minimum number of bad edges which result from an equitable near proper coloring of G is defined as *equitable defective number* and is denoted by $b^k_{\chi_e}(G)$.

In a defective coloring there is a deficiency of the available number of colors to color a graph. In an equitable near proper coloring, we have k number of colors available and we consider all possible values for k. That is, k takes the values from 2 to $\chi_e(G) - 1$.

Graphs with equitable chromatic number 2 are excluded from this discussion. Hence, paths, even cycles, ladder graphs, gear graphs, Heawood graph etc. are omitted. We only consider graphs with $\chi_e(G) \geq 3$. For odd cycles, we have only one case $k = 2$ and we end up with only one bad edge.

Theorem 1. *The equitable defective number of a star graph $K_{1,n}$ is given by*

$$b^k_{\chi_e}(K_{1,n}) = \left\lfloor \frac{n+1}{k} \right\rfloor - 1.$$

Proof. Let $K_{1,n}$ be the star graph with $n+1$ vertices. Let v_0 be the central vertex and $\{v_1, v_2, \dots, v_n\}$ be the pendant vertices. Let $c_1, c_2, \dots, c_k$ be the available colors and $V_1, V_2, \dots, V_k$ be the corresponding color classes. In an equitable near proper coloring, r color classes consist of $\lceil \frac{n+1}{k} \rceil$ vertices and the remaining $(k-r)$ color classes consist of $\lfloor \frac{n+1}{k} \rfloor$ vertices. Also, we can color the central vertex in such a way that it belongs to a color class with $\lfloor \frac{n+1}{k} \rfloor$ elements (see Figure 1 for illustration). Therefore, we will get $\lfloor \frac{n+1}{k} \rfloor - 1$ bad edges. Thus, the equitable defective number $b^k_{\chi_e}(K_{1,n}) = \lfloor \frac{n+1}{k} \rfloor - 1$. □

A 4-equitable near proper coloring of star graphs is illustrated in Figure 1.

The following theorem discusses the k-equitable near proper coloring of a complete graph.

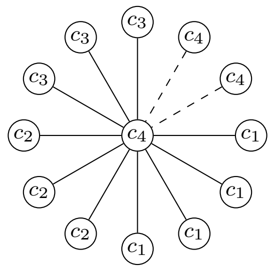

Figure 1. $K_{1,12}$ **and its 4-equitable near proper coloring.**

Theorem 2. *For a complete graph K_n, the equitable defective number is given by*

$$b^k_{\chi_e}(K_n) = \frac{r\lceil\frac{n}{k}\rceil\lceil\frac{n}{k}-1\rceil}{2} + \frac{(k-r)\lfloor\frac{n}{k}\rfloor\lfloor\frac{n}{k}-1\rfloor}{2}.$$

Proof. Let $G = K_n$ be a complete graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. Assume that $\mathcal{C} = \{c_1, c_2, \ldots, c_k\}$ be the set of colors available for coloring the vertices of K_n in an equitable manner, where $k < \chi_e(K_n)$. With respect to this equitable near proper coloring, exactly r color classes in K_n contain $\lceil\frac{n}{k}\rceil$ vertices of K_n and remaining $(k-r)$ color classes contain $\lfloor\frac{n}{k}\rfloor$ vertices, where $n \equiv r \pmod k$. Since the subgraphs induced by each color class will also be complete subgraphs (cliques) of K_n, we have r cliques with $\frac{(\lceil\frac{n}{k}\rceil)(\lceil\frac{n}{k}\rceil-1)}{2}$ number of bad edges and $(k-r)$ cliques with $\frac{(\lfloor\frac{n}{k}\rfloor)(\lfloor\frac{n}{k}\rfloor-1)}{2}$ number of bad edges. Hence, the equitable defective number of a complete graph is given by $\frac{r\lceil\frac{n}{k}\rceil\lceil\frac{n}{k}-1\rceil}{2} + \frac{(k-r)\lfloor\frac{n}{k}\rfloor\lfloor\frac{n}{k}-1\rfloor}{2}$. □

3. Equitable Near Proper Coloring of Some Cycle Related Graphs

A *wheel graph* denoted by $W_{1,n}$, is a graph obtained by joining every vertex of a cycle C_n to an external vertex. That is, $W_{1,n} = C_n + K_1$. The vertex K_1 is called the *central vertex* of $W_{1,n}$, where as the vertices of the cycle are known as the *rim vertices* of $W_{1,n}$. The edges connecting the central vertex and the rim vertices of a wheel graph are called its *spokes*. The following theorem discusses the equitable near proper coloring of a wheel graph.

Theorem 3. *The equitable defective number of a wheel graph $W_{1,n}$ where $n \geq 4$, is given by*

$$b^k_{\chi_e}(W_{1,n}) = \begin{cases} \lceil\frac{n}{2}\rceil & \text{if } k = 2 \\ \lfloor\frac{n+1}{k}\rfloor - 1 & \text{otherwise.} \end{cases}$$

Proof. Let $\{c_i : 1 \leq i \leq k\}$ be the set of colors available in an equitable near proper coloring of the wheel graph $W_{1,n}$, where $2 \leq k \leq \chi_e(G) - 1$. Let v_0 be the central vertex and $\{v_1, v_2, \ldots, v_n\}$ be the rim vertices of $W_{1,n}$.

Case 1. $k = 2$.
Here, we have to consider two subcases as mentioned below.

Subcase 1.1. n is even.
In this case, we can assign all the rim vertices with the available colors c_1 and c_2 alternatively. Now assign v_0 with color c_1 or c_2, which results in exactly $\frac{n}{2}$ bad edges.

Subcase 1.2. n is odd.
In this case, as mentioned above, the vertices $v_1, v_2, \ldots, v_{n-1}$ can have the colors c_1 and c_2 alternatively and the vertex v_n can take the color c_1 (or c_2), which results in a bad edge on the rim. Now, the central vertex can be assigned the color c_2 (or c_1) which makes $\lfloor \frac{n}{2} \rfloor$ spokes bad edges. Therefore, the number of bad edges in this case is $1 + \lfloor \frac{n}{2} \rfloor = \lceil \frac{n}{2} \rceil$. In both cases, we have $b^k_{\chi_e}(W_{1,n}) = \lceil \frac{n}{2} \rceil$.

Case 2. $k \geq 3$.
In an equitable near proper coloring of $W_{1,n}$, r color classes consist of $\lceil \frac{n+1}{k} \rceil$ vertices and the remaining $(k - r)$ color classes contain $\lfloor \frac{n+1}{k} \rfloor$ vertices. We can place the central vertex v_0 in a color class with minimum cardinality, (that is, $\lfloor \frac{n+1}{k} \rfloor$), we obtain $\lfloor \frac{n+1}{k} \rfloor - 1$ bad edges in this case. □

Figure 2 illustrates the 3-equitable near proper coloring of wheel graphs. The bad edges in each case are represented by dashed lines.

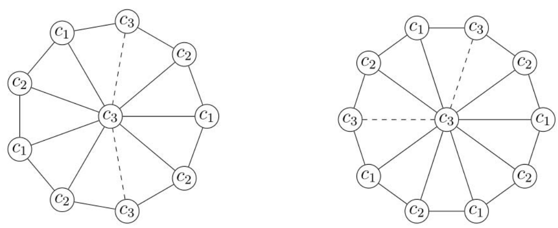

Figure 2. **Wheel graphs with 3-equitable near proper coloring**

A *double wheel graph* DW_n is obtained by joining all vertices of two disjoint cycles of order n to an external vertex. Hence, we can denote the double wheel graph as $DW_n = K_1 + 2C_n$. The following result provides the equitable defective number of a double wheel graph.

Theorem 4. *The equitable defective number of a double wheel graph DW_n where $n \geq 4$ is given by*

$$b^k_{\chi_e}(DW_n) = \begin{cases} 2\lceil \frac{n}{2} \rceil & \text{if } k = 2 \\ \lfloor \frac{2n+1}{k} \rfloor - 1 & \text{otherwise.} \end{cases}$$

Proof. Let $DW_n = K_1 + 2C_n$ be the double wheel graph with $2n+1$ vertices. Let v_0 denotes the central vertex and $v_1, v_2, \ldots, v_n$ be the vertices of the first cycle. Let $u_1, u_2, \ldots, u_n$ denote the vertices of the second cycle and let $c_1, c_2, \ldots, c_k$ be the available colors in an equitable near proper coloring of a double wheel graph. Here we consider two cases.

Case 1. Let $k = 2$.
We consider two subcases as below.

Subcase 1.1. n is even.
Assign all u_i's and v_i's with the available colors c_1 and c_2 in a cyclic order. Here in each cycle we have n vertices and hence $\frac{n}{2}$ of vertices will receive color c_1 and also $\frac{n}{2}$ of vertices will receive color c_2. Thus, altogether n vertices receive color c_1 and n vertices receive color c_2. Now assign the central vertex v_0 with color c_1 or c_2. Since v_0 is adjacent with all other vertices in a double wheel graph we get exactly n of bad edges.

Subcase 1.2. n is odd.
In an equitable near proper coloring of a double wheel graph, the two color classes V_1 and V_2 contains $\lceil \frac{2n+1}{2} \rceil$ and $\lfloor \frac{2n+1}{2} \rfloor$ vertices. Now as in Subcase-1.1 assign all u_i's and v_i's with colors c_1 and c_2. Here in each cycle, we get an additional bad edge since to properly color an odd cycle we require minimum three colors. Now place v_0 in the color class having minimum cardinality, that is the color class with $\lfloor \frac{2n+1}{2} \rfloor$ vertices. As a result of this we obtain $\lfloor \frac{2n+1}{2} \rfloor - 1$ bad edges among the spokes. Thus, we obtain $\lfloor \frac{2n+1}{2} \rfloor - 1 + 2 = \lceil \frac{2n+1}{2} \rceil$ bad edges in this subcase. Hence, combining the above two subcases we can conclude that when $k = 2$, the equitable defective number is given by $2\lceil \frac{n}{2} \rceil$.

Case 2. $k \geq 3$.
We say that for any $k \geq 3$, r color classes contain $\lfloor \frac{2n+1}{k} \rfloor$ vertices and $(k-r)$ color classes contain $\lceil \frac{2n+1}{k} \rceil$ vertices where $2n+1 \equiv r \pmod{k}$. If we place the central vertex v_0 in the color class with minimum cardinality, that is $\lfloor \frac{2n+1}{k} \rfloor$ we end up with $\lfloor \frac{2n+1}{k} \rfloor - 1$ bad edges. □

Figure 3 illustrates the 4-equitable near proper coloring of some double wheel graphs.

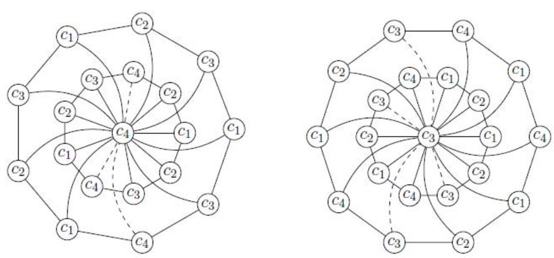

Figure 3. **Double Wheel graphs with 4-equitable near proper coloring**

A *helm graph* $H_{1,n}$ is a graph obtained by attaching pendant edges to all the rim vertices of a wheel graph $W_{1,n}$. The following theorem discusses the equitable near proper coloring of a helm graph $H_{1,n}$.

Theorem 5. *For a helm graph $H_{1,n}$, the equitable defective number is given by*

$$b^k_{\chi_e}(H_{1,n}) = \begin{cases} 1 & \text{if } k = 3,\ n \text{ is odd} \\ \lceil \frac{n}{k} \rceil & \text{otherwise.} \end{cases}$$

Proof. Let $G = H_{1,n}$ be the Helm graph with $2n+1$ vertices. Let v_0 denotes the central vertex of the graph. Let $v_1, v_2, \ldots, v_n$ be the rim vertices and $u_1, u_2, \ldots, u_n$ be the pendant vertices such that v_i is adjacent to u_i $\forall i$. Let $c_1, c_2, \ldots, c_k$ be the available colors and $V_1, V_2, \ldots, V_k$ be the corresponding color classes. In an equitable near proper coloring, the number of vertices in each color class differ by at most one. Hence, for a helm graph r color classes contain $\lfloor \frac{n+1}{k} \rfloor$ vertices and $(k-r)$ color classes contain $\lceil \frac{n+1}{k} \rceil$ vertices. Here we consider two cases.

Case 1. n is even.
We recall that when n is even, the equitable chromatic number of a helm graph is 3. Hence, in an equitable near proper coloring we consider only one case $k = 2$. When $k = 2$, we have two available colors say c_1 and c_2. Assign c_1 and c_2 to the rim vertices alternatively. Since n is even, we can properly color the cycle C_n with two colors. Now assign the central vertex v_0 with color c_1 or c_2 resulting in exactly $\frac{n}{2}$ bad edges among the spokes. And assign the colors for the pendant vertices in such a way that if v_1 is assigned with color c_1, then assign u_1 with color c_2 and if v_1 is assigned with color c_2, then assign u_1 with color c_1. Continue this process until all u_i's are assigned with some color resulting no bad edges. Hence, the equitable defective number is given by $\frac{n}{2}$.

Case 2. n is odd.
Note that when n is odd, the equitable chromatic number of a helm graph is 4 and hence we consider two sub cases $k = 2$ and $k = 3$ as below.

Subcase 2.1. $k = 2$.
As in Case 1, color all vertices with available colors c_1 and c_2 except the central vertex. Here we end up with one bad edge on the cycle since to properly color an odd cycle we need minimum three colors. Now assign the central vertex v_0 with color c_1 or c_2 by placing v_0 in the color class with minimum cardinality. Hence, in this case we obtain $\lfloor \frac{n}{2} \rfloor + 1 = \lceil \frac{n}{2} \rceil$ bad edges.

Subcase 2.2. $k = 3$.
We can color the rim vertices $v_1, v_2, \ldots, v_{n-1}$ with two colors c_1 and c_2 alternatively. Now assign v_n and the central vertex v_0 with color c_3 which results in only bad edge among the spokes. Now all pendent vertices can be assigned with any of the three colors without leaving any bad edges and also satisfying the equitability condition. Hence, $b^k_{\chi_e}(G) = 1$ in this case. □

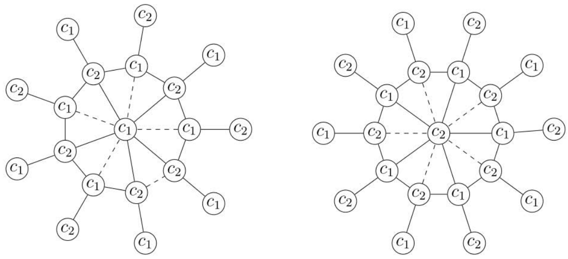

Figure 4. **Helm graphs with 2-equitable near proper coloring**

A 2-equitable near proper coloring of helm graphs is illustrated in Figure 4.
A *flower graph* denoted by F_n is obtained from a helm graph $H_{1,n}$ by joining all the pendent vertices of the helm graph $H_{1,n}$ to its central vertex. The following theorem discusses the equitable near proper coloring of the flower graph.

Theorem 6. *The equitable defective number of a flower graph F_n is given by*

$$b^k_{\chi_e}(F_n) = \begin{cases} n & \text{if } k = 2,\ n \text{ is even} \\ n+1 & \text{if } k = 2,\ n \text{ is odd} \\ \lfloor \frac{2n+1}{k} \rfloor - 1 & \text{otherwise.} \end{cases}$$

Proof. Let F_n be a flower graph on $2n+1$ vertices. Let v_0 denotes the central vertex and $v_1, v_2, \ldots, v_n$ be the rim vertices of F_n. Let $u_1, u_2, \ldots, u_n$ be the vertices adjacent to the rim vertices such that u_i is adjacent to v_i $\forall i$. Let $c_1, c_2, \ldots, c_k$ be the available colors in an equitable near proper coloring of a flower graph and we consider three different cases here.

Case 1. $k = 2$ and n is even.
We have two available colors say c_1 and c_2. Assign all rim vertices with c_1 and c_2 alternatively. Since n is even, we can properly color the cycle C_n with two colors. Now assign the colors to all u_i's such that if v_i is assigned with color c_1(or c_2), then assign u_i with color c_2(or c_1). Now assign the central vertex v_0 with either c_1 or c_2. This results in exactly n bad edges since in a flower graph the central vertex v_0 is adjacent with all other vertices.

Case 2. $k = 2$ and n is odd.
Here as in Case 1 we assign the colors to all vertices of F_n. Observe that along with the n bad edges we obtain another bad edge on the cycle since to properly color an odd cycle we require minimum three colors. Hence, the equitable defective number is given by $n + 1$.

Case 3. $k \geq 3$.
In an equitable near proper coloring, r color classes of F_n contain $\lceil \frac{2n+1}{k} \rceil$ vertices and

remaining $(k-r)$ color classes contain $\lfloor \frac{2n+1}{k} \rfloor$ vertices. By placing the central vertex v_0 in the color class with minimum cardinality we obtain $\lfloor \frac{2n+1}{k} \rfloor - 1$ bad edges. □

A 5-equitable near proper coloring of flower graphs is depicted in Figure 5.

A *sunflower graph* SF_n is a graph obtained by replacing each edge of the rim of a wheel graph $W_{1,n}$ by a triangle such that two triangles share a common vertex if and only if the corresponding edges in $W_{1,n}$ are adjacent in $W_{1,n}$. The following theorem discusses the equitable near proper coloring of a sunflower graph SF_n.

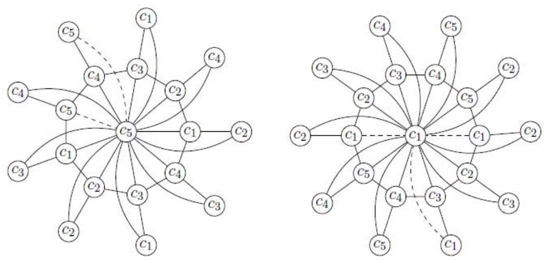

Figure 5. **Flower graphs with 5-equitable near proper coloring**

Theorem 7. *For a Sunflower graph SF_n, the equitable defective number is given by*

$$b^k_{\chi_e}(SF_n) = \begin{cases} n + \lfloor \frac{n}{k} \rfloor & \text{if } k = 2 \\ \lceil \frac{n}{k} \rceil & \text{if } k = 3. \end{cases}$$

Proof. Let SF_n be the Sunflower graph with $2n+1$ vertices. Let v_0 denotes the central vertex of the graph. Let $v_1, v_2, \ldots, v_n$ be the rim vertices and $u_1, u_2, \ldots, u_n$ be the vertices connected to the rim vertices such that each u_i is adjacent to v_i and v_{i+1} . Let $c_1, c_2, \ldots, c_k$ be the available colors and $V_1, V_2, \ldots, V_k$ be the corresponding color classes. We observe that the equitable chromatic number of a sunflower graph is 4. Hence, in an equitable near proper coloring we consider only two cases say $k = 2$ and $k = 3$.

Case 1. $k = 2$.
Then, we have only two available colors say c_1 and c_2. Assign all u_i's and v_i's $(1 \le i \le n)$ alternatively with colors c_1 and c_2. Now each u_i contributes one bad edge resulting n bad edges. Then assign the central vertex v_0 with either c_1 or c_2. This results $\lfloor \frac{n}{k} \rfloor$ bad edges. Hence, the total number of bad edges is given by $n + \lfloor \frac{n}{k} \rfloor$.

Case 2. Let $k = 3$.
Here we consider three subcases as below.

Subcase 2.1. $n \equiv 0 \pmod{k}$.
We assign the three available colors to the rim vertices in a cyclic order. Now we

assign colors to the u_i's where $1 \leq i \leq n$ in such a way that each triangle with base as the rim edges can be properly colored using the three colors. Now assign the central vertex v_0 satisfying the equatability condition, we observe that we obtain $\frac{n}{k}$ bad edges among the spokes.

Subcase 2.2. $n \equiv 1 \pmod{k}$.
We assign colors to the vertices as in Subcase 2.1. Here we observe that we obtain $\lfloor \frac{n}{k} \rfloor$ bad edges among the spokes. Along with that we obtain another bad edge on the cycle since $n \equiv 1 \pmod{k}$. Hence, the resulting number of bad edges in this case is given by $\lfloor \frac{n}{k} \rfloor + 1 = \lceil \frac{n}{k} \rceil$.

Subcase 2.3. $n \equiv 2 \pmod{k}$.
We follow the same coloring pattern as in the above two subcases. In this case along with the $\lfloor \frac{n}{k} \rfloor$ bad edges among the spokes we get another u_iv_i bad edge and hence, the resulting number of bad edges is given by $\lceil \frac{n}{k} \rceil$. Combining the above three subcases we conclude that when $k = 3$, the equitable defective number is $\lceil \frac{n}{k} \rceil$. □

Figure 6 depicts the 3-equitable near proper coloring of sunflower graphs.

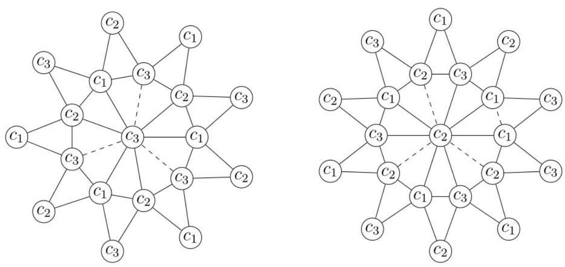

Figure 6. **Sunflower graphs with 3-equitable near proper coloring**

A *closed sunflower graph* denoted by CSF_n is obtained by joining the independent vertices of a sunflower graph SF_n which are not adjacent to its central vertex so that these vertices induces a cycle on n vertices. The following result provides the equitable near proper coloring of a closed sunflower graph.

Theorem 8. *For a closed sunflower graph CSF_n, the equitable defective number is given by*

$$b^k_{\chi_e}(CSF_n) = \begin{cases} \lceil \frac{3n}{2} \rceil & \text{if } k = 2 \\ \lceil \frac{n}{3} \rceil & \text{if } k = 3. \end{cases}$$

Proof. Let CSF_n be the closed sunflower graph with $2n+1$ vertices. We observe that the equitable chromatic number of a closed sunflower graph is 4. So in an equitable near proper coloring we consider two cases $k = 2$ and $k = 3$.

Case 1. When $k = 2$ and n is even, assign all u_i's and v_i's $(1 \leq i \leq n)$ alternatively with the available colors c_1 and c_2. Now each u_i contributes one bad edge resulting

n bad edges. Then assign v_0 with either c_1 or c_2. This results in exactly $\frac{n}{2}$ bad edges. Hence, we obtain $n + \frac{n}{2}$ bad edges. When $k = 2$ and n is odd, continue the same coloring pattern as above, we obtain$\lfloor \frac{n}{2} \rfloor$ bad edges among the spokes of the wheel and also we get $(n-1)$ number of u_iv_i bad edges. Along with this we obtain two bad edges one from each cycle since to properly color an odd cycle we require minimum three colors. Hence, the equitable defective number is given by $\lfloor \frac{n}{2} \rfloor + (n-1) + 2 = n + \lceil \frac{n}{2} \rceil = \lceil \frac{3n}{2} \rceil$.

Case 2. $k = 3$.
We consider three sub cases as below.

Subcase 2.1. $n \equiv 0 \pmod 3$.
As in Subcase 2.1 of Theorem 3.6 we obtain $\frac{n}{3}$ bad edges.

Subcase 2.2. $n \equiv 1 \pmod 3$.
We can color the rim vertices in such a way that $\lfloor \frac{n}{3} \rfloor - 2$ vertices receive color c_1 and the remaining rim vertices receive colors c_2 and c_3 equal number of times. Now, assign the central vertex v_0 with color c_1 resulting $\lfloor \frac{n}{3} \rfloor - 1$ bad edges among the spokes. Further, assign the colors to the vertices on the outer cycle considering the equitable manner we get $2u_iv_i$ bad edges. Hence, the equitable defective number is given by $\lfloor \frac{n}{3} \rfloor - 1 + 2 = \lceil \frac{n}{3} \rceil$.

Subcase 2.3. $n \equiv 2 \pmod 3$.
We observe that as in Subcase 2.3 of Theorem 7, we get $\lceil \frac{n}{3} \rceil$ bad edges. Combining the above three subcases, we conclude that when $k = 3$, the equitable defective number is given by $\lceil \frac{n}{3} \rceil$. □

A 3-equitable near proper coloring of some closed sunflower graphs is illustrated in Figure 7.

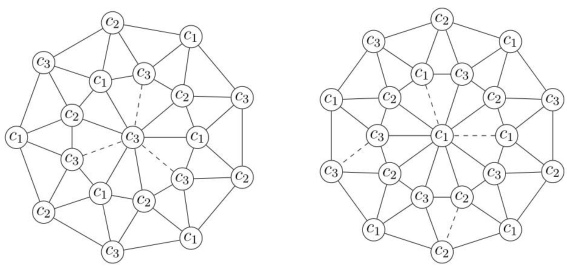

Figure 7. **Closed sunflower graphs with 3-equitable near proper coloring**

A *blossom graph* is the graph obtained by joining all vertices of the outer cycle of a closed sunflower graph CSF_n to its central vertex. The following theorem discusses the equitable near proper coloring of a blossom graph.

Theorem 9. *The equitable defective number of a blossom graph Bl_n is given by* *i. If* $k = 2$*, then* $b^k_{\chi_e}(Bl_n) = \begin{cases} 2n & \text{if } n \text{ is even} \\ 2n+1 & \text{if } n \text{ is odd.} \end{cases}$

ii. If $k = 3$*, then* $b^k_{\chi_e}(Bl_n) = \begin{cases} \frac{2n}{3} & \text{if } n \equiv 0 \pmod 3 \\ \frac{2n+1}{3} + 1 & \text{if } n \equiv 1 \pmod 3 \\ \lceil \frac{2n+1}{3} \rceil & \text{if } n \equiv 2 \pmod 3). \end{cases}$

iii. If $k \geq 4$*, then* $b^k_{\chi_e}(Bl_n) = \lfloor \frac{2n+1}{k} \rfloor - 1$.

Proof. Let Bl_n be the blossom graph with $2n+1$ vertices. Let v_0 denotes the central vertex and $v_1, v_2, \ldots, v_n$ be the rim vertices. Let $u_1, u_2, \ldots, u_n$ be the vertices of the outer cycle such that each u_i is adjacent with v_i and v_{i+1}. Also each u_i is adjacent with u_{i-1} and u_{i+1}. And in a blossom graph the central vertex v_0 is adjacent with all other vertices. We observe that the equitable chromatic number of blossom graph Bl_n is $n+1$. Hence, in an equitable near proper coloring we need to consider the cases from $k = 2$ to $k = n$.

Case 1. $k = 2$.
In this case, we need to consider the two subcases given below.

Subcase 1.1. $k = 2$ and n is even.
We have only two available colors c_1 and c_2. Thus, among the two color classes one of the color class contains $\lfloor \frac{2n+1}{2} \rfloor$ vertices and the other color class consists of $\lceil \frac{2n+1}{2} \rceil$ vertices. Since the central vertex v_0 is adjacent with all u_i's and v_i's where $1 \leq i \leq n$ we get n bad edges. Also each vertex in the outer cycle u_i contributes one $u_i - v_i$ bad edge and thus, we get n bad edges. Hence, we obtain $2n$ bad edges in this case.

Subcase 1.2. $k = 2$ and n is odd.
As in Subcase 1.1 we get $\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil = n$ bad edges. Along with that $(n-1)$ number of u_i's contribute one bad edge each and also on each cycle we get one bad edge since to properly color a cycle we need minimum three colors. Hence, the equitable defective number is $n + (n-1) + 2 = 2n + 1$.

Case 2. $k = 3$.
Here, we have to consider the following subcases.

Subcase 2.1. $n \equiv 0 \pmod 3$.
We see that $2n \equiv 0 \pmod 3$ and hence we assign the available three colors to $2n$ vertices on the cycles in an equitable manner. Now all the three color classes contain exactly $\frac{2n}{3}$ vertices. Now assign the central vertex v_0 with any of the three colors. We observe that we obtain $\frac{2n}{k}$ bad edges in this case (See Figure 8 for illustration).

Subcase 2.2. $n \equiv 1 \pmod 3$.
We observe that $2n + 1 \equiv 0 \pmod 3$. Hence, in an equitable near proper coloring each color class contains equal number of vertices. That is $\frac{2n+1}{3}$ vertices. By placing the central vertex v_0 in any of the three color classes we obtain $\frac{2n+1}{3} - 1$ bad edges. Along with that we get two u_iv_i bad edges where $1 \leq i \leq n$. Hence, the equitable defective number is $\frac{2n+1}{3} - 1 + 2 = \frac{2n+1}{3} + 1$ (See Figure 8 for illustration).

Subcase 2.3. $n \equiv 2 \pmod 3$.
In this case we have two color classes having $\lceil \frac{2n+1}{k} \rceil$ vertices and one color class contains $\lfloor \frac{2n+1}{k} \rfloor$ vertices. Now place the central vertex v_0 in the color class with cardinality $\lfloor \frac{2n+1}{3} \rfloor$ we obtain $\lfloor \frac{2n+1}{k} \rfloor - 1$ bad edges and also we obtain two $u_i - v_i$ bad edges. Hence, we have $\lfloor \frac{2n+1}{3} \rfloor - 1 + 2 = \lceil \frac{2n+1}{3} \rceil$ bad edges.

Case 3. $k \geq 4$.
When $4 \leq k \leq n$ we observe that r color classes contain $\lceil \frac{2n+1}{k} \rceil$ vertices and $(k - r)$ color classes contain $\lfloor \frac{2n+1}{k} \rfloor$ vertices, where $2n + 1 \equiv r \pmod k$. Now place the central vertex v_0 in the color class with $\lfloor \frac{2n+1}{k} \rfloor$ vertices, we end up with $\lfloor \frac{2n+1}{k} \rfloor - 1$ bad edges. $\square$

Figure 8 illustrates a 3-equitable near proper coloring of blossom graphs.

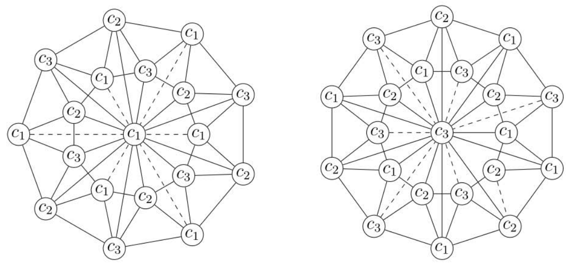

Figure 8. **Blossom graphs with 3-equitable near proper coloring**

4. Conclusion

In this paper, we introduced the notion of equitable near proper coloring of graphs and determined the equitable defective number of some graph classes. The results can be extended to many other graph classes, graph operations, graph products, derived graph classes and graph powers.

Acknowledgements. The authors would like to gratefully acknowledge the comments and suggestions of their co-researchers Ms. Merlin Thomas Ellumkalayil and Dr Johan Kok which improved the content of the paper significantly.

Conflict of interest. The authors declare that they have no conflict of interest.

Data Availability. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

References

[1] F. Harary, *Graph Theory*, Narosa Publ. House, New Delhi, 1996.

[2] J. Kok and N.K. Sudev, $\delta^{(k)}$- *colouring of cycle related graphs*, Adv. Stud. Contemp. Math. **32** (2022), no. 1, 113–120.

[3] S. Naduvath and M. Ellumkalayil, *A note on* $\delta^{(k)}$*-colouring of the cartesian product of some graphs*, Commun. Comb. Optim. **7** (2022), no. 1, 113–120 https://doi.org/10.22049/cco.2021.27114.1211.

[4] D.B. West, *Introduction to graph theory*, Prentice Hall of India, New Delhi, 2001.