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Research Article

On chromatic number and clique number in k-step Hamiltonian graphs

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Abstract: A graph G of order n is called $k-$step Hamiltonian for $k \geq 1$ if we can label the vertices of G as $v_1, v_2, \ldots, v_n$ such that $d(v_n, v_1) = d(v_i, v_{i+1}) = k$ for $i = 1, 2, \ldots, n-1$. The (vertex) chromatic number of a graph G is the minimum number of colors needed to color the vertices of G so that no pair of adjacent vertices receive the same color. The clique number of G is the maximum cardinality of a set of pairwise adjacent vertices in G. In this paper, we study the chromatic number and the clique number in $k-$step Hamiltonian graphs for $k \geq 2$. We present upper bounds for the chromatic number in $k-$step Hamiltonian graphs and give characterizations of graphs achieving the equality of the bounds. We also present an upper bound for the clique number in $k-$step Hamiltonian graphs and characterize graphs achieving equality of the bound.

Keywords: Hamiltonian graph, $k-$step Hamiltonian graph, chromatic number, clique number

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1. Introduction

Throughout this paper, $G = (V(G), E(G))$ is a simple graph with $V(G)$ as its vertex set and $E(G)$ as its edge set. The open *neighborhood* of a vertex $v \in V(G)$, denoted by $N_G(v)$(or just $N(v)$) is the set $\{u : uv \in E(G)\}$. The *degree* of a vertex v, $\deg_G(v)$(or just $\deg(v)$) is the number of neighbors of v in G, that is, $\deg(v) = |N_G(v)|$. We refer to $\delta(G)$ and $\Delta(G)$ as the minimum and maximum degree among all vertices of G, respectively. Also, a vertex $v \in V(G)$ is called a *pendant vertex* if $\deg(v) = 1$. Let K_n, C_n and P_n be a complete graph, a path and a cycle with n vertices, respectively. The *distance* between two vertices u and v in G, $d(u, v)$, is the minimum length among all paths between u and v and the maximum distance $d(u, v)$ among two vertices u, v of G is the *diameter* of G and is denoted by $\mathrm{diam}(G)$. For a set $A \subseteq V(G)$, $G[A]$ is the subgraph of G induced by A. A *circulant graph* $C_m(a_1, a_2, \ldots, a_k)$ with $0 < a_1 < a_2 < \ldots < a_k < \frac{m+1}{2}$ is a graph of order m with vertices $\{v_1, v_2, \ldots, v_m\}$ such that v_i is adjacent to v_{i+a_j} for all $a_j \in \{a_1, a_2, \ldots, a_k\}$, where the summation $i + a_j$ is taken modulo m. For a graph G, the *corona graph* of G, $\mathrm{cor}(G)$, is the graph obtained by adding a pendant vertex to every vertex of G. For other notation and terminology not defined here, we refer to [15].

A proper *vertex coloring* of a graph G is an assignment of colors to the vertices of G such that every pair of adjacent vertices receives different colors. The *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimum number of colors required in a proper vertex coloring of G. If G has a proper vertex coloring of k colors, then $\chi(G) \leq k$. The study of chromatic number of graph is an active area of research, see for example [6, 11–14]. A *clique* in a graph G is a set S of pairwise adjacent vertices and the number of vertices in the maximum clique is referred to as the *clique number* of G, denoted by $\omega(G)$.

A graph G is said to be *Hamiltonian* if G has a spanning cycle referred as a *Hamiltonian cycle*. Although the Hamiltonicity problem is a widely studied subject in graph theory, no exact characterization for the existence of the Hamiltonian cycle has been found. A good survey on the developments of Hamiltonicity problem can be found in [4]. The concept of Hamiltonicity has been extended by Lau et al. [9] to $k-$step Hamiltonicity as follows: For a graph G of order n, if we can arrange the vertices as $v_1, v_2, \ldots, v_n$ such that $d(v_n, v_1) = d(v_i, v_{i+1}) = k$ for $i = 1, 2, \ldots, n-1$ and $k \geq 1$, then we call G a $k-$step Hamiltonian (or just $k-$SH) graph with $v_1, v_2, \ldots, v_n, v_1$ as the $k-$step Hamiltonian (or just $k-$SH) walk of G. The $k-$step Hamiltonicity of some family of graphs including trees, tripartite graphs, cycles, grid graphs, torus graphs, cubic graphs and subdivision of cycles, have been studied, see [1, 2, 5, 7–10].

In this paper, we continue the study of $k-$SH graphs by proving bounds for the chromatic number and the clique number in $k-$SH graphs, where $k \geq 2$. In Section 2 we give a proof for the fact that a $k-$SH graph has at least $2k + 1$ vertices. In

Section 3, we present upper bounds for the chromatic number in $k-$SH graphs and give characterizations of graphs achieving the equality of the bounds. In Section 4, we present an upper bound for the clique number in $k-$SH graphs and characterize graphs achieving equality of the bound. We make use of the following known results.

Theorem 1 (Brooks' Theorem). *For every connected graph G other than an odd cycle or a complete graph, $\chi(G) \leq \Delta(G)$.*

Theorem 2 (Chartrand et al. [3]). *If G is a connected graph of order n and diameter d, then $\chi(G) \leq n - d + 1$.*

2. Preliminary

The following theorem has played an important role in several works on the subject of $k-$step Hamiltonian graphs, while the proof given in [8] does not have any argument for the bound $n \geq 2k + 1$.

Theorem 3 (Lau et al. [8]). *The cycle C_n for $n \geq 3$ is $k-$SH for $k \geq 2$ if and only if $n \geq 2k + 1$ and $gcd(n, k) = 1$.*

We provide in the following a proof for the above bound.

Theorem 4. *If G is a $k-$SH graph of order n for $k \geq 1$, then $n \geq 2k + 1$.*

Proof. The result is obvious if $k = 1$. Thus assume that $k \geq 2$. Let G be a k-SH graph of order n, and let $W : v_1, v_2, \ldots, v_n, v_1$ be a $k-$SH walk in G. Thus $d(v_n, v_1) = d(v_i, v_{i+1}) = k$ for each $i = 1, 2, \ldots, n - 1$. Since $d(v_1, v_2) = k$, let $x_0, x_1, \ldots, x_k$ be a shortest path between v_1 and v_2, where $x_0 = v_1$ and $x_k = v_2$. Clearly each of x_i, $(i = 1, 2, \ldots, k - 1)$ lies on W. We follow the walk W starting from v_1. We relabel the vertices $x_1, \ldots, x_{k-1}$ according to their place in W. Let the relabeled vertices be $x_{j_1}, x_{j_2}, \ldots, x_{j_{k-1}}$, where x_{j_r} is before x_{j_s} if $r < s$. For each $r \in \{1, 2, \ldots, k - 1\}$, x_{j_r} has two consecutive vertices on W namely x'_{j_r} and x''_{j_r}, and without loss of generality, assume that x'_{j_r} is on the left side of x_{j_r} and x''_{j_r} is on the right side of x_{j_r} in W. Clearly $\{x'_{j_1}, x''_{j_1}\} \cap \{v_1, v_2\} = \emptyset$. For each $r = 2, \ldots, k - 1$, $\{x'_{j_r}, x''_{j_r}\} \not\subseteq \{v_1, v_2, x_{j_s}, x'_{j_s}, x''_{j_s}\}$ for $s < r$. So for each $r = 2, \ldots, k - 1$, $\{x'_{j_r}, x''_{j_r}\} - \{v_1, v_2, x_{j_s}, x'_{j_s}, x''_{j_s} : s < r\} \neq \emptyset$. So $n \geq k + 1 + 2 + k - 2 = 2k + 1$. $\square$

For the sharpness of the bound in Theorem 4, consider the graph $G = C_{2k+1}$ for $k \geq 1$. By Theorem 3, G is $k-$SH.

3. Chromatic number

We begin with the following bound.

Theorem 5. *If G is a $k-SH$ graph of order n for $k \geq 2$, then $\chi(G) \leq \lceil \frac{n}{2} \rceil$. If equality holds, then $k = 2$.*

Proof. Let G be a $k-$SH graph of order n for $k \geq 2$. Without loss of generality, assume that $v_1, v_2, \ldots, v_n, v_1$ is a $k-$SH walk of G. Clearly, $d(v_i, v_{i+1}) = d(v_n, v_1) = k$ for $i = 1, 2, \ldots, n-1$. We define a vertex coloring c of G as follows: If n is even, then for $i = 0, 1, \ldots, \frac{n}{2} - 1$, we let $c(v_{2i+1}) = c(v_{2i+2}) = i + 1$. If n is odd, then for $i = 0, 1, \ldots, \frac{n-1}{2} - 1$, we let $c(v_{2i+1}) = c(v_{2i+2}) = i + 1$ and $c(v_n) = \frac{n-1}{2} + 1$. Clearly no pair of adjacent vertices receive the same color. The number of colors used in this proper vertex coloring c is $\lceil \frac{n}{2} \rceil$. Therefore, $\chi(G) \leq \lceil \frac{n}{2} \rceil$, as required.

Assume now that $\chi(G) = \lceil \frac{n}{2} \rceil$. Since G is $k-$SH for $k \geq 2$, clearly G is connected and G is not a complete graph. If G is an odd cycle, then $\chi(G) = 3$, that is, $\frac{n+1}{2} = 3$. This means $G = C_5$. By Theorem 3, C_5 is $2-$SH. Thus assume that G is not an odd cycle. Since G is not a complete graph or an odd cycle, by Brooks' Theorem, $\chi(G) \leq \Delta(G)$. Therefore we have $\lceil \frac{n}{2} \rceil \leq \Delta(G)$. Let v be a vertex of maximum degree, that is $\deg(v) = \Delta(G)$ and let $A = V(G) - N[v]$.

Assume that n is even. Then, $n \leq 2\Delta(G)$ and $|A| = n - \Delta(G) - 1 \leq \Delta(G) - 1$. Since G is $k-$SH, there exist two vertices $y, z \in A$ such that y, v, z are consecutive vertices in a $k-$SH walk of G. Let W be such $k-$SH walk and $W' = W - \{y, v, z\}$. Then it remains $\Delta(G)$ vertices from $N(v)$ in W' and some vertices of A. Thus, clearly there exist two consecutive vertices α, β in W with $\alpha, \beta \in N(v)$. Therefore, $k = 2$. The case n odd is similarly verified. □

We next show that for each $n \geq 5$, there exists a graph achieving equality of the bound in Theorem 5.

Proposition 1. *For each $n \geq 5$, there exists a $2-SH$ graph G of order n with $\chi(G) = \lceil \frac{n}{2} \rceil$.*

Proof. If G is a graph of order $n \leq 4$, then by Theorem 4, G is not 2-SH. Thus, we consider $n \geq 5$. Let $G = G_n$ be a graph obtained from the complete graph K_n, $(n \geq 5)$ with $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ by removing the edges of the Hamiltonian cycle $v_1, v_2, \ldots, v_n, v_1$. The graph G is connected since K_n, $n \geq 5$ has $\lfloor \frac{n-1}{2} \rfloor$ edge-disjoint Hamiltonian cycles. Note that for each $i = 1, 2, \ldots, n$, v_i is adjacent to every v_j for $j \notin \{i+1, i-1\}$ with the summations $i+1$ and $i-1$ are taken in modulo n and so $d(v_i, v_{i+1}) = 2$. Therefore, $v_1, v_2, \ldots, v_n, v_1$ is a $2-$SH walk of G and thus G is $2-$SH. By Theorem 5, $\chi(G) \leq \lceil \frac{n}{2} \rceil$.

We next prove that $\chi(G) \geq \lceil \frac{n}{2} \rceil$. For even n, clearly $\{v_{2i} : 1 \leq i \leq \frac{n}{2}\}$ is a clique. Therefore, $\chi(G) \geq \omega(G) \geq \frac{n}{2}$. Now, we prove by induction on n that for each odd $n \geq 5$, $\chi(G) \geq \lceil \frac{n}{2} \rceil = \frac{n+1}{2}$. For the base step assume that $n = 5$. Then $G = G_5 = C_5$. Clearly, $\chi(G) = 3 \geq \frac{n+1}{2}$. Assume the result holds for all odd n' with $5 \leq n' < n$.

Now, consider the graph $G = G_n$ for odd n. Let c be a proper vertex coloring of G. Since G is not a complete graph, there exist i, j such that $c(v_i) = c(v_j)$. Clearly, $j = i - 1$ or $j = i + 1$. Without loss of generality, we assume that $j = i + 1$. Now, remove v_i and v_{i+1} and also the edge $v_{i-1}v_{i+2}$ to obtain G_{n-2}. Clearly, the restriction of c on G_{n-2} is a proper vertex coloring of G_{n-2}. By the induction hypothesis, we have $|\{c(v) : v \in V(G_{n-2})\}| \geq \lceil \frac{n-2}{2} \rceil = \frac{n-1}{2}$. Therefore, for the graph $G = G_n$, we have $|\{c(v) : v \in V(G)\}| \geq 1 + \lceil \frac{n-2}{2} \rceil = \frac{n+1}{2}$, as desired. □

As another example of families of graphs achieving the equality in the bound in Theorem 5, consider the complete graph $K_{\frac{n}{2}}$ for even $n \geq 6$ with vertices $v_1, v_2, \ldots, v_{\frac{n}{2}}$. Let $G = \mathrm{cor}(K_{\frac{n}{2}})$ with $V(G) = V\left(K_{\frac{n}{2}}\right) \cup \{u_1, u_2, \ldots, u_{\frac{n}{2}}\}$ such that u_i is adjacent to v_i in G for $i = 1, 2, \ldots, \frac{n}{2}$. Clearly, $d(v_i, u_j) = 2$ for $i \neq j$. Also, it is clear that $\chi(G) = \chi\left(K_{\frac{n}{2}}\right) = \frac{n}{2}$ because we can color each vertex u_i for $i = 1, 2, \ldots, \frac{n}{2}$ with one of the color in the set $\{1, 2, \ldots, \frac{n}{2}\}$ that is different from the color of v_i. The $2-$SH walk is then given by the sequence of vertices $v_1, u_2, v_3, u_4, \ldots, u_{\frac{n}{2}-1}, v_{\frac{n}{2}}, u_1, v_2, \ldots, v_{\frac{n}{2}-1}, u_{\frac{n}{2}}, v_1$ when $\frac{n}{2}$ is odd and $u_1, v_2, u_3, v_4, \ldots, u_{\frac{n}{2}-1}, v_{\frac{n}{2}}, u_2, v_1, u_{\frac{n}{2}}, v_{\frac{n}{2}-1}, \ldots, u_4, v_3, u_1$ when $\frac{n}{2}$ is even.

Now, we propose the following problem.

Problem 1. Characterize all $2-$SH graphs G of order n with $\chi(G) = \lceil \frac{n}{2} \rceil$.

Theorem 6. *There is no forbidden induced subgraph characterization for $2-$SH graphs of order n with chromatic number $\lceil \frac{n}{2} \rceil$.*

Proof. Let G be a graph of order a. The result is obvious if $a \leq 2$. Thus, assume that $a \geq 3$. Let $b = 2\lceil \frac{a}{2} \rceil$. Then we form the graph $\mathrm{cor}(K_b)$. Identify each vertex of G with a pendant vertex of $\mathrm{cor}(K_b)$ to obtain a graph H of order $2b$. Since the adding edges are between pendant vertices of $\mathrm{cor}(K_b)$, clearly $\chi(H) = \chi(\mathrm{cor}(K_b)) = b$. As before, one can easily see that the graph $\mathrm{cor}(K_b)$ is $2-$SH and no two pendant vertices of $\mathrm{cor}(K_b)$ are consecutive in the $2-$SH walk. Therefore, a $2-$SH walk in $\mathrm{cor}(K_b)$ is also a $2-$SH walk in H and thus H is $2-$SH. Thus G is an induced subgraph of H, where H is a $2-$SH graph with $\chi(H) = \left\lceil \frac{|V(H)|}{2} \right\rceil$. □

Proposition 2. *The difference $\left\lceil \frac{|V(G)|}{2} \right\rceil - \chi(G)$ can be arbitrarily large in a 2-SH graph G.*

Proof. Let $n \geq 7$ be an odd integer, and $r = \frac{n+1}{2} - 3$. Consider the graph $G = C_{2(r+3)-1}$. By Theorem 3, G is $2-$SH. Then, $\left\lceil \frac{|V(G)|}{2} \right\rceil - \chi(G) = \left\lceil \frac{2(r+3)-1}{2} \right\rceil - 3 = r = \frac{n-5}{2}$. □

Theorem 7. *For each $k \geq 3$, there exists a $k-$SH graph G of order n with $\chi(G) = \lceil \frac{n}{k} \rceil$.*

Proof. Let $k \geq 3$ and consider the graph $G = C_{2k+1}$. By Theorem 3, G is $k-$SH. Since G is an odd cycle of order $2k+1$, then $\chi(G) = 3 = \lceil \frac{2k+1}{k} \rceil$. □

We propose the following conjecture.

Conjecture 1. If G is a $k-$SH graph of order n for $k \geq 1$, then $\chi(G) \leq \lceil \frac{n}{k} \rceil$.

We next present another upper bound for the chromatic number in a $k-$SH graph.

Theorem 8. *If G is a $k-$SH graph of order n for $k \geq 2$, then $\chi(G) \leq n-k$, with equality if and only if $k = 2$ and $G = C_5$.*

Proof. Let G be a $k-$SH graph of order n for $k \geq 2$. Clearly, G is connected. Since G is $k-$SH, we have $\mathrm{diam}(G) \geq k$ and thus, by Theorem 2, we have $\chi(G) \leq n-k+1$. If $\chi(G) = n-k+1$, then we obtain the contradiction $2k+1 \leq n \leq 2k-1$ from Theorems 4 and 5. Thus $\chi(G) \leq n-k$.

We next prove the equality part. Assume that $\chi(G) = n-k$. Since G is $k-$SH, it follows by Theorem 4 that $n \geq 2k+1$. If $k \geq 3$, then by Theorem 5, $\chi(G) < \lceil \frac{n}{2} \rceil$ and so $n < 2k+1$, a contradiction. Therefore, $k = 2$ and thus $n \geq 5$. Now, we have $\chi(G) = n-2 \leq \lceil \frac{n}{2} \rceil$. If n is even, then $n \leq 4$, a contradiction. If n is odd, then $n \leq 5$ and thus $n = 5$. Therefore, G is a $2-$SH graph of order 5. We can easily check that $G = C_5$.

The converse is clear. □

Let $C_m(1,2)$ be the circulant graph of order m. Abd Aziz et al. [1] obtained the following sufficient condition for the graph $C_m(1,2)$ to be $k-$SH.

Theorem 9 (Abd Aziz et al. [1]). *If $gcd(m, 2j-1) = 1$ for $m \geq 6$ and $2 \leq j \leq \lceil \frac{m-1}{4} \rceil$, then $C_m(1,2)$ is $j-$SH.*

They then gave a construction namely $B-$construction that produces a $(k+1)-$SH graph from any given $k-$SH graph G. The construction is as follows:

***B*-Construction.** Let G be a $k-$SH graph of order n for $k \geq 1$ with a given $k-$SH walk $v_1, v_2, \ldots, v_n, v_1$. Consider the graph $\mathrm{cor}(G)$ with the new n vertices $u_1, u_2, \ldots, u_n$ such that u_i is adjacent to v_i for $i = 1, 2, \ldots, n$. Then, the B-construction produces a graph $B(G)$ from G as follows:

(i) For odd n, $B(G) = \mathrm{cor}(G)$.

(ii) For even n, $B(G)$ is obtained from $\mathrm{cor}(G)$ by the following scheme:

Step 1. For an integer m, $m \geq 6$ and $k \leq \lceil \frac{m-1}{4} \rceil - 1$ with $\gcd(m, 2k+1) = 1$, the circulant graph $C_m(1,2)$ is $(k+1)-$SH by Theorem 9. Let $C_m^1(1,2)$ and $C_m^2(1,2)$ be two copies of $C_m(1,2)$. Without loss of generality, assume that

$u_{1,1}, u_{1,2}, \ldots, u_{1,m}, u_{1,1}$ (respectively $u_{2,1}, u_{2,2}, \ldots, u_{2,m}, u_{2,1}$) is a $(k+1)-$SH walk of $C_m^1(1,2)$ (respectively $C_m^2(1,2)$). Note that $d(u_{i,j}, u_{i,j+1}) = k+1$ for $i = 1,2$ and for $j = 1, 2, \ldots, m$, where the summation $j+1$ is taken in modulo m.

Step 2. Identify the vertices $u_{1,1}, u_{1,m}, u_{2,1}$ and $u_{2,m}$ to the vertices u_n, u_1, v_n and v_1, respectively.

The i-th iterated construction B of G, $B^i(G)$ for any $i \geq 1$ is defined recursively by $B^1(G) = B(G)$, $B^2(G) = B(B(G))$ and $B^i(G) = B(B^{i-1}(G))$ for $i \geq 2$.

Theorem 10 (Abd Aziz et al. [1]). *If G is $k-SH$ for $k \geq 1$, then $B(G)$ is $(k+1)-SH$.*

From the B-construction above, we can obtain the following two results.

Lemma 1. *If G is a $k-SH$ graph of order n with $\chi(G) \geq 5$ and H is a graph obtained from G by B-construction, then $\chi(H) = \chi(G)$.*

Proof. Let G be a k-SH graph of order n with $\chi(G) \geq 5$ and H be a graph obtained from G by applying the B-construction. If n is odd, then by the above construction, $H = B(G) = \mathrm{cor}(G)$. Clearly, $\chi(H) = \chi(G)$ because in H, we can color every vertex u_i for $i = 1, 2, \ldots, n$ with one of the color in the set $\{1, 2, \ldots, \chi(G)\}$ that is not the color of v_i.

If n is even, then $H = B(G)$ is obtained from $\mathrm{cor}(G)$ as described above. Since the circulant graph $C_m(1,2)$ contains a triangle, $\chi(C_m(1,2)) \geq 3$ and it is not difficult to see that for $m \geq 6$, $\chi(C_m(1,2)) = 3$ when $m \equiv 0 (\bmod 3)$ and $\chi(C_m(1,2)) = 4$ when $m \not\equiv 0 (\bmod 3)$. Note that in $B(G)$, we have $v_n = u_{2,1}$, $u_n = u_{1,1}$, $v_1 = u_{2,m}$ and $u_1 = u_{1,m}$. As before, we can color $\mathrm{cor}(G)$ with $\chi(G)$ colors. Let c be this coloring. Now, we can color $C_m^1(1,2)$ and $C_m^2(1,2)$ in such a way that the vertices $u_{1,1}$ and $u_{1,m}$ receive $c(u_n)$ and $c(u_1)$, respectively, and the vertices $u_{2,1}$ and $u_{2,m}$ receive $c(v_n)$ and $c(v_1)$, respectively. Therefore, we have $\chi(H) = \chi(G)$. □

Theorem 11. *For each $l \geq 5$, there exists a chain of graphs $H_2 \subseteq H_3 \subseteq H_4 \subseteq \ldots$ such that $\chi(H_i) = l$ for each $i \geq 2$, H_i is $i-SH$, and $\dfrac{n(H_{i+1})}{n(H_i)} > 2$.*

Proof. Let $l \geq 5$ and $H_2 = G_{2l}$ be the graph defined in the proof of Proposition 1. Then, we know that H_2 is $2-$SH with $\chi(H_2) = l$. Now, for each $i > 2$, let $H_i = B^{i-2}(H_2)$. By Theorem 10 and Lemma 1, for each $i > 2$, H_i is an $i-$SH graph with $\chi(H_i) = l$. Clearly, from the construction of H_i for $i \geq 2$, we have $\dfrac{n(H_{i+1})}{n(H_i)} > 2$. □

Corollary 1. *For each $k \geq 3$, there exists a k−SH graph G such that $\chi(G) < \frac{n(G)}{2^{k-1}}$.*

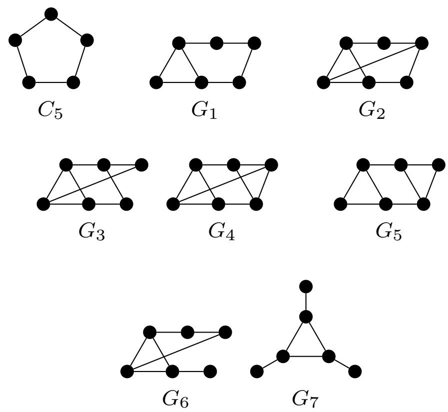

Figure 1. **The graphs in $\mathcal{F}$.**

4. Clique number

In this section, we present an upper bound for the clique number in a k−SH graph and characterize the graphs achieving the equality of the bound. Let $\mathcal{F}$ be the family of graphs shown in Figure 1.

Theorem 12. *If G is a k−SH graph of order n for $k \geq 2$, then $\omega(G) \leq n - k - 1$, with equality if and only if $k = 2$ and $G \in \mathcal{F}$.*

Proof. Let G be a k−SH graph of order n for $k \geq 2$ and S be a maximum clique in G. Let v_1 and v_2 be two consecutive vertices on a k−SH walk of G and assume that $P : x_0, x_1, \ldots, x_{k-1}, x_k$ is a shortest path in G from v_1 to v_2, where $v_1 = x_0$ and $v_2 = x_k$. Clearly, $|S \cap \{x_0, \ldots, x_k\}| \leq 2$, otherwise we will have a shorter v_1, v_2-path. Therefore, $\omega(G) \leq n - k + 1$. Suppose that $\omega(G) = n - k + 1$. Then, $|S \cap \{x_0, \ldots, x_k\}| = 2$. Let $x_i, x_{i+1} \in S \cap \{x_0, \ldots, x_k\}$. Without loss of generality, we can assume that $x_i \neq v_1$. Then, there is no vertex at distance k from x_i, a contradiction. Thus, $\omega(G) \leq n - k$.

Suppose that $\omega(G) = n - k$. Clearly $1 \leq |S \cap \{x_0, \ldots, x_k\}| \leq 2$. Suppose that $|S \cap \{x_0, \ldots, x_k\}| = 1$. Let $x_i \in S \cap \{x_0, \ldots, x_k\}$. If $x_i = v_1$, then v_2 is the only vertex at distance k from x_i in G, a contradiction. Thus $x_i \neq v_1$. Similarly, $x_i \neq v_2$. But then there is no vertex at distance k from x_i in G, a contradiction. Next suppose that $|S \cap \{x_0, \ldots, x_k\}| = 2$. Let $x_i, x_{i+1} \in S \cap \{x_0, \ldots, x_k\}$. Since $|S| = n - k$, clearly there exists a vertex $y \notin S \cup V(P)$. Without loss of generality, assume that $x_i \neq v_1$.

Then there exists at most one vertex at distance k from x_i (possibly $d(x_i, y) = k$), a contradiction.

We conclude that, $\omega(G) \leq n - k - 1$, as desired. We next prove the equality part. Assume that $\omega(G) = n-k-1$. Let $v_1, v_2, x_0, x_1, \ldots, x_k$ and S be as described above.

Claim 1. $k = 2$.
Proof of Claim 1. Suppose $k \geq 3$. Clearly $|S| \geq 2$. According to $|S \cap \{x_0, \ldots, x_k\}| \leq 2$, we have three possibilities.

Suppose that $|S \cap \{x_0, \ldots, x_k\}| = 0$. If there exists a vertex x_i for $i = 1, 2, \ldots, k-1$ such that x_i is adjacent to some vertex $y \in S$, then there is no vertex at distance k from x_i in G, a contradiction. Therefore, every vertex x_i for $i = 1, 2, \ldots, k-1$ has no neighbor in S. But then, v_2 is the only vertex at distance k from v_1 in G, a contradiction.

Next suppose that $|S \cap \{x_0, \ldots, x_k\}| = 1$. Let $x_i \in S \cap \{x_0, \ldots, x_k\}$. Since $|S| = n-k-1$, there exists a vertex $y \notin S \cup V(P)$. Suppose $x_i = v_1$. If $d(v_1, y) \neq k$, then v_2 is the only vertex at distance two from v_1 in G, a contradiction. Therefore, $d(v_1, y) = k$ and thus y is adjacent to x_{k-1}. Clearly, y is not adjacent to $x_0, x_1, \ldots, x_{k-2}$. But now, x_1 is at distance at most $k-1$ to other vertices of G, a contradiction. Therefore, $x_i \neq v_1$ and similarly $x_i \neq v_2$. But then, x_i is at distance k to at most one vertex of G (possibly $d(x_i, y) = k$), a contradiction.

Next, suppose that $|S \cap \{x_0, \ldots, x_k\}| = 2$. Let $x_i, x_{i+1} \in S \cap \{x_0, \ldots, x_k\}$. Since $|S| = n-k-1$, there exist two vertices $y, z \notin S \cup V(P)$. Assume that $x_i = v_1$. Since G is $k-$SH, there is another vertex at distance k from v_1 different from v_2. Clearly, that vertex is either y or z. Without loss of generality, assume that $d(v_1, y) = k$. Since $k \geq 3$, y has no neighbor in S. Also, it is clear that y is not adjacent to $x_1, x_2, \ldots, x_{k-2}$. Thus, y is adjacent to x_{k-1}. But now, x_{i+1} is at distance k to at most one vertex of G (possibly $d(x_{i+1}, z) = k$), a contradiction. Therefore, $x_i \neq v_1$. Similarly, $x_i \neq x_{k-1}$. Now, assume that $x_i = x_1$. Clearly $d(x_i, y) = d(x_i, z) = k$ since G is $k-$SH. Again, since $k \geq 3$, y and z have no neighbor in S. Also, y and z are not adjacent to $v_1, x_1, \ldots, x_{k-1}$. Thus, both y and z are adjacent to v_2. But now, v_2 is the only vertex at distance k from v_1, a contradiction. Therefore, $x_i \neq x_1$. Similarly, $x_i \neq x_{k-2}$ ($k \geq 4$). Next, consider $x_i \neq x_1$ or $x_i \neq x_{k-2}$ ($k \geq 5$). Again $d(x_i, y) = d(x_i, z) = k$. Clearly, y and z are adjacent to some vertex in $\{v_1, x_1, \ldots, x_{k-1}, v_2\} - \{x_i\}$. But, every $x_i, y-$path and every $x_i, z-$path created by joining y and z to any of those vertices has length at most $k-1$, a contradiction. So the proof of Claim 1 is complete. $\diamondsuit$

We now prove that $G \in \mathcal{F} = \{C_5, G_1, G_2, G_3, G_4, G_5, G_6, G_7\}$. Note that $|S| = \omega(G) = n-3$. Let v_1 and v_2 be two consecutive vertices on a $2-$SH walk of G and assume that $P : x_0, x_1, x_2$ is a shortest path in G from v_1 to v_2, where $v_1 = x_0$ and $v_2 = x_2$. Since $k = 2$, clearly $|S| \geq 2$. According to $|S \cap \{x_0, x_1, x_2\}| \leq 2$, we have three cases.

Case 1. $|S \cap \{x_0, x_1, x_2\}| = 0$.

Since G is $2-$SH, there exist two vertices y_1, y_2 at distance two from x_1 and clearly, $y_1, y_2 \in S$. Therefore, x_1 is not adjacent to y_1 and also not adjacent to y_2. Assume that $|S| \geq 3$ and consider $y_3 \in S$. Since v_1, v_2 are consecutive vertices in the $2-$SH walk, we have at least four other vertices x_1, y_1, y_2, y_3 in the $2-$SH walk of G. Clearly, two vertices in S will be consecutive in the $2-$SH walk, a contradiction. Thus, we assume that $|S| = 2$. Since G is $2-$SH, there exists a vertex at distance two from v_1 different from v_2 and clearly, that vertex is from S. Without loss of generality, let $d(v_1, y_2) = 2$. Thus v_1 is adjacent to y_1. Since $d(x_1, y_2) = 2$ and y_2 is not adjacent to v_1, clearly y_2 is adjacent to v_2. If y_1 is adjacent to v_2, then x_1 is the only vertex at distance two from y_1, a contradiction. Therefore, y_1 is not adjacent to v_2. Thus, we have $G = C_5$.

Case 2. $|S \cap \{x_0, x_1, x_2\}| = 1$.
Let $x_i \in S \cap \{x_0, x_1, x_2\}$. Since $|S| = n - 3$, there exists a vertex $y \notin S \cup V(P)$. If $x_1 \in S$, then there exists at most one vertex at distance two from x_1 in G (possibly $d(x_1, y) = 2$), a contradiction. Therefore, $x_1 \notin S$. Without loss of generality, assume that $x_0 \in S$. Clearly, $d(x_0, y) = 2$ since G is $2-$SH. Now, follow the $2-$SH walk starting from x_2, x_0, y. Assume that $|S| \geq 4$ and consider $y_1, y_2, y_3 \in S \setminus \{x_0\}$. The next vertex after y in the $2-$SH walk is either x_1 or some vertex in $S \setminus \{x_0\}$. Suppose the next vertex after y is x_1. Then the next vertex after x_1 should be in $S \setminus \{x_0\}$, say y_1. But then the next vertex after y_1 in the $2-$SH walk does not exist, a contradiction. Therefore, the next vertex after y in the $2-$SH walk is from $S \setminus \{x_0\}$, say y_1. Then the next vertex after y_1 should be x_1 and the next vertex after x_1 is from $S \setminus \{x_0, y_1\}$, say y_2. But again there is no next vertex after y_2 in the $2-$SH walk, a contradiction. Therefore $2 \leq |S| \leq 3$.

Assume that $|S| = 2$. Let $S = \{x_0, y_1\}$. Since G is $2-$SH, the two vertices at distance two from x_1 are y_1 and y. Therefore, x_1 is not adjacent to y_1 and also not adjacent to y. Since $d(x_0, y) = 2$, y is adjacent to both y_1 and x_2. And since $\omega(G) = 2$, x_2 is not adjacent to y_1. Thus we have $G = C_5$.

Next assume that $|S| = 3$. Let $S = \{x_0, y_1, y_2\}$.

Assume that y is adjacent to x_1. Then, the next vertex after y in the $2-$SH walk is either y_1 or y_2. Without loss of generality, assume that the next vertex after y is y_1. The next vertex after y_1 should be x_1. Thus y_1 is not adjacent to x_1. Then, the next vertex after x_1 should be y_2 and thus x_1 is not adjacent to y_2. Now, the vertices $x_2, x_0, y, y_1, x_1, y_2$ are consecutive in the $2-$SH walk. Assume that y is adjacent to x_2. Suppose y is not adjacent to y_2. Since $d(y, y_1) = 2$, y_1 is adjacent to x_2. If y_2 is adjacent to x_2, then x_0 is the only vertex at distance two from x_2 in G, a contradiction. Therefore y_2 is not adjacent to x_2 and thus we have $G = G_5$. Suppose next y is adjacent to y_2. If y_2 is adjacent to x_2, then x_1 is the only vertex at distance two from y_2 in G, a contradiction. Therefore y_2 is not adjacent to x_2. If y_1 is adjacent to x_2, then we have $G = G_4$, otherwise we have $G = G_5$. Assume next y is not adjacent to x_2. Since $d(y, y_1) = 2$, y is adjacent to y_2. If y_2 is adjacent to x_2, then x_1 is the only vertex at distance two from y_2 in G, a contradiction. Therefore y_2 is not adjacent to x_2. If y_1 is not adjacent to x_2, again x_1 is the only vertex at distance two from y_2 in G, a contradiction. Therefore y_1 is adjacent to x_2. Thus we have $G = G_3$.

Next, assume that y is not adjacent to x_1. Then y is adjacent to at least one of y_1 or y_2 since $d(x_0, y) = 2$. Without loss of generality, assume that y is adjacent to y_2. If x_1 is adjacent to y_2, then y_2 is at distance two to at most one vertex of G, a contradiction. Therefore, x_1 is not adjacent to y_2. If y_2 is adjacent to x_2, then x_1 is the only vertex at distance two from y_2 in G, a contradiction. Thus, y_2 is not adjacent to x_2. Suppose y is adjacent to y_1. Then clearly y is adjacent to x_2, for otherwise there is no next vertex after y in the $2-$SH walk. Then, x_1 is the next vertex after y in the $2-$SH walk. Then, the next vertex after x_1 is either y_1 or y_2. But then the next vertex in the $2-$SH walk does not exist, a contradiction. Therefore, y is not adjacent to y_1. Assume that y is not adjacent to x_2. Then the next vertex after y in the $2-$SH walk is y_1. Then the next vertex after y_1 is x_1. Thus y_1 is not adjacent to x_1. Then the next vertex after x_1 is y_2. If y_1 is not adjacent to x_2, then, x_1 is the only vertex at distance two from y_2 in G, a contradiction. Therefore, y_1 is adjacent to x_2. Thus we have $G = G_6$. Now, assume that y is adjacent to x_2. Then, the next vertex after y in the $2-$SH walk is either x_1 or y_1. Suppose the next vertex after y is x_1. Then, the next vertex after x_1 is either y_1 or y_2. But in either case, there exists no next vertex in the $2-$SH walk, a contradiction. Therefore, the next vertex after y is y_1. Then, the next vertex after y_1 must be x_1 and so y_1 is not adjacent to x_1. Then clearly the next vertex after x_1 is y_2. If y_1 is adjacent to x_2, then $G = G_2$, otherwise $G = G_1$.

Case 3. $|S \cap \{x_0, x_1, x_2\}| = 2$.
Without loss of generality, assume that $x_0, x_1 \in S$. Since $|S| = n - 3$, there exist two vertices $y_1, y_2 \notin S \cup V(P)$. Clearly, the vertices y_1, x_1, y_2 are consecutive in the $2-$SH walk. Also, x_0 is consecutive with one of y_1 or y_2 in the $2-$SH walk. Without loss of generality, assume that x_0 is consecutive with y_1. Now, follow the $2-$SH walk starting from x_2, x_0, y_1, x_1, y_2. Assume that $|S| \geq 4$ and consider $z_1, z_2 \in S \setminus \{x_0, x_1\}$. Then the next vertex after y_2 must be in $S \setminus \{x_0, x_1\}$. Without loss of generality, assume that the next vertex after y_2 in the $2-$SH walk is z_1. But then the next vertex after z_1 in the $2-$SH walk does not exist, a contradiction. Therefore $2 \leq |S| \leq 3$.

Assume that $|S| = 2$. Since $d(x_0, y_1) = d(x_1, y_1) = d(x_1, y_2) = 2$, it follows that x_0 is adjacent to y_2, x_2 is adjacent to y_1 and y_1 is adjacent to y_2. If x_2 is adjacent to y_2, then x_1 is the only vertex at distance two from y_2 in G, a contradiction. Therefore, x_2 is not adjacent to y_2. Thus we have $G = C_5$.

Next assume that $|S| = 3$. Let $S = \{x_0, x_1, z\}$. Then, clearly the next vertex after y_2 in the $2-$SH walk is z and so y_2 is not adjacent to z. If z is adjacent to x_2, then y_2 is the only vertex at distance two from z in G, a contradiction. Therefore, z is not adjacent to x_2.

Assume that y_1 is not adjacent to y_2. Since $d(x_0, y_1) = d(z, y_2) = 2$, clearly y_1 is adjacent to z and y_2 is adjacent to x_0. Suppose now y_2 is adjacent to x_2. If y_1 is adjacent to x_2, then $G = G_3$, otherwise $G = G_6$. Suppose next y_2 is not adjacent to x_2. If y_1 is adjacent to x_2, then $G = G_6$, otherwise $G = G_7$.

Assume next y_1 is adjacent to y_2. Now, look at the adjacency between x_0 and y_2. First, assume that x_0 is adjacent to y_2. Consider the adjacency between y_1 and z. Suppose y_1 is not adjacent to z. Since $d(x_1, y_1) = 2$, y_1 is adjacent to x_2. If x_2 is adjacent to y_2, then $G = G_5$, otherwise $G = G_1$. Suppose next y_1 is adjacent to z. Now, assume that y_1 is not adjacent to x_2. If x_2 is adjacent to y_2, then $G = G_2$, otherwise $G = G_6$. Next, assume that y_1 is adjacent to x_2. If x_2 is adjacent to y_2, then $G = G_4$, otherwise $G = G_3$. Now, assume that x_0 is not adjacent to y_2. Since $d(x_0, y_1) = d(x_1, y_2) = 2$, it follows that y_1 is adjacent to z and y_2 is adjacent to x_2. If y_1 is adjacent to x_2, then $G = G_5$, otherwise $G = G_1$.

For the converse, it is not difficult to show that any graph $G \in \mathcal{F}$ is $2-$SH with $\omega(G) = n - 3$. □

Conflict of interest. The authors declare that they have no conflict of interest.

Data Availability. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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